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Gough, John E.; Obreskov, Oleg; Smolyanov, Oleg

Published in:

Izvestiya: Mathematics

DOI:

[10.1070/IM2005v069n06ABEH002290](https://doi.org/10.1070/IM2005v069n06ABEH002290)

Publication date:

2005

Citation for published version (APA):

Gough, J. E., Obreskov, O., & Smolyanov, O. (2005). Randomized Hamiltonian Feynman integrals and Schrödinger-Ito stochastic equations. *Izvestiya: Mathematics*, 69(6), 1081-1098.
<https://doi.org/10.1070/IM2005v069n06ABEH002290>

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tel: +44 1970 62 2400

email: is@aber.ac.uk

Randomized Hamiltonian Feynman integrals and Schrödinger-Ito stochastic equations

J. Gough, O. O. Obrezkov, and O. G. Smolyanov

Abstract

In this paper, we consider stochastic Schrödinger equations with two-dimensional white noise. Such equations are used to describe the evolution of an open quantum system undergoing a process of continuous measurement. Representations are obtained for solutions of such equations using a generalization to the stochastic case of the classical construction of Feynman path integrals over trajectories in the phase space.

AMS 2000 Mathematics Subject Classification. 35A8, 35C15, 35C20, 35R15, 35R60, 46L45, 46N50, 60H15, 60H99, 81Q05, 81Q10, 81Q30, 81S40.

Introduction

In this paper we derive a representation for solutions of Schrödinger equations with white-noise type coefficients (stochastic Schrödinger-Ito equations) using randomized Feynman path integrals over trajectories in the phase space (Hamiltonian Feynman integrals). These equations describe the limit behavior of a quantum system observed at discrete instants of time under the condition that the precision of measurement and the time intervals between measurements are proportional and tend to zero. Continuous observation of a quantum system can be defined (informally) as the limit of these observations. This enables one to assume that Schrödinger-Ito equations describe the evolution of a quantum system [20], [22] undergoing a process of continuous measurement. At the same time, they describe the so-called Markov approximation for the evolution of an open quantum system¹ as opposed to the approximation given by Hudson Parthasarathy type quantum stochastic equations (see [12], [4], and [5]).

An equation describing the evolution of a quantum system subject to a process of continuous measurement for some fixed observable (the operator of multiplication by a coordinate for an appropriate realization of the Hilbert state space in the form of $L^2(\mathbb{R}^1)$) was first postulated in [25] to describe the spontaneous reduction of the wave function. It was derived independently by Belavkin [17] in the general situation. (In [25], use was made of the Hudson Parthasarathy

¹By an open quantum system we mean one which is part of a larger quantum system. The evolution of such a system is described not by the Schrödinger equation but by a master-equation implied by it. Since master-equations are integro-differential, it is rather difficult to investigate them, and therefore use is made of various approximate equations. The Schrödinger-Ito equation can be regarded as one of these.

quantum stochastic equations [26]; see also [18].) In the most important special case, this equation was derived independently by Diosi [21] at the physical level of rigor. For a derivation on the basis of the standard axioms of quantum mechanics, see [8] (also [3], [15], and [16]). In the same paper [8] was announced the stochastic Schrödinger equation with two-dimensional white noise. For a full derivation, see [32]. Apart from the local approach to the description of the behavior of a continuously observed quantum system resulting in an evolution equation that generalizes the Schrödinger equation and takes into account the interaction between the quantum system and the measurement equipment and the effect of this equipment on the state of the system, there also exists a global approach developed in [30] and [31]. For the global description of the process of continuous measurement in the latter approach, a linear propagator for the quantum system is introduced in the form of an heuristic Feynman path integral over phase-space trajectories.

For the relationship between these two approaches, see [15] and [16]. Representations for solutions of stochastic Schrödinger equations via Feynman path integrals were first obtained in [8], [2] and [16]. In these papers, the Feynman integral was defined as the analytic continuation of the integral with respect to the Wiener measure (see [9]), as a result of which the analytic constraints imposed on the initial condition and on the potential turned out to be rather restrictive. Furthermore, a representation for the solution of the stochastic Schrödinger equation with one-dimensional white noise was obtained in [13] under the condition that the potential and the initial condition in the Cauchy problem are the Fourier transforms of countably additive measures. In this approach, use is made of the definition of the Feynman path integral via Parseval's equation [7], [14], [10], [9]. We use Feynman's original definition [11], [23] of the functional integral via the limit of finitely multiple integrals and extend the approach based on Chernoff's theorem [19] to the probabilistic case. This approach was first used in [35] to obtain a representation for the solution of the heat equation on a compact Riemannian manifold and then in [34] for the representation of the solution of the Schrödinger equation using a Feynman path integral over trajectories in the phase space. In the present paper, we obtain a representation for the solution of the stochastic Schrödinger equation using randomized Feynman path integrals over trajectories in the phase space (randomized Hamiltonian Feynman integrals).

1 Pseudo-differential operators and the stochastic Schrödinger equation

Definition 1. For an arbitrary $\tau \in [0, 1]$ we define a map $\wedge: \mathcal{H} \mapsto \hat{\mathcal{H}}$ from the space of complex-valued functions on $\mathbb{R}^1 \times \mathbb{R}^1$ to the space of linear operators on $L^2(\mathbb{R})$ as follows: the action of the operator $\hat{\mathcal{H}}: D(\hat{\mathcal{H}}) \subset L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$

on a function φ is given by the formula (see [34] and [24])

$$\left(\hat{\mathcal{H}}\varphi\right)(q) = \frac{1}{2\pi} \lim_{z \rightarrow 0} \int_{-z}^z \int_{-z}^z \mathcal{H}((1-\tau)q + \tau q_0, p_0) e^{ip_0(q-q_0)} \varphi(q_0) dq_0 dp_0, \quad (1)$$

where the limit is taken in $L^2(\mathbb{R})$. Let $D(\hat{\mathcal{H}})$ be the set of all $\varphi \in L^2(\mathbb{R})$ such that $\hat{\mathcal{H}}\varphi$ exists.

The function $\mathcal{H}(\cdot, \cdot)$ is called the τ -symbol (or the classical Hamiltonian on the phase space $\mathbb{R}^1 \times \mathbb{R}^1$) for the pseudo-differential operator $\hat{\mathcal{H}}$. The map \wedge determines the τ -quantization.

The operator \wedge for $\tau=0$ is called the *operator of qp-quantization* and, for $\tau=1$, the *operator of pq-quantization*. This terminology is related to the fact that the pseudo-differential operator corresponding to the τ -symbol $H(p, q) = pq$ is equal to qp for $\tau=0$ and to pq for $\tau=1$. Here and henceforth, \hat{q} and \hat{p} are the coordinate and momentum operators given by the formulae

$$\begin{aligned} \hat{q} & : f \in \text{Dom}(\hat{q}) \subset L^2(\mathbb{R}) \mapsto [q \rightarrow qf(q)], \\ \hat{p} & : f \in \text{Dom}(\hat{p}) \subset L^2(\mathbb{R}) \mapsto [q \rightarrow -if'(q)], \end{aligned}$$

respectively.

The operation of τ -quantization for $\tau=1/2$ is called the *Weyl quantization*. We also note that if $H(q, p) = f(q) + g(p)$ for all $p, q \in \mathbb{R}$ and some functions f and g , then the result of quantization does not depend on the parameter τ .

In what follows, we consider the stochastic Schrödinger equation with two-dimensional white noise and interpret it as the Itô stochastic equation

$$\begin{aligned} d\varphi(t) & = \left[\left(-i\hat{\mathcal{H}} - \frac{\mu_1}{2} k^2(\hat{q}) - \frac{\mu_2}{2} h^2(\hat{p}) \right) (\varphi(t)) \right] dt \\ & \quad - \sqrt{\mu_1} k(\hat{q})(\varphi(t)) dW_1(t) - \sqrt{\mu_2} h(\hat{p})(\varphi(t)) dW_2(t), \quad (2) \end{aligned}$$

where $\hat{\mathcal{H}}$ is the (internal) Hamiltonian obtained for the observed system by the τ -quantization of the classical Hamiltonian \mathcal{H} and where $k(\hat{q})$ and $h(\hat{p})$ are the (non-commuting) differential operators corresponding to the real-valued symbols $(q, p) \mapsto k(q)$ and $(q, p) \mapsto h(p)$, respectively. Furthermore, W_1 and W_2 are independent standard Wiener processes and $\varphi(t) \in L^2(\mathbb{R})$ is a random (wave) function describing the evolution of mixed states for the observed system. Equation (2) describes the evolution of an open quantum system undergoing a continuous measurement of the observables $k(q)$ and $h(p)$. It was considered in [8] and [32] in the special case when $k(q) = q$, $h(p) = p$ and $H(q, p) = p^2/2 + V(q)$ for all $p, q \in \mathbb{R}$ and some real-valued function V .

For arbitrary functions h, k and H , equation (2) can be obtained by the method used in [8] and [32]. For this, it suffices to choose an appropriate realization of the Hilbert state space.

2 Feynman path integrals over trajectories in the phase space

Let E be a real vector space, $G \subset E^*$ the space of linear functionals on E (it is assumed that E and G satisfy the duality relation) and b a quadratic functional on G . Then, for all $a \in E$ and $\alpha \in \mathbb{C}$, the Feynman α -pseudo-measure with correlation functional b and mean value a is a generalized measure $\Phi_{b,a,\alpha}$ on E whose Fourier transform is given by the formula

$$(F\Phi_{b,a,\alpha})(g) = \exp \left\{ \frac{\alpha b(g)}{2} + \alpha g(a) \right\},$$

for all $g \in G$.

Let Q and P be infinite-dimensional locally convex spaces which, as vector spaces, satisfy the relations $Q = P^*$ and $P = Q^*$. If $E = Q \times P$ is the phase space and $G = P \times Q$, then the zero-mean Feynman α -pseudo-measure on E with correlation operator given by the formula $b(p, q) = 2p(q)$ for all $(p, q) \in G$ is called the *Hamiltonian Feynman integral* (or the Feynman path integral over trajectories in the phase space).

In this case, the value of the Feynman pseudo-measure on a given function $f : Q \times P \mapsto \mathbb{C}$ (the Feynman integral of f) can be defined using the limit of finitely multiple integrals. Let $\{Q_n\}_n$ and $\{P_n\}_n$ be sequences of finite-dimensional subspaces (with $\dim Q_n = \dim P_n = n$) in Q and P , respectively. The sequential Feynman integral of the function f over trajectories in the phase space is defined as the limit (if it exists) of the expressions

$$\left(\int_{Q_n \times P_n} e^{ip(q)} dq dp \right)^{-1} \int_{Q_n \times P_n} f(q, p) e^{ip(q)} dq dp$$

as $n \rightarrow \infty$, where the integration is with respect to an arbitrary Lebesgue measure.

For the representation of solutions of Schrödinger type equations by means of sequential Feynman integrals, special importance is attached to the case of the Feynman path integral over phase-space trajectories for some spaces Q and P of real-valued functions on the closed interval $[0, t]$. Let $z \in \mathbb{R}^2$, $\tau \in [0, 1]$ and $t > 0$. Let Q be the vector space of all real-valued functions on $[0, t]$ whose generalized derivatives are measures on $[0, t]$ and let $P = \{f \in C([0, t], \mathbb{R}) : f(t) = 0\}$. In this case, the duality relation between Q and P is given by the formula

$$\xi_p(\xi_q) = \int_0^t \xi_p(s) \xi_q'(s) ds$$

for all $\xi_p \in P$ and $\xi_q \in Q$, where $\xi_q'(s)$ denotes the measure equal to the generalized derivative of the function $\xi_q(\cdot)$.

Definition 2. The sequential Feynman integral

$$I(F, z) = \int_{Q \times P^0} F(\xi_q, \xi_p) \Phi^{\tau, t, z}(d\xi_q, d\xi_p)$$

of a function $F : Q \times P^0 \in \mathbb{C}$ over trajectories in the phase space $Q \times P^0$ is defined as the limit (if it exists) of the finitely multiple integrals

$$\begin{aligned} I_n(F, z) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} F(J_\tau(q_0, \dots, q_n), J_1(p_0, \dots, p_n)) \\ &\quad \times \exp \left\{ i \sum_{k=0}^{n-1} p_k (q_{k+1} - q_k) \right\} dq_0 dp_0 \cdots dq_{n-1} dp_{n-1} \quad (3) \end{aligned}$$

as $n \rightarrow \infty$. Here $p_n = 0$, $q_n = z$ and, for each $r \in [0, 1]$, the expression J_r is an (injective) map from \mathbb{R}^{n+1} onto the space consisting of functions that are constant on each of the intervals $\left(\frac{(k-1)t}{n}, \frac{kt}{n}\right)$, $k = 1, \dots, n$. Furthermore, very n -tuple $(q_0, \dots, q_n) \in \mathbb{R}^{n+1}$, the function $J_r(q_0, \dots, q_n)$ assumes the value $(1 - \tau)q_k + \tau q_{k-1}$ on the interval $\left(\frac{(k-1)t}{n}, \frac{kt}{n}\right)$, $k = 1, \dots, n$.

Remark 1 (see [9] and [34]). For $\tau = 0$, the sequential Feynman integral defined above can be interpreted as an integral over the space of right-continuous functions. The case $\tau = 1$ corresponds to the Hamiltonian Feynman integral over the space of left-continuous functions and, for $\tau \in (0, 1)$, the set of functions that forms the convex hull (with weights $(1 - \tau)$ and τ) of the above function spaces should be regarded as the phase space.

3 Feynman path integrals and representations of solutions of evolution equations

The existence of Feynman path integrals over phase-space trajectories (that is, the convergence of the corresponding finitely multiple integrals) was proved in some special cases in [1] and [6] using the finite-difference method. The definitions of the Feynman integral via the analytic continuation of the integral with respect to the Wiener measure and via Parseval's equation were applied in [9] in the representation of solutions of Schrödinger equations.

A new approach based on Chernoff's theorem was first applied in [34]. This made it possible to extend substantially the area of application of Feynman's formulae (that is, the representation of solutions of Schrödinger equations using Feynman integrals). It was noted in [34] that if $\varphi : \mathbb{R}_+ \mapsto L^2(\mathbb{R})$ is the solution of the Cauchy problem for the Schrödinger equation with initial datum φ_0 and Hamiltonian equal to the pseudo-differential operator with τ -symbol \mathcal{H} ,

$$\frac{d\varphi}{dt} = -i\hat{\mathcal{H}}\varphi, \quad (4)$$

then the relation

$$\varphi(t) \equiv e^{-it\hat{\mathcal{H}}}\varphi_0 = \lim_{n \rightarrow \infty} \left(\widehat{e^{-i\frac{t}{n}\mathcal{H}}} \right)^n \varphi_0 \quad (5)$$

is a representation for the solution of the Schrödinger equation via the Feynman path integral over trajectories in the phase space. Indeed, it can be verified that the right-hand side of (5) is a function whose value at a point z coincides with the limit of finitely multiple approximations of the Feynman integral

$$\int \exp \left\{ -i \int_0^t \mathcal{H}(\xi_q(s), \xi_p(s)) ds \right\} \varphi_0(\xi_q(0)) \Phi^{\tau, t, z}(d\xi_q, d\xi_p).$$

The formula (5) was proved in [34] for a rather wide class of Hamiltonians using Chernoff's theorem.

In the present paper, we extend this approach to stochastic differential equations of type (2). In this case, the corresponding Feynman formula changes. We shall show that if a random function $\varphi : \mathbb{R}_+ \mapsto L^2(\mathbb{R})$ is the solution of the Cauchy problem for the stochastic equation

$$d\varphi = \hat{A}\varphi dt + \hat{B}\varphi dW(t) \quad (6)$$

with initial datum φ , where \hat{A} and \hat{B} are pseudo-differential operators on $L^2(\mathbb{R})$ and W is the standard Wiener process, then, under certain conditions on A and B , the relation

$$\varphi(t) = \lim_{n \rightarrow \infty} \text{hat} \left(\exp \left\{ -\frac{tB^2}{2n} + B\Delta W_{k,n} + \frac{t}{n}A \right\} \right) \varphi_0 \quad (7)$$

holds, where $\Delta W_{k,n} = W(tk/n) - W(t(k-1)/n)$ for all $k, k = 1, \dots, n$, and $\text{hat}(M) = \hat{M}$. The additional factors $\exp \left\{ -\frac{tB^2}{2n} \right\}$ under the product sign correspond to Itô's formula. The right-hand side of (7) can be interpreted as a randomized Hamiltonian Feynman integral.

4 A stochastic analogue of Feynman's formula

The approach used in [34] to find solutions of non-stochastic Schrödinger equations is based on the construction of a family of operators approximating in the sense of Chernoff (see [36]) the resolvent operator semigroup for the Schrödinger equation. Let D_1 be an essential domain for the operator $\hat{\mathcal{H}}$, which means that the operator $\left(\hat{\mathcal{H}}, D \left(\hat{\mathcal{H}} \right) \right)$ is the closure of $\left(\hat{\mathcal{H}}, D_1 \right)$. The one-parameter operator family $\{S(t)\}_{t>0}$ approximates the semigroup with generator $-i\hat{\mathcal{H}}$ (by definition, it is precisely the resolvent semigroup of equation (2) for $\mu_1 = \mu_2 = 0$)

in the sense of Chernoff if the relation $S(t)f = f - it\hat{\mathcal{H}}f + o(t)$, $t \rightarrow 0$, holds for all $f \in D_1$. Then Chernoff's theorem implies that the relation

$$e^{-it\hat{\mathcal{H}}}\varphi_0 = \lim_{n \rightarrow \infty} \left(S\left(\frac{t}{n}\right) \right)^n \varphi_0$$

holds for all $t > 0$ and $\varphi_0 \in L^2(\mathbb{R})$. If $\widehat{e^{-it\hat{\mathcal{H}}}}$ is taken as $S(t)$, then we obtain the representation (5).

This method cannot be used for the representation of solutions in the case of stochastic Schrödinger type equations (with $\mu_1, \mu_2 > 0$ in (2)) since the solution is a random function in $L^2(\mathbb{R})$ and the family of operators approximating the resolvent family for equation (2) is non-deterministic. We also note that, since the right-hand side involves a random process, the semigroup property does not hold for the resolvent family corresponding to the stochastic equation. Nevertheless, if $\{T_r^s\}_{s \geq r \geq 0}$ is the resolvent family of (random) operators on $L^2(\mathbb{R})$ that corresponds to the Cauchy problem for the stochastic equation (2) (that is, T_r^s is defined by the formula $T_r^s\varphi_0 = \varphi(s)$ for arbitrary s and $r, s \geq r \geq 0$, where φ is the solution of (2) with $\varphi(r) = \varphi_0$, under the assumption that such a solution exists and is unique), then, since W_1 and W_2 are processes with independent increments, we have the following stochastic semigroup property: the distribution of T_r^s depends only on the difference $s - r$. This enables us to generalize the approach based on the notion of equivalence in the sense of Chernoff to the stochastic case.

In what follows, we consider equation (2) for the case $\tau = 0$, that is, the operator $\hat{\mathcal{H}}$ is obtained from \mathcal{H} by means of qp-quantization. It is assumed that the formula (5) holds for the Hamiltonian $\hat{\mathcal{H}}$. A sufficient condition [34] for the fulfillment of (5) is that the relation

$$H(q, p) = k_0(q) + h_0(p) + l(q, p) \quad (8)$$

hold for all $q, p \in \mathbb{R}$ and some real-valued functions $k_0, h_0, l \in L^2(\mathbb{R})$. In addition, we assume that $-i\hat{\mathcal{H}}$ is the generator of a strongly continuous operator semigroup.

We consider the family $\{Q_r^s\}_{0 \leq r \leq s}$ of random operators on $L^2(\mathbb{R})$ defined by the formula

$$(Q_r^s f)(q) = \exp \left\{ -\sqrt{\mu_1} k(q) (W_1(s) - W_1(r)) - \mu_1 (s - r) k^2(q) \right\} f(q), \quad q \in \mathbb{R}. \quad (9)$$

We claim that the function $s \mapsto Q_r^s f$ is the solution of the Cauchy problem for the equation

$$d\varphi(t) = -\frac{\mu_1}{2} k^2(\hat{q})(\varphi(t)) dt - \sqrt{\mu_1} k(\hat{q})(\varphi(t)) dW_1(t) \quad (10)$$

with initial condition $\varphi(r) = f$.

We take the total derivative of $Q_r^s f$ with respect to s ,

$$\begin{aligned} d_s(Q_r^s f)(q) &= \{-\sqrt{\mu_1 k(q)} dW_1(s) + \frac{1}{2}\mu_1 k^2(q) (dW_1(s))^2 \\ &\quad - \mu_1(s-r)k^2(q) ds\} (Q_r^s f)(q). \end{aligned}$$

According to Itô's formula $(dW_1(s))^2 = ds$, it follows that $s \mapsto Q_r^s f$ is the solution of equation (10). The fulfillment of the initial condition is obvious.

It can be similarly shown that if $\{P_r^s\}_{0 \leq r \leq s}$ is the operator family given by the formula

$$\begin{aligned} (P_r^s f)(q) &= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\{-\sqrt{\mu_2} h(p_0) (W_2(s) - W_2(r)) - \mu_2(s-r) h^2(p_0)\} \\ &\quad \times e^{ip_0(q-q_0)} dq_0 dp_0, \quad q \in \mathbb{R}, \end{aligned}$$

then $s \mapsto P_r^s f$ is the solution of the Cauchy problem for the equation

$$d\varphi(t) = -\frac{\mu_2}{2} h^2(\hat{p})(\varphi(t)) dt - \sqrt{\mu_2} h^2(\hat{p})(\varphi(t)) dW_2(t) \quad (11)$$

with initial condition $\varphi(r) = f$.

As shown in [34], if $Y_t = \widehat{e^{-it\mathcal{H}}}$ for all t , then the function $t \mapsto Y_t \varphi_0$ approximates the solution of the equation

$$d\varphi(t) = -i\hat{\mathcal{H}}\varphi(t) dt \quad (12)$$

in the sense of Chernoff.

We note that the sum of the right-hand sides of the (linear) equations (10), (12) and (13) coincides with the right-hand side of (2), and it is therefore to be expected that the operator family $\{U_r^s\}_{0 \leq r \leq s}$ determined by the formula $U_r^s = Q_r^s Y_{s-r} P_r^s$ approximates, in a sense, the resolvent family for (2).

The following relation will be called the *Feynman stochastic formula*:

$$T_0^t \varphi_0 = w - \lim_{n \rightarrow \infty} U_{(n-1)t/n}^t \cdots U_0^{t/n} \varphi_0, \quad (13)$$

where $w - \lim$ denotes a kind of convergence (defined below) for $L^2(\mathbb{R})$ -valued random variables. To see an analogy with the usual Feynman formula, it suffices to note that $t \mapsto T_0^t \varphi_0$ is the solution of equation (2) with initial datum φ_0 and that the right-hand side of (14) is a finite-dimensional approximation to the Feynman integral of a random function (see (7)). Finally, for $\mu_1 = \mu_2 = 0$, we obtain the standard Feynman formula in [34].

Definition 3. Let $\{\xi_n\}$ be random variables defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and having values in $L^2(\mathbb{R})$. A random variable ξ on the same space

is the w -limit of ξ_n as $n \rightarrow \infty$ if and only if $\mathbb{E} \|\xi - \xi_n\|_{L^2(\mathbb{R})}^2$ tends to zero as $n \rightarrow \infty$. Here \mathbb{E} denotes the operator of mathematical expectation in the probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

To prove formula (14), it is necessary to consider asymptotic properties of the operator U_r^s as $s - r \rightarrow 0$. In what follows, we use a well-known result of Lévy [28] (see also [29]) on the local smoothness of a Wiener process.

Lemma 1. Let $\{W(r)\}_{0 \leq r \leq 1}$ be the standard Wiener process. Then the relation

$$\overline{\lim}_{v \rightarrow 0} \sup_{r \in [0,1]} \frac{|W(r+v) - W(r)|}{\sqrt{-2v \ln v}} = 1$$

holds with probability 1.

It follows from Lemma 1 that the relation

$$\overline{\lim}_{v \rightarrow 0} \sup_{r \in [0,1]} \frac{|W(r+v) - W(r)|}{v^{\frac{1}{2}-\varepsilon}} = 0$$

holds for each $\varepsilon > 0$ with probability 1.

For $q \in \mathbb{R}$ we write $\varepsilon_2(q) = e^q - 1 - q - \frac{q^2}{2}$. It is clear that there is a constant $A_1 > 0$ such that the inequality $|\varepsilon_q(q)| \leq A_1 |q|^3$ holds as $q \rightarrow 0$. For brevity, in what follows we shall use the notation $\Delta W_j^{r,v} = W_j(r+v) - W_j(r)$ for $j = 1, 2$, and therefore the relation

$$\begin{aligned} & \exp \left\{ -\sqrt{\mu_1} k(q) (W_1(r+v) - W_1(r)) - \mu_1 v k^2(q) \right\} \\ &= 1 - \sqrt{\mu_1} k(q) \Delta W_1^{r,v} - \mu_1 v k^2(q) \\ &+ \frac{1}{2} \left[\sqrt{\mu_1} \Delta W_1^{r,v} k(q) + \mu_1 v k^2(q) \right]^2 + \varepsilon_2 \left(\sqrt{\mu_1} \Delta W_1^{r,v} k(q) + \mu_1 v k^2(q) \right) \\ \equiv & 1 - \sqrt{\mu_1} k(q) \Delta W_1^{r,v} - \mu_1 v k^2(q) + \frac{1}{2} \left[\sqrt{\mu_1} \Delta W_1^{r,v} k(q) + \mu_1 v k^2(q) \right]^2 + c(v, r, q) \end{aligned}$$

will hold for the function defining the action of the operator Q_r^{r+v} (see (9)).

It is assumed below that the function k is bounded ($\sup_{q \in \mathbb{R}} |k(q)| = K1 < \infty$) and belongs to $L^2(\mathbb{R})$. We claim that the relation $\mathbb{E} \|c(v, r, \cdot)\|_{L^2(\mathbb{R})} = O(v^{3/2})$ then holds uniformly with respect to r as $v \rightarrow 0$. By definition,

$$\begin{aligned} c(v, r, q) &= \frac{1}{2} \mu_1^2 v^2 k^4(q) + (\mu_1)^{3/2} v \Delta W_1^{r,v} k^3(q) \\ &+ \varepsilon_2 \left(\sqrt{\mu_1} \Delta W_1^{r,v} k(q) + \mu_1 v k^2(q) \right). \end{aligned}$$

Therefore we have the inequalities

$$\begin{aligned}
\|v^2 k^4\|_{L^2} &\leq \|k\|_{L^2} K_1^3 v^2 = O(v^{3/2}), \\
\mathbb{E} \|v \Delta W_1^{r,v} k^3\|_{L^2} &\leq \|k\|_{L^2} K_1^2 v \mathbb{E} |\Delta W_1^{r,v}| \\
&= \|k\|_{L^2} K_1^2 v \frac{1}{\sqrt{2\pi v}} \int_{\mathbb{R}} |z| e^{-z^2/(2v)} dz \\
&= \|k\|_{L^2} K_1^2 v \frac{2}{\sqrt{2\pi}} \sqrt{v} = O(v^{3/2}), \\
\mathbb{E} \|\varepsilon_2 (\sqrt{\mu_1} \Delta W_1^{r,v} k(q) + \mu_1 v k^2(q))\|_{L^2} &\leq \\
A_1 \mathbb{E} \|\varepsilon_2 (\sqrt{\mu_1} \Delta W_1^{r,v} k(q) + \mu_1 v k^2(q))^3\|_{L^2} &
\end{aligned}$$

The expression on the right-hand side of the third of these inequalities has an upper bound equal to the sum $\sum_{j=0}^3 b_j \mathbb{E} |\Delta W_1^{r,v}|^j \|k^j\|_{L^2} v^{3-j} \|k^{2(3-j)}\|_{L^2}$ with some constants b_j . Since $\mathbb{E} |\Delta W_1^{r,v}|^j$ is proportional to $v^{j/2}$ and we have $\|k^\alpha\|_{L^2} \leq \|k\|_{L^2} K_1^{\alpha-1}$ for all $\alpha > 1$, each of the terms in this sum is of order $O(v^{3/2})$. Consequently, $\mathbb{E} \|c(v, r, \cdot)\|_{L^2} = O(v^{3/2})$, that is, $\mathbb{E} \|c(v, r, \cdot)\|_{\mathcal{L}(L^2)} = O(v^{3/2})$, where $\mathcal{L}(L^2)$ is the space of continuous linear operators in $L^2(\mathbb{R})$. The uniformity of the estimates in r follows from the fact that the distribution of $\Delta W_1^{r,v}$ does not depend on r .

Everywhere below, \mathcal{Q}_r^s with arbitrary s and $r, s > r > 0$, denotes the minimal σ -algebra relative to which the random variables $\{W_1(t) - W_1(r)\}$ are measurable. We note that, since W_1 is a process with independent increments, the σ -algebras $\mathcal{Q}_{r_1}^{s_1}$ and $\mathcal{Q}_{r_2}^{s_2}$ are independent if the intervals (r_1, s_1) and (r_2, s_2) are disjoint.

Lemma 2. Let k be a bounded function belonging to $L^2(\mathbb{R})$. Then for arbitrary $r, v > 0$, we have the following asymptotic expansion of the random operator Q_r^{r+v} with respect to the parameter v :

$$Q_r^{r+v} = 1 - \sqrt{\mu_1} k(\hat{q}) \Delta W_1^{r,v} - \frac{\mu_1}{2} k^2(\hat{q}) + \frac{\mu_1}{2} v k^2(\hat{q}) \zeta^{r,v} + O(v^{3/2}), \quad v \rightarrow 0, \quad (14)$$

where $O(v^{3/2})$ denotes a random operator in $L^2(\mathbb{R})$ measurable relative to \mathcal{Q}_r^{r+v} such that the mathematical expectation of its norm is of order $O(v^{3/2})$ uniformly with respect to r . Furthermore, $\zeta^{r,v}$ is a zero-mean random variable measurable relative to \mathcal{Q}_r^{r+v} whose distribution does not depend on r or v , and 1 denotes the identity operator in $L^2(\mathbb{R})$.

Proof. It follows from the previous argument that

$$\begin{aligned}
Q_r^{r+v} &= 1 - \sqrt{\mu_1} k(\hat{q}) \Delta W_1^{r,v} - \mu_1 v k^2 + \frac{\mu_1}{2} k^2 (\Delta W_1^{r,v})^2 + O(v^{3/2}) \\
&= 1 - \sqrt{\mu_1} k(\hat{q}) \Delta W_1^{r,v} - \mu_1 v k^2 + \frac{\mu_1}{2} v k^2 \frac{(\Delta W_1^{r,v})^2 - v}{v} + O(v^{3/2}).
\end{aligned}$$

It remains to note that $\zeta^{r,v} = \frac{(\Delta W_1^{r,v})^2 - v}{v}$ is a zero-mean random variable distributed according to the law $\chi^2(1) - 1$.

An asymptotic expansion for P_r^{r+v} can be obtained in a similar way (everywhere below, \mathcal{P}_r^s denotes the minimal σ -algebra relative to which the random variables $\{W_2(t) - W_2(r)\}_{r \leq t \leq s}$ are measurable). If $h \in L^2(\mathbb{R})$ and $\sup_{q \in \mathbb{R}} |h(q)| < 1$, then the relation

$$P_r^{r+v} = 1 - \sqrt{\mu_2} h(\hat{p}) \Delta W_2^{r,v} - \frac{\mu_2}{2} v h^2(\hat{p}) + \frac{\mu_2}{2} v h^2(\hat{p}) \eta^{r,v} + O(v^{3/2}), \quad v \rightarrow 0, \quad (15)$$

holds, where $O(v^{3/2})$ denotes a random operator in $L^2(\mathbb{R})$ measurable relative to \mathcal{P}_r^{r+v} such that the mathematical expectation of its norm is of order $O(v^{3/2})$ uniformly with respect to r . Furthermore, $\eta^{r,v}$ is a zero-mean random variable measurable relative to \mathcal{P}_r^{r+v} whose distribution does not depend on r or v .

To prove (16), it suffices to note that the operator P_r^{r+v} can be written in the using the Fourier transform F in L^2 :

$$P_r^{r+v} = F \exp \left\{ -\sqrt{\mu_2} h \Delta W_2^{r,v} - \mu_2 v h^2 \right\} F^{-1}.$$

An expansion similar to (15) can be obtained for the exponential in this formula, and the application of the Fourier transform operator on the right and left of the exponential leads to the asymptotic expansion (16) since this operator is unitary and non-random.

The asymptotic expansions (15) and (16), the definition of the Hamiltonian symbol (8) and Lemma 1 imply an asymptotic expansion for the operator $U_r^{r+v} = Q_r^{r+v} \widehat{e^{-iv\mathcal{H}}} P_r^{r+v}$:

$$\begin{aligned} U_r^{r+v} &= 1 - \sqrt{\mu_1} k(\hat{q}) \Delta W_1^{r,v} - \frac{\mu_1}{2} v k^2(\hat{q}) + \frac{\mu_1}{2} v k^2(\hat{q}) \zeta^{r,v} - \sqrt{\mu_2} h(\hat{p}) \Delta W_2^{r,v} \\ &\quad - \frac{\mu_2}{2} v h^2(\hat{p}) + \frac{\mu_2}{2} v h^2(\hat{p}) \eta^{r,v} - iv \left(k_0(\hat{q}) + h_0(\hat{p}) + \hat{l} \right) \\ &\quad + \sqrt{\mu_1 \mu_2} k(\hat{q}) h(\hat{p}) \Delta W_1^{r,v} \Delta W_2^{r,v} + o(v^{3/2-\varepsilon}), \end{aligned}$$

where $\varepsilon > 0$ is an arbitrary number.

To derive a similar asymptotic expansion for the operator T_r^{r+v} , we need the auxiliary assertion below.

Lemma 3. Let A and B be operators in a Banach space X , let W denote the standard Wiener process and let $\varphi : \mathbb{R}_+ \mapsto X$ be a random function satisfying the Itô stochastic equation $d\varphi = A\varphi dt + B\varphi dW(t)$. Then for any $\varepsilon > 0$, the asymptotic expansion

$$\varphi(t) = \varphi(0) + B\varphi(0)W(t) + A\varphi(0)t + \frac{1}{2}B^2\varphi(0) \left(W(t)^2 - t \right) + o\left(t^{\frac{3}{2}-\varepsilon}\right), \quad t \rightarrow 0,$$

holds. Here $o\left(t^{\frac{3}{2}-\varepsilon}\right)$ denotes a random variable ξ_t such that the relation

$$\lim_{t \rightarrow 0} \frac{|\xi_t|}{t^{\frac{3}{2}-\varepsilon}} = 0$$

holds with probability 1.

Proof. By the definition of a solution of the Itô stochastic differential equation, the function φ satisfies the integral relation

$$\varphi(t) - \varphi(0) = \int_0^t A\varphi(r) dr + \int_0^t B\varphi(r) dW(r), \quad (16)$$

where $\int_0^t B\varphi(r) dW(r)$ is the Itô stochastic integral. We shall find an asymptotic expansion for $\varphi(t)$ as $t \rightarrow 0$ using the method of indeterminate coefficients. Let

$$\varphi(t) = \varphi(0) + \beta W(t) + \alpha t + \frac{1}{2}\gamma \left(W(t)^2 - t\right) + o\left(t^{\frac{3}{2}-\varepsilon}\right).$$

We substitute the right-hand side of this equation into the right- and left-hand sides of equation (18). Since we are interested in an asymptotic expansion to within $o\left(t^{\frac{3}{2}-\varepsilon}\right)$, we can take into account only the constant in the asymptotic expansion obtained when $\varphi(r)$ is substituted in $\int_0^t A\varphi(r) dr$ (by Lemma 1, the other terms are of order $o\left(r^{\frac{1}{2}-\varepsilon}\right)$). Consequently,

$$\int_0^t A\varphi(r) dr = A\varphi(0)t + o\left(t^{\frac{3}{2}-\varepsilon}\right).$$

We similarly conclude that

$$\begin{aligned} \int_0^t B\varphi(r) dW(r) &= B\varphi(0)W(t) + \int_0^t B\beta W(r) dW(r) + o\left(t^{\frac{3}{2}-\varepsilon}\right) \\ &= B\varphi(0)W(t) + \frac{1}{2}B\beta \left[W(t)^2 - 1\right] + o\left(t^{\frac{3}{2}-\varepsilon}\right). \end{aligned}$$

Equating the left- and right-hand sides of (18), we arrive at the formula

$$\begin{aligned} &\beta W(t) + \alpha t + \frac{\gamma}{2} \left[W(t)^2 - 1\right] + o\left(t^{\frac{3}{2}-\varepsilon}\right) \\ &= A\varphi(0)t + B\varphi(0)W(t) + \frac{1}{2}B\beta \left[W(t)^2 - 1\right] + o\left(t^{\frac{3}{2}-\varepsilon}\right), \end{aligned}$$

whence we obtain the relations $\alpha = A\varphi(0)$, $\beta = B\varphi(0)$ and $\gamma = B\beta$, and the assertion of the lemma follows.

Using Lemma 3 in the case of equation (2), we derive an asymptotic expansion for the operator T_r^{r+v} :

$$\begin{aligned}
T_r^{r+v} &= 1 - \sqrt{\mu_1} k(\hat{q}) \Delta W_1^{r,v} - \frac{\mu_1}{2} v k^2(\hat{q}) \\
&\quad + \frac{\mu_1}{2} v k^2(\hat{q}) \zeta^{r,v} - \sqrt{\mu_2} h(\hat{p}) \Delta W_2^{r,v} - \frac{\mu_2}{2} v h^2(\hat{p}) \\
&\quad + \frac{\mu_2}{2} v h^2(\hat{p}) \eta^{r,v} - i v \left(k_0(\hat{q}) + h_0(\hat{p}) + \hat{l} \right) + o\left(v^{3/2-\varepsilon}\right),
\end{aligned}$$

where, as usual, $\zeta^{r,v} = \frac{(\Delta W_1^{r,v})^2 - v}{v}$ and $\eta^{r,v} = \frac{(\Delta W_2^{r,v})^2 - v}{v}$. We note that the direct application of Lemma 3 proves the expansion (19) only for $r = 0$. However, since W_1 and W_2 are Wiener processes, the distribution of the operator-valued random variable T_r^{r+v} does not depend on r , and the resulting expansion holds for any r . Moreover, the expression $o(v^{3/2-\varepsilon})$ on the right-hand side of (19) is an operator in $L^2(\mathbb{R})$ such that the mathematical expectation of its norm is of order $o(v^{3/2-\varepsilon})$ uniformly with respect to r .

As can be seen from the asymptotic expansions (19) and (17), the difference between T_r^{r+v} and U_r^{r+v} is equal to the term $\sqrt{\mu_1 \mu_2} k(\hat{q}) h(\hat{p}) \Delta W_1^{r,v} \Delta W_2^{r,v} + o(v^{3/2-\varepsilon})$ which plays a key role in the proof of the randomized Feynman formula (14). We also note that, in its standard form, the Chernoff theorem used in the derivation of Feynman type formulae involves, apart from the approximation requirement for the operator family, the condition that the norm $\|S(t)\|$ of the operator in the approximating family should not exceed e^{Ct} , where $C > 0$ is some constant (common to all values of $t > 0$).

It turns out that a similar technical constraint is also necessary for proving the randomized Feynman formula.

Lemma 4. Let h and k be bounded functions belonging to $L^2(\mathbb{R})$. Let the map $t \mapsto \mathbb{E} \|T_0^t\|$ be differentiable at zero. Then there is a constant $C > 0$ such that, for all $r, v > 0$, the estimates

$$\mathbb{E} \|U_r^{r+v}\| \leq e^{Cv}, \mathbb{E} \|T_r^{r+v}\| \leq e^{Cv}$$

hold, where $\|\cdot\|$ denotes the norm on the space of linear operators in $L^2(\mathbb{R})$.

Proof. It suffices to consider the case $r = 0$ (the estimates for arbitrary values of $r > 0$ will be the same as those for $r = 0$ since the distributions of the operator-valued random variables under consideration depend only on the difference between the superscript and the subscript). We first derive an estimate for U_0^v . If $K_1 = \sup_{q \in \mathbb{R}} |k(q)| < \infty$, then $\|Q_0^v\| \leq e^{K_1 \sqrt{\mu_1} \Delta W_1(v)}$. Therefore $\mathbb{E} \|Q_0^v\| \leq e^{\frac{1}{2} K_1^2 \mu_1 v}$.

Similarly, if $K_2 = \sup_{q \in \mathbb{R}} |h(q)| < \infty$, then $\mathbb{E} \|P_0^v\| \leq e^{\frac{1}{2} K_2^2 \mu_2 v}$. To see this, it suffices to note that, by the definition of the quantization operator, we have

$$P_0^v = F^{-1} \exp \left\{ \sqrt{\mu_2} h(\cdot) W_2(v) - \mu_2 v h^2(\cdot) \right\} F,$$

where $\exp \left\{ \sqrt{\mu_2} h(\cdot) W_2(v) - \mu_2 v h^2(\cdot) \right\}$ is the operator of multiplication by the corresponding function in $L^2(\mathbb{R})$ and F is the Fourier transform operator in

$L^2(\mathbb{R})$. Then the inequality $\mathbb{E} \|P_0^v\| \leq e^{\frac{1}{2}K_2^2\mu_2v}$ follows from the fact that the Fourier transform operator is unitary along with the estimate for the mathematical expectation of the norm of the operator $\exp\{\sqrt{\mu_2}h(\cdot)W_2(v) - \mu_2vh^2(\cdot)\}$.

We note that, since the operator-valued random variable Q_0^v (P_0^v) depends only on the realization of the process W_1 (W_2) respectively and the processes W_1 and W_2 are independent, we have the relation

$$\mathbb{E} \|U_0^v\| \leq \mathbb{E} \|Q_0^v\| \left\| \widehat{e^{-iv\mathcal{H}}} \right\| \mathbb{E} \|P_0^v\| \leq e^{K_1^2\mu_1v} e^{K_2^2\mu_2v}$$

We shall now prove a similar estimate for the operator T_0^v . Since $\{T_r^s\}_{0 \leq r \leq s}$ is the resolvent family of random operators that corresponds to equation (2), the relation $T_0^v = T_r^v T_0^r$ holds for all $r \in (0, v)$. The random variables T_r^v and T_0^r are independent, and therefore the inequality

$$\mathbb{E} \|T_0^v\| \leq \mathbb{E} \|T_r^v\| \mathbb{E} \|T_0^r\|$$

holds. Since the distribution of T_r^v depends only on $v - r$, the function f determined by the formula $f(v - r) = \mathbb{E} \|T_0^v\|$ for all r and v , $0 \leq r \leq v$, is well defined.

This inequality can be rewritten in terms of f :

$$f(v) \leq f(r)f(v - r).$$

In a similar way, the inequality $f(v) \leq (f(v/n))^n$ can be established for all $v > 0$ and $n \in \mathbb{N}$. It follows from the hypotheses of the lemma that $f'(0)$ exists and $f'(0) < 1$. Consequently, the relation

$$f(v) \leq \lim_{n \rightarrow \infty} (f(v/n))^n = e^{f'(0)v}$$

holds. The lemma is proved.

Theorem 1 (the randomized Feynman formula). Let h and k be bounded functions belonging to $L^2(\mathbb{R})$. Let the map $t \mapsto \mathbb{E} \|T_0^v\|$ be differentiable at zero. Then, for an arbitrary function $\varphi_0 \in L^2(\mathbb{R})$ and $t > 0$, the relation

$$T_0^t \varphi_0 = w - \lim_{n \rightarrow \infty} U_{(n-1)t/n}^t U_{(n-2)t/n}^{(n-1)t/n} \cdots U_0^{t/n} \varphi_0$$

holds.

Proof. Everywhere below, $t > 0$ is arbitrary and fixed and $v = t/n$. We write the difference between the operators on the left- and right-hand sides of (14) in the form

$$\begin{aligned} T_0^{nv} - U_{(n-1)v}^{nv} U_{(n-2)v}^{(n-1)v} \cdots U_0^v &= \Pi_{j=n}^1 T_{(j-1)v}^{jv} - \Pi_{j=n}^1 U_{(j-1)v}^{jv} \\ &= \frac{t}{n} \sum_{j=n}^1 \Pi_{k=n}^{j+1} T_{(k-1)v}^{kv} \frac{T_{(j-1)v}^{jv} - U_{(j-1)v}^{jv}}{v} \Pi_{k=j-1}^1 U_{(k-1)v}^{kv}. \end{aligned}$$

For all $v > 0$ and $j = 1, \dots, n$ (of course, v and n are related by the formula $nv = t$), we define $L^2(\mathbb{R})$ -valued random variables ξ_j^v by putting

$$\xi_j^v = \prod_{k=n}^{j+1} T_{(k-1)v}^{kv} \frac{T_{(j-1)v}^{jv} - U_{(j-1)v}^{jv}}{v} \prod_{k=j-1}^1 U_{(k-1)v}^{kv} \varphi_0.$$

Then the proof of the randomized Feynman formula (14) becomes equivalent to the verification of the fact that

$$\mathbb{E} \left\| \frac{1}{n} \sum_{j=1}^n \xi_j^r \right\|_{L^2(\mathbb{R})}^2 \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (17)$$

To prove (20), it suffices to show that the expressions $\mathbb{E} \|\xi_j^r\|_{L^2(\mathbb{R})}^2$ are uniformly bounded with respect to j in some neighbourhood of the point $v = 0$ and that the limit relation $\mathbb{E} (\xi_j^v, \xi_k^v)_{L^2(\mathbb{R})} \rightarrow 0$ as $v \rightarrow 0$ holds uniformly with respect to all $j \neq k$. For $r, v > 0$, let \mathcal{Y}_r^{r+v} be the minimal $\frac{3}{4}$ -algebra containing \mathcal{Q}_r^{r+v} and \mathcal{P}_r^{r+v} .

Furthermore, for an arbitrary random variable ξ , we denote by $\mathbb{E}_r^{r+v} \xi$ the conditional mathematical expectation $\mathbb{E} [\xi | \mathcal{Y}_0^r \cup \mathcal{Y}_{r+v}^t]$. Thus, \mathbb{E}_r^{r+v} is the averaging operator over the σ -algebra \mathcal{Y}_r^{r+v} .

Consider the expression

$$\begin{aligned} \|\xi_j^v\|_{L^2(\mathbb{R})}^2 &= \int_{\mathbb{R}} \left(\prod_{k=n}^{j+1} T_{(k-1)v}^{kv} \frac{T_{(j-1)v}^{jv} - U_{(j-1)v}^{jv}}{v} \prod_{k=j-1}^1 U_{(k-1)v}^{kv} \varphi_0 \right) (q) \\ &\quad \times \overline{\left(\prod_{k=n}^{j+1} T_{(k-1)v}^{kv} \frac{T_{(j-1)v}^{jv} - U_{(j-1)v}^{jv}}{v} \prod_{k=j-1}^1 U_{(k-1)v}^{kv} \varphi_0 \right)} (q) dq. \end{aligned}$$

It follows from the asymptotic expansions (19) and (17) that the relation

$$\frac{T_{(j-1)v}^{jv} - U_{(j-1)v}^{jv}}{v} = \frac{1}{v} \sqrt{\mu_1 \mu_2} k(\hat{q}) h(\hat{p}) \Delta W_1^{(j-1)v, v} \Delta W_2^{(j-1)v, v} + o\left(v^{\frac{1}{2}-\varepsilon}\right)$$

holds. This relation and the estimates for the mathematical expectations of the operator norms in Lemma 4 imply the inequalities

$$\begin{aligned} \|\xi_j^v\|_{L^2(\mathbb{R})}^2 &= \mathbb{E} \left(\mathbb{E}_{(j-1)v}^{jv} \|\xi_j^v\|_{L^2(\mathbb{R})}^2 \right) \\ &\leq e^{Cv(n-j)} [\mu_1 \mu_2 \|k\|^2 \|h\|^2] \mathbb{E} \left(\Delta W_1^{(j-1)v, v} / \sqrt{v} \right)^2 \\ &\quad \times \mathbb{E} \left(\Delta W_2^{(j-1)v, v} / \sqrt{v} \right)^2 + o(v^{1-2\varepsilon}) e^{2Cv(j-1)} \|\varphi_0\|^2 \\ &\leq e^{Ct} [\mu_1 \mu_2 \|k\|^2 \|h\|^2 + o(v^{1-2\varepsilon})] \|\varphi_0\|^2. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the second of these inequalities provides the desired estimate.

It remains to show that the covariances $\mathbb{E}(\xi_j^v, \xi_k^v)_{L^2(\mathbb{R})}$ tend to zero as $v \rightarrow 0$ uniformly with respect to $j \neq l$. Consider the case $j = n, l = n - 1$, for which the relation

$$\begin{aligned} (\xi_j^v, \xi_k^v)_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left(\frac{T_{(j-1)v}^{jv} - U_{(j-1)v}^{jv} \prod_{k=j-1}^1 U_{(k-1)v}^{kv}}{v} \right) \varphi_0(q) \\ &\quad \times \left(\frac{T_{(n-1)v}^{nv} - U_{(n-1)v}^{nv} \prod_{k=n-2}^1 U_{(k-1)v}^{kv}}{v} \right) \varphi_0(q) dq \end{aligned}$$

holds.

We first find $\mathbb{E}_{(n-1)v}^{nv}(\xi_n^v, \xi_{n-1}^v)_{L^2(\mathbb{R})}$, that is, we carry out averaging over the σ -algebra $\mathcal{Y}_{(n-1)v}^{nv}$. The scalar product (ξ_n^v, ξ_{n-1}^v) contains only two random variables depending on $\mathcal{Y}_{(n-1)v}^{nv}$, namely, $\frac{T_{(n-1)v}^{nv} - U_{(n-1)v}^{nv}}{v}$ and $T_{(n-1)v}^{nv}$.

If $\psi = \prod_{k=j-1}^1 U_{(k-1)v}^{kv}$, then the asymptotic expansions for these two random variables imply the relation

$$\begin{aligned} (\xi_n^v, \xi_{n-1}^v)_{L^2(\mathbb{R})} &= ([1 - \sqrt{\mu_1} k(\hat{q}) \Delta W_1^{r,v} - \frac{\mu_1}{2} vk^2(\hat{q}) + \frac{\mu_1}{2} vk^2(\hat{q}) \zeta^{r,v} \\ &\quad - \sqrt{\mu_2} h(\hat{p}) \Delta W_2^{r,v} - \frac{\mu_2}{2} vh^2(\hat{p}) + \frac{\mu_2}{2} vh^2(\hat{p}) \eta^{r,v} \\ &\quad - iv(k_0(\hat{q}) + h_0(\hat{p}) + \hat{l}) + o(v^{\frac{3}{2}-\varepsilon})] \frac{T_{(n-1)v}^{nv} - U_{(n-1)v}^{nv}}{v} \psi, \\ &\quad \left[\frac{1}{v} \sqrt{\mu_1 \mu_2} k(\hat{q}) h(\hat{p}) \Delta W_1^{(j-1)v,v} \Delta W_2^{(j-1)v,v} + o(v^{\frac{1}{2}-\varepsilon}) \right] U_{(n-2)v}^{(n-1)v} \psi]_{L^2(\mathbb{R})}. \end{aligned}$$

To calculate $\mathbb{E}_{(n-1)v}^{nv}(\xi_n^v, \xi_{n-1}^v)_{L^2(\mathbb{R})}$, it is necessary to remove the two pairs of square brackets in the above relation. It should be noted that the mathematical expectations of some of the terms resulting from the removal of the square brackets are zero. Therefore

$$\begin{aligned} \mathbb{E}_{(n-1)v}^{nv}(\xi_n^v, \xi_{n-1}^v)_{L^2(\mathbb{R})} &= \mathbb{E}_{(n-1)v}^{nv}([1 - \sqrt{\mu_1} k(\hat{q}) \Delta W_1^{r,v} - \frac{\mu_1}{2} vk^2(\hat{q}) \\ &\quad + \frac{\mu_1}{2} vk^2(\hat{q}) \zeta^{r,v} - \sqrt{\mu_2} h(\hat{p}) \Delta W_2^{r,v} - \frac{\mu_2}{2} vh^2(\hat{p}) + \frac{\mu_2}{2} vh^2(\hat{p}) \eta^{r,v} \\ &\quad - iv(k_0(\hat{q}) + h_0(\hat{p}) + \hat{l}) + o(v^{\frac{3}{2}-\varepsilon})] \frac{T_{(n-1)v}^{nv} - U_{(n-1)v}^{nv}}{v} \psi, o(v^{\frac{1}{2}-\varepsilon}) U_{(n-2)v}^{(n-1)v} \psi]_{L^2(\mathbb{R})} \\ &\quad + ((1 + o(1)) \frac{T_{(n-1)v}^{nv} - U_{(n-1)v}^{nv}}{v} \psi, o(v^{\frac{1}{2}-\varepsilon}) U_{(n-2)v}^{(n-1)v} \psi]_{L^2(\mathbb{R})} \\ &\quad + \left(o(v^{\frac{3}{2}-\varepsilon}) \frac{T_{(n-1)v}^{nv} - U_{(n-1)v}^{nv}}{v} \psi, \sqrt{\mu_1 \mu_2} k(\hat{q}) h(\hat{p}) U_{(n-2)v}^{(n-1)v} \psi \right)_{L^2(\mathbb{R})}. \end{aligned}$$

It remains to note that $\|\psi\| \leq e^{C(n-2)v} \|\varphi_0\| \leq e^{Ct} \|\varphi_0\|$, whence it follows that the relation

$$\begin{aligned} |\mathbb{E}(\xi_n^v, \xi_{n-1}^v)| &= |\mathbb{E}\left\{\mathbb{E}_{(n-2)v}^{(n-1)v} \mathbb{E}_{(n-1)v}^{nv}(\xi_n^v, \xi_{n-1}^v)\right\}| \\ &\leq (1 + o(1)) o\left(v^{\frac{1}{2}-\varepsilon}\right) e^{2Ct} \|\varphi_0\|^2 \\ &+ (1 + o(1)) o\left(v^{\frac{3}{2}-\varepsilon}\right) \|k\| \|h\| \sqrt{\mu_1 \mu_2} e^{2Ct} \|\varphi_0\|^2 \rightarrow 0 \end{aligned}$$

holds as $v \rightarrow 0$.

The situation with arbitrary j and l is treated in a similar way. In this case, the convergence of $\mathbb{E}(\xi_j^v, \xi_l^v)$ to zero is uniform with respect to $j \neq l$ since the asymptotic expansions for the operators $T_{(j-1)v}^{jv}$ and $U_{(j-1)v}^{jv}$ as $v \rightarrow 0$ do not depend on j .

5 Stochastic Feynman path integrals over trajectories in the phase space

If a function on the phase space depends on a random parameter, then the Hamiltonian Feynman integral of that function can be defined in a natural way

by extending the classical definition in [24]. Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. Everywhere below, the random Feynman path integral of a function $F : C([0, t], \mathbb{R}) \times C([0, t], \mathbb{R}) \times \Omega \mapsto \mathbb{C}$ such that $F(\xi_p, x_{i_p}, \cdot)$ is measurable relative to (Ω, \mathcal{G}) for all $\xi_q, \xi_p \in C([0, t], \mathbb{R})$ over trajectories in the phase space is a random function in $L^2(\mathbb{R})$ equal to the w -limit as $n \rightarrow \infty$ of the random functions $z \mapsto I_n(F, z)$ defined by equation (3).

Theorem 2. Let h and k be bounded real-valued functions belonging to $L^2(\mathbb{R})$. Let h_0 and k_0 be functions from \mathbb{R} to \mathbb{R} , let $l \in L^2(\mathbb{R})$ be a real-valued function and let $\mathcal{H}(q, p) = k_0(q) + h_0(p) + l(q, p)$ for all $q, p \in \mathbb{R}$. Furthermore, it is assumed that if $\{T_r^t\}_{0 \leq r \leq t}$ is the resolvent operator family corresponding to equation (2) with Hamiltonian $\hat{\mathcal{H}}$ obtained by the qp-quantization of \mathcal{H} , then the map $t \mapsto \mathbb{E} \|T_0^t\|$ is differentiable at zero. Under these conditions, the relation

$$\begin{aligned} T_0^t \varphi_0 &= \int \exp \left\{ - \int_0^t (i\mathcal{H}(\xi_q(s), \xi_p(s)) + \mu_1 k^2(\xi_q(s)) + \mu_2 h^2(\xi_p(s))) ds \right\} \\ &\times \exp \left\{ -\sqrt{\mu_1} \int_0^t k(\xi_q(s)) dW_1(s) - \sqrt{\mu_2} \int_0^t h(\xi_p(s)) dW_2(s) \right\} \\ &\times \varphi_0(\xi_q(0)) \Phi^{0,t,\cdot}(d\xi_q, d\xi_p) \end{aligned} \quad (18)$$

holds for an arbitrary function $\varphi_0 \in L^2(\mathbb{R})$ and all values of $t > 0$.

Proof. By the definition of the stochastic Hamiltonian Feynman integral, the right-hand side of (21) is equal to $w - \lim_{n \rightarrow \infty} B_{t(n-1)/n}^t B_{t(n-2)/n}^{t(n-1)/n} \cdots B_0^{t/n} \varphi_0$,

where for all $r, v > 0$ the action of the operator B_r^{r+v} on a function $\varphi \in L^2(\mathbb{R})$ is defined as follows. Let

$$f(q, p) = \exp\{- (i\mathcal{H}(q, p) + \mu_1 k^2(q) + \mu_2 h^2(p)) v - \sqrt{\mu_1} k(q) \Delta W_1^{r,v} - \sqrt{\mu_2} \int_0^t h(p) \Delta W_2^{r,v}\}$$

for all $q, p \in \mathbb{R}$. Then $B_r^{r+v}\varphi = \hat{f}\varphi$, where \wedge is the operation of qp-quantization.

We claim that B_r^{r+v} coincides with U_r^{r+v} . Let $f_1(q, p) = \exp\{\mu_1 k^2(q) v - \sqrt{\mu_1} k(q) \Delta W_1^{r,v}\}$, $f_2(q, p) = \exp\{-iv\mathcal{H}(q, p)\}$ and $f_3(q, p) = \exp\{\mu_2 h^2(p) v - \sqrt{\mu_2} \int_0^t h(p) \Delta W_2^{r,v}\}$ for all $q, p \in \mathbb{R}$. Then, by the definitions of the corresponding operator families, we have the relations $U_r^{r+v} = \hat{f}_1 \hat{f}_2 \hat{f}_3$ and $B_r^{r+v} = \widehat{f_1 f_2 f_3}$. The second of these was proved in [34] under the condition that f_1, f_2 and f_3 are non-random functions. The proof in the stochastic case is similar.

For every function $g \in L^2(\mathbb{R} \times L^2(\mathbb{R}))$, we define an operator $\mathcal{J}(g) : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ by putting

$$[\mathcal{J}(g)\varphi](q) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(q, p) e^{iqp} \varphi(p) dp.$$

Then the qp-quantization of g can be written as

$$\hat{g} = \mathcal{J}(g) F^{-1}.$$

The relations

$$\widehat{f_1 f_2 f_3} = \mathcal{J}(f_1 f_2 f_3) F^{-1} = \hat{f}_1 \mathcal{J}(f_2) F^{-1} \hat{f}_3 = \hat{f}_1 \hat{f}_2 \hat{f}_3$$

follow from the fact that f_1 (f_3) does not depend on p (q), respectively.

Thus, the relation $B_v^{r+v} = U_v^{r+v}$ holds for all $r, v > 0$, and (21) follows from Theorem 1.

Remark 2. The result in Theorem 2 can be extended to the case of τ -quantization for an arbitrary $\tau \in [0, 1]$. For this, it suffices to find asymptotic expansions as $v \rightarrow 0$ for the operator $B_v^{r+v} = \hat{f}$ in the case when \wedge is equal to the operator of τ -quantization. It can be shown that these expansions will have the same form as (17). (Of course, in this case, the operators \wedge on the right-hand side of this equation will correspond to τ -quantization.) Then an analogue of Theorem 1 can be proved which will imply the representation of the solution using the Hamiltonian Feynman integral. The resulting formula for this representation will coincide with (21) except that the symbol $\Phi^{0,t,\cdot}$ is replaced by the pseudo-measure $\Phi^{t\tau,t,\cdot}$ corresponding to the case of τ -quantization.

We also note that the use of methods based on Chernoff's theorem in the stochastic case also leads to randomized analogues of formulae in [9], [33]. For this, suitable expansions into Dyson series must be considered for random measures depending on realizations of the Wiener processes W_1 and W_2 .

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