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An approximate JKR model of elliptical contact between thin incompressible elastic coatings covering rigid cylinders

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Abstract In the framework of the Johnson–Kendall–Roberts (JKR) theory, the adhesive contact between thin incompressible elastic coatings covering rigid cylinders is studied. The leading-order asymptotic model of three-dimensional non-axisymmetric JKR-type adhesive contact for a thin incompressible coating bonded to a rigid substrate is applied, and an approximate solution has been obtained under the assumption that the contact area remains elliptical.

Keywords Adhesive contact · JKR model · incompressible coating · asymptotic model

1 Introduction

An approximate JKR (Johnson–Kendall–Roberts) theory [1] for the general elliptical Hertzian contact, which, in particular, covers the case of contact between two elastic cylinders was developed by Johnson and Greenwood [2]. Their approach assumes that the contact area remains elliptical, whereas the eccentricity is not that predicted by the Hertzian theory, and varies continuously as the contact load is varied. Note also that the three-dimensional Hertz theory was extended for coated elastic bodies in [3].

In many engineering and biomedical applications (see, e.g, [4,5]), one is faced with the contact problem for thin adhesive coatings covering relatively stiff cylinders, which are arbitrarily oriented. In this case, the mathematical models of adhesive contact listed above cannot be applied, especially in the case of thin coatings. On the other hand, it is known that a very thin compressible elastic coating responses to deformation as a Winkler foundation (see, in particular, [6–12]). The leading-order asymptotic solution of the axisymmetric JKR-type adhesive contact problem for a thin elastic compressible layer was obtained in [13]. Approximate theories of adhesive contact for a Winkler-type elastic foundation were recently developed in a number of works [14–16].

In the present paper, we consider the JKR-type adhesive contact between coated cylinders, provided the cylinders are absolutely rigid while the thin elastic coatings are assumed to be incompressible. In the symmetrical configuration, when two identical coated cylinders are crossed at right angle, the contact area will be circular and the asymptotic solution constructed by Yang [17] can be used. However, when the angle between the cylinders changes or the coating radii vary, the axisymmetric solution becomes no longer valid.

The leading-order asymptotic model of three-dimensional non-axisymmetric JKR-type adhesive contact for a thin incompressible coating bonded to a rigid substrate was recently proposed in [18], and here we apply

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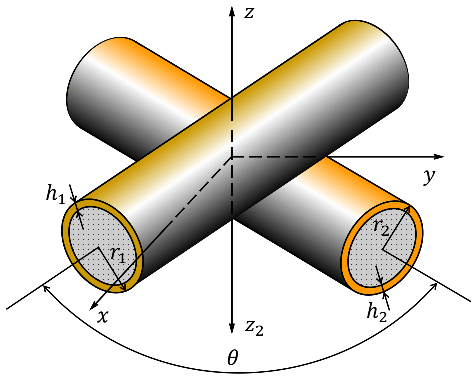

Fig. 1 Contact configuration.

this model to study the adhesive contact between thin incompressible elastic coatings covering rigid cylinders.

2 Asymptotic model for adhesive contact of thin incompressible coatings

Consider two coated rigid cylinders in frictionless adhesive contact, which in the underformed state touch each other at a single point taken as the origin of the Cartesian coordinate system $Oxyz$ (Fig. 1), and thus forming the gap

$$\varphi(x,y) = \frac{x^2}{2R_1} + \frac{y^2}{2R_2}, \tag{1}$$

where R_1 and R_2 depend on the radii of the cylinders r_1 and r_2 and the angle between the cylinder axes θ (see, e.g., [19]). It is assumed that $R_2 \leq R_1$.

Let $w_0^{(1)}(x,y)$ and $w_0^{(2)}(x,y)$ be the vertical elastic displacements of the surface points of the coating layers, while the contact approach of the rigid cylinders will be denoted by δ. Given that the materials of the coatings are elastic, incompressible and transversely isotropic with out-of-plane shear moduli G_1' and G_2', under the assumption that the coating thicknesses h_1 and h_2 are small compared to the characteristic sizes of the contact area, ω, the following asymptotic model for the frictionless JKR-type adhesive contact can be established [18]:

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = m\big(\varphi(x,y) - \delta\big), \quad (x,y) \in \omega, \tag{2}$$

$$p(x,y) = 0, \quad \frac{\partial p}{\partial \nu}(x,y) = \sqrt{2m\Delta\gamma}, \quad (x,y) \in \Gamma. \tag{3}$$

Here, $p(x,y)$ is the contact pressure, $\Delta\gamma$ is the surface energy, Γ is the contour of the contact area ω, $\partial/\partial\nu$ is the outward normal derivative, and the parameter m is given by

$$m = \left(\frac{h_1^3}{3G_1'} + \frac{h_2^3}{3G_2'}\right)^{-1}. \tag{4}$$

Note that in the non-adhesive case (when $\Delta\gamma = 0$), the contact problem (1)–(3) was solved in [8,20,11]. We emphasize that, in spite of the elliptic gap $(1)_1$, the contact area ω as it is determined by the adhesive contact problem (2), (3) is not elliptic. In the next section we construct a simple elliptical approximation for the solution of the problem (2), (3), based on an analogous idea to that of Johnson and Greenwood [2].

3 Approximate solution of the non-axisymmetric adhesive contact problem

Looking for the simplest form of approximate solution, we write

$$p(x,y) = p_0\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\big(1 - Ax^2 - By^2\big), \tag{5}$$

where p_0 is the maximum contact pressure, a and b are the semiaxes of the approximate elliptical contact area ω_0. Note that this form of solution incorporates the case of non-adhesive contact (when $\Delta\gamma = 0$) as well as the case of contact with a fixed elliptical region, which is constituted by Eq. (2) and the first boundary condition (3).

By substituting (5) into Eq. (2), we arrive at the system of three equations

$$\begin{aligned} &\frac{1}{a^2} + \frac{1}{b^2} + A + B = \frac{m\delta}{2p_0}, \\ &\quad\frac{A}{b^2} + \frac{6A}{a^2} + \frac{B}{a^2} = \frac{m}{4p_0R_1}, \\ &\quad\frac{A}{b^2} + \frac{6B}{b^2} + \frac{B}{a^2} = \frac{m}{4p_0R_2}. \end{aligned} \tag{6}$$

If the contour Γ_0 of the domain ω_0 coincides with the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{7}$$

then the approximation (5) satisfies not only Eq. (2) (this fact follows from (6)), but also the first boundary condition (3).

Integrating the contact pressure density (5) over the elliptical domain ω_0 delineated by the ellipse (7), we evaluate the contact force

$$F = \frac{\pi ab p_0}{12}(6 - Aa^2 - Bb^2). \tag{8}$$

The remaining boundary condition $(3)_2$ will be approximately satisfied if we ensure that it holds at least at certain points of the ellipse Γ_0 described by Eq. (7). Observe that the contact pressure approximation (5) contains five parameters (namely, p_0, a, b, A, and B), for which we have derived only three equations (6) so far. Hence, the second boundary condition (3) should provide yet two equations in order to close the problem. Following [2], we make use of the collocation method and satisfy the mentioned boundary only at four points (with symmetry taken into account).

It can be easily checked that the normal derivative of the contact pressure approximation (5) on the ellipse (7) is given by

$$\frac{\partial p}{\partial \nu}\bigg|_{\Gamma_0} = \frac{2p_0}{ab}\sqrt{a^2\sin^2 t + b^2\cos^2 t} \times \big(Aa^2\cos^2 t + Bb^2\sin^2 t - 1\big). \qquad (9)$$

By putting

$$A = \frac{1}{a^2}\Big(1+\frac{1}{\xi}\Big), \quad B = \frac{1}{b^2}\Big(1+\frac{b}{a\xi}\Big), \qquad (10)$$

we assure that the values of the function (9) at the apexes of the ellipse (7) coincide with each other.

The substitution of (10) into (9) yields

$$\frac{\partial p}{\partial \nu}\bigg|_{\Gamma_0} = \frac{2p_0}{sa\xi}\sqrt{\sin^2 t + s^2\cos^2 t}\big(\cos^2 t + s\sin^2 t\big), \qquad (11)$$

where s denotes the aspect ratio of the contact area, i.e.,

$$s = \frac{b}{a}. \qquad (12)$$

Note that, while Johnson and Greenwood [2] by equalizing the Stress Intensity Factors (SIFs) at the ends of the axes obtained local maxima and so evaluated the maximum (critical) SIF. In contrast, here the values of the normal derivative $\partial p/\partial \nu$ given by (11) on the axes are local minima. This means that we must then determine what the maximum value of $\partial p/\partial \nu$ on Γ_0 is, and thus, by equating the maximum of (11) to the right-hand side of $(3)_2$, we arrive at the equation

$$\xi = \frac{4p_0}{3^{3/2}a\sqrt{2m\Delta\gamma}}\frac{(s^2+s+1)^{3/2}}{s(s+1)}. \qquad (13)$$

Fig. 2 shows the variation of the quantity

$$\frac{\max \partial_\nu p - \min \partial_\nu p}{\min \partial_\nu p} \times 100\%. \qquad (14)$$

We note that the analogous characteristic for the SIF's approximation in [2] is less than 4% for $s \geq 0.6$, whereas the quantity (14) is less than 10% in the same range.

Then, taking into account (10)–(12), we rewrite Eqs. (6) after some algebra as follows:

$$\xi = \frac{1}{2(s^2+1)}\big(\eta(s)s^2\bar{\delta}\bar{a} - s - s^2\big),$$
$$\xi = \frac{\eta(s)s^2\bar{a}^3 - 2(6s^2+s+1)}{4(3s^2+1)}, \qquad (15)$$
$$\xi = \frac{\eta(s)s^4\bar{a}^3 - 2\rho s(s^2+s+6)}{4\rho(s^2+3)}.$$

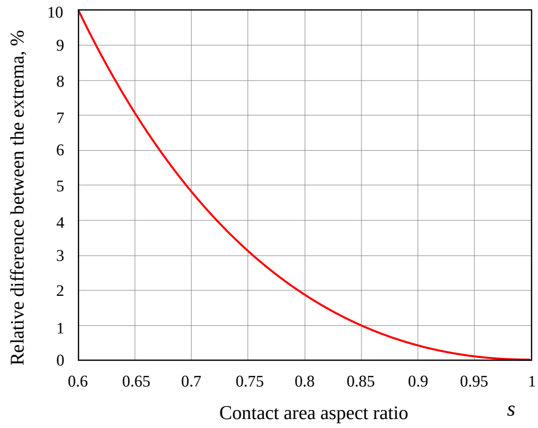

Fig. 2 Error in satisfying the normal derivative constant boundary condition $(3)_2$.

Here we have introduced the notation

$$\eta(s) = \frac{2(s^2+s+1)^{3/2}}{3^{3/2}s(s+1)}, \qquad (16)$$

$$\bar{\delta} = \Big(\frac{mR_1}{2\Delta\gamma}\Big)^{1/3}\delta, \quad \bar{F} = \frac{F}{2R_1\Delta\gamma}, \quad \bar{a} = \Big(\frac{m}{2R_1^2\Delta\gamma}\Big)^{1/6}a. \qquad (17)$$

Correspondingly, formula (8), after the substitution of $(15)_1$ for ξ, takes the following form:

$$\bar{F} = \frac{\pi\sqrt{3}}{16}\frac{\bar{a}^3\big[2s^2\bar{a}\bar{\delta} - (s+1)^3\big]s^2(s+1)}{(s^2+1)(s^2+s+1)^{3/2}}. \qquad (18)$$

At the same time, Eqs. $(15)_2$, $(15)_3$ with $(15)_1$ taken into account transform as follows:

$$\bar{a}^3 - \frac{2\bar{\delta}(3s^2+1)}{s^2+1}\bar{a} - \frac{2(3s^4-2s^3+6s^2+1)}{s^2(s^2+1)\eta(s)} = 0 \qquad (19)$$

$$\bar{a}^3 - \frac{2\rho\bar{\delta}(s^2+3)}{s^2(s^2+1)}\bar{a} - \frac{2\rho(s^4+6s^2-2s+3)}{s^3(s^2+1)\eta(s)} = 0. \qquad (20)$$

Here we have introduced the notation

$$\rho = \frac{R_2}{R_1}. \qquad (21)$$

Note that if $R_2 \leq R_1$, we will have $a \geq b$, i.e., $s \leq 1$.

Thus, the problem of evaluating the five parameters p_0, a, b, A, and B in the approximation (5) is reduced to a system of two equations (19) and (20) with respect to the two dimensionless variables $\bar{a}$ and s.

4 Exact solution to the axisymmetric adhesive contact problem

In the axisymmetric case (when two identical coated cylinders are pressed at right angle), we have $R_1 = R_2 = R$, and the contact approach and the contact force are related to the contact radius as follows:

$$\delta = \frac{a^2}{4R} - \frac{2}{a}\sqrt{\frac{2\Delta\gamma}{m}}, \qquad (22)$$

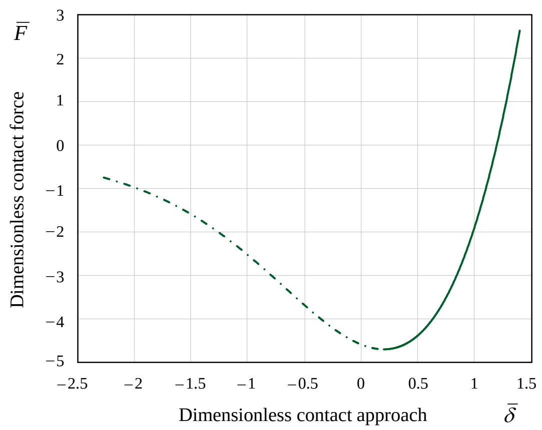

Fig. 3 Force-displacement relationship in the axisymmetric case. The dash-dotted line line corresponds to the non-stable part of the force-displacement diagram.

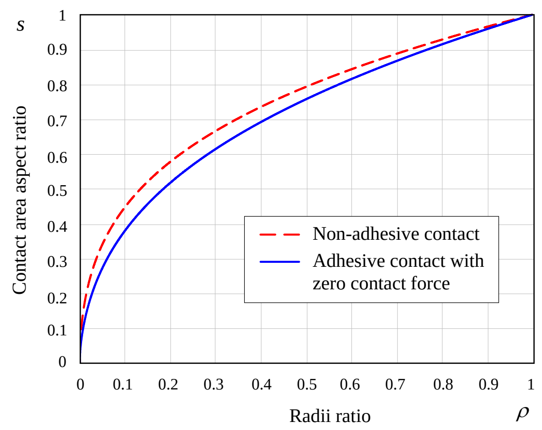

Fig. 4 Contact area aspect ratio s as a function of the radii ratio $\rho = R_2/R_1$ according to Eq. (28) (solid line), which holds for the case of zero load only.

$$F = \frac{\pi m}{48} a^4 \left(\frac{a^2}{2R} - \frac{12}{a} \sqrt{\frac{2\Delta\gamma}{m}} \right). \tag{23}$$

Apart from the notation, formulas (22) and (23) (see also [18]) coincide with the solution previously derived by Yang [17] in the isotropic case.

Let us introduce dimensionless variables by the following formulas:

$$\delta = \sqrt[3]{\frac{2\Delta\gamma}{mR}} \bar{\delta}, \quad F = 2R\Delta\gamma \bar{F}, \quad a = \sqrt[6]{\frac{2R^2\Delta\gamma}{m}} \bar{a}. \tag{24}$$

Then, by the substitution of (24) into Eqs. (22) and (23), we transform them into the following ones:

$$\bar{\delta} = \frac{\bar{a}^2}{4} - \frac{2^{4/3}}{\bar{a}}, \quad \bar{F} = \frac{\pi}{96} \bar{a}^4 \left(\bar{a}^2 - \frac{24}{\bar{a}} \right). \tag{25}$$

Fig. 3 presents the force-displacement relationship given by two parametric equations (25).

The minimum value of the normalized contact force $\bar{F}$ is given by $\bar{F}_c = -3\pi/2$, and the corresponding values of the contact radius and the contact displacement are $\bar{a}_c = 2^{2/3}3^{1/3} \approx 2.2894$, $\bar{\delta}_c = 2^{-2/3}3^{2/3} - 2^{2/3}3^{-1/3} \approx 0.2097$, respectively.

5 Case of zero contact force

By setting $\bar{F} = 0$ in Eq. (18), we obtain

$$\bar{a}\bar{\delta} = \frac{1}{2s^2}(s+1)^3. \tag{26}$$

The substitution of (26) into Eqs. (19) and (20) yields

$$\bar{a}^3 = \frac{3}{s^2}(s^3 + 5s^2 + s + 1), \tag{27}$$

$$\frac{s^2(s^3 + 5s^2 + s + 1)}{s^3 + s^2 + 5s + 1} = \rho. \tag{28}$$

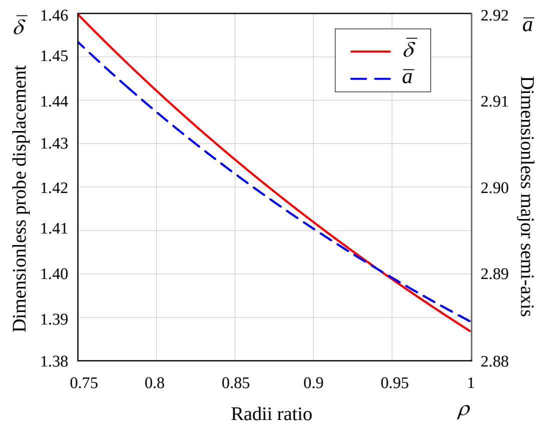

Fig. 5 Relative contact displacement and contact semi-axis (in the zero contact force case) as functions of the radii ratio.

Fig. 4 illustrates the variation of the solution of Eq. (28) as a function of its right-hand side. The dashed line corresponds to the aspect ratio of the elliptical contact area in the non-adhesive contact (see, in particular, [8,11]), which is given by the formula

$$s^2 = \frac{1}{6}\left[\sqrt{(1-\rho)^2 + 36\rho} - 1 + \rho\right]. \tag{29}$$

A good approximation to the solution of Eq. (28) is given by the following formula:

$$s \simeq \left(\sqrt{\left(\frac{3}{10}(1-\rho)\right)^2 + \rho} - \frac{3}{10}(1-\rho) \right)^{1/2}. \tag{30}$$

Observe that formula (30) is asymptotically exact for ρ close to 1. At the same time, for $\rho \in (0.4, 1)$, the error of the approximation (30) is less than 0.01 %.

Fig. 5 shows the behavior of $\bar{\delta}$ and $\bar{a}$ as functions of the radii ratio $\rho = R_2/R_1$ according to Eqs. (26) and (27) in the case of zero contact force.

Finally, note that achieving the zero load state requires positive action by the experimenter (bringing the two cylinders into contact and after that relaxing the external compressing load). So, Fig. 4 and 5 do not represent the initial configuration, when the cylinder surfaces touch at a single point. Indeed, since the JKR model neglects long range forces [1,19], nothing can happen until the cylinders touch; and then with an infinitesimally small approach the contact force jumps to a certain tensile tensile value.

6 Numerical implementation of the approximate model

Returning back to the original problem (2), (3), observe that the approximate solution (5) contains five unknown parameters p_0, a, b, A, and B. Recall that the substitution of the approximation (5) into the differential equation (2) yielded three equations (6), while the boundary condition $(3)_2$ implied also three equations (10) and (13) plus one new unknown ξ.

Further, by using (10) and (12), the system (6) was reduced to the system (19), (20) of two equations with respect to unknowns $\bar{a}$ and s. The last system can be now solved numerically for given values of ρ and $\bar{\delta}$. After that Eq. (15) yields the value of x, Eq. (12) gives the minor semiaxis $b = sa$, Eqs. (13) provides the maximum contact pressure in the form

$$p_0 = \frac{3^{3/2}\xi a\sqrt{2m\Delta\gamma}s(s+1)}{4(s^2+s+1)^{3/2}}.$$

Finally, Eq. (8) delivers the corresponding value of the contact force F, while, according to (5) and (10), the pressure distribution over the contact area is given by

$$p = p_0\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)\left(1 - \frac{x^2}{a^2}\left(1+\frac{1}{\xi}\right) - \frac{y^2}{b^2}\left(1+\frac{s}{\xi}\right)\right).$$

Figs. 6 and 7 show the variation of the aspect ratio s of the approximate contact area ω_0 as functions of its dimensionless major semi-axis $\bar{a}$ and the corresponding dimensionless load $\bar{F}$, respectively. The horizontal asymptote (dashed line) in Fig. 6 corresponds to the non-adhesive case (see formula (29)). At the same time, Fig. 7 demonstrates the continuous shape change as the load increases (cf. Fig. 5 in [2]).

Fig. 8 shows the relative contact force $\bar{F}$ as a function of the relative contact approach $\bar{\delta}$. Again, the dashed line indicates the non-adhesive case. An additional comparison with the axisymmetric adhesive case is presented in Fig. 9, from where it is readily seen that the value of pull-off force decreases with ρ.

As it is seen from (11), the variation of the normal derivative $\partial p/\partial\nu$ on the ellipse Γ_0, which is defined by Eq. (7), solely depends on the ellipse aspect ratio

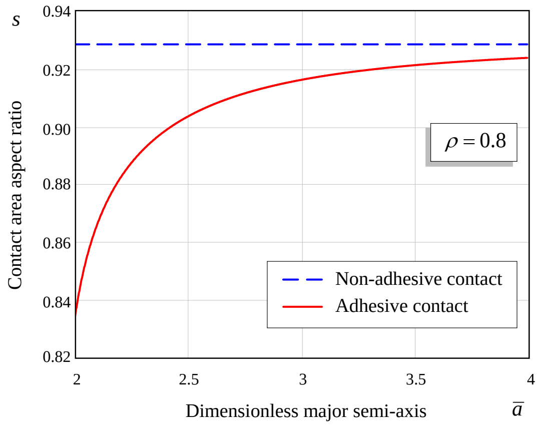

Fig. 6 Variation of the ratio s with variation of the dimensionless $\bar{a}$, for the fixed value 0.8 of the curvature radii ratio $\rho = R_2/R_1$.

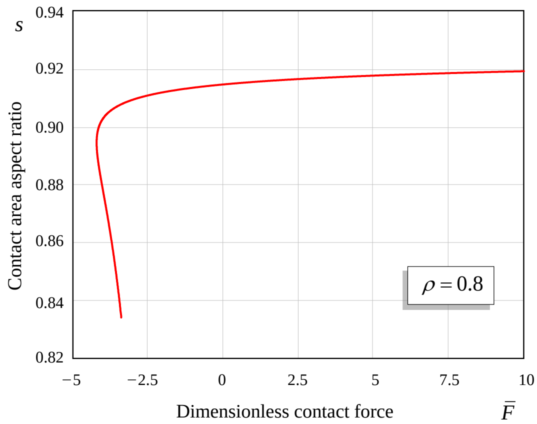

Fig. 7 Variation of the ratio s with variation of the dimensionless $\bar{F}$, for the fixed curvature radii ratio 0.8.

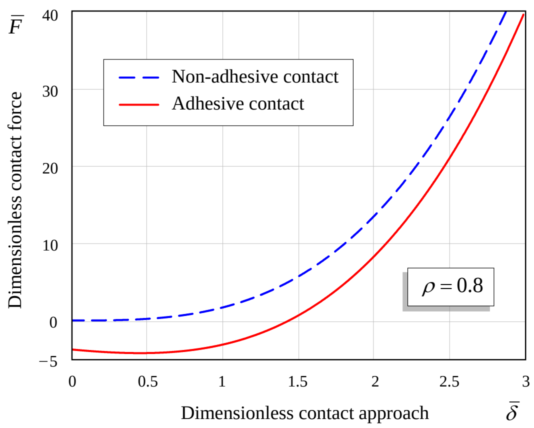

Fig. 8 Force-displacement relation for the fixed curvature radii ratio $\rho = 0.8$.

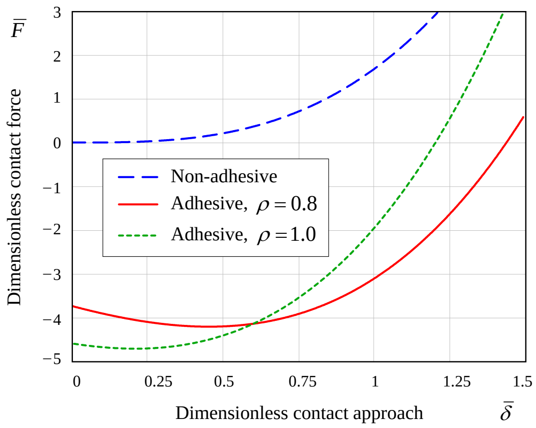

Fig. 9 Comparison of the force-displacement relationships.

s. Therefore, the relative error in satisfying the second boundary condition (3) is also governed by the parameter s. Now, from Eqs. (18)–(20), it is seen that the value of s depends on the value of the radii ratio ρ as well as on the dimensionless contact approach $\bar{\delta}$ (or on the value of the dimensionless contact force $\bar{F}$). Further, in a special case of $\rho = 0.8$, from Fig. 6, it is readily seen that the case of non-adhesion contact (29) provides the upper bound for s, whereas a lower bound for s is required to obtain an *a priori* estimate of the solution error. On the other hand, Fig. 7 shows that for $\rho = 0.8$, the aspect ratio s varies between 0.895 at pull-off, when $\bar{F} = -4.21$, and 0.93 under relatively heavy load, when $\bar{F} = 10$. Therefore, according to Fig. 2, the solution error for $\rho \geq 0.8$ will be less than 0.5% for any loading scenario. In the case when $\rho < 0.8$, it is recommended to perform the *a posteriori* error analysis using (14) and to apply the constructed analytical solution up to an admissible level of accuracy.

Finally, as it is known [21], the JKR theory is applicable only for relatively large values of the Tabor parameter, $\mu = \left[R(\Delta\gamma)^2/(E^{*2}z_0^3)\right]^3$, where $E^* = 4G$ is the reduced elastic modulus, z_0 is the interatomic spacing. On the other hand, the asymptotic model developed in [18] requires that the ratio h/a of the maximum coating thickness, h, to typical contact area dimension, a, to be small enough. In light of (17) and Fig. 6, it can be shown that the last dimensionless parameter is proportional to $h^{1/2}(z_0R)^{-1/4}\mu^{-1/4}$.

7 Conclusion

The JKR-type frictionless contact between thin incompressible elastic coating layers bonded to rigid cylinders is considered in the framework of the leading-order asymptotic model, which assumes that $h_n \ll r_n$, $n = 1, 2$, (each coating thickness is small compared to the corresponding cylinder radius) and $h_n \ll a$, $n = 1, 2$, (each coating thickness is small compared to the characteristic sizes of the contact area). It should be emphasized that the obtained analytical solution (5), which is based on the elliptical contact area ω_0, to the adhesive contact problem (2), (3) is approximate, because the second boundary condition (3) is satisfied only at certain points of the ellipse Γ_0. At the same time, this solution turns out to be exact in the symmetric case, when $\rho = 1$, while its accuracy decreases when the radii ratio ρ decreases and, correspondingly, the contact area aspect ratio s decreases as well. The present solution is limited to a very narrow range $0.8 < \rho < 1$.

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Appendix. Nearly circular contacts

Let the interface gap be represented in the form

$$\varphi(x,y) = \frac{x^2+y^2}{2R} + \mu\frac{y^2-x^2}{2R}, \tag{31}$$

where we have introduced the notation

$$R = \frac{2R_1R_2}{R_1+R_2}, \quad \mu = \frac{R_1-R_2}{R_1+R_2}. \tag{32}$$

On the other hand, we will have

$$R_1 = \frac{R}{1-\mu}, \quad R_2 = \frac{R}{1+\mu}. \tag{33}$$

The two-term asymptotic expansion for the solution of the adhesive contact problem under the assumption that μ is a small positive parameter was constructed in [18]. In particular, the polar equation for the contact contour was obtained in the form $r \simeq a_0 + \mu a_2 \cos 2\theta$, where r, θ are polar coordinates, so that the major and minor semi-axes are

$$a \simeq a_0 + \mu a_2, \quad b \simeq a_0 - \mu a_2, \tag{34}$$

respectively, whereas the aspect ratio of the contact area is given by

$$s \simeq 1 - 2\mu\frac{a_2}{a_0}. \tag{35}$$

Here, a_0 is the leading order approximation as $\mu \to 0$, and a_2 is given by the formula

$$a_2 = \frac{a_0}{3}\left(1 - \frac{4R}{a_0^3}\sqrt{\frac{2\Delta\gamma}{m}}\right)^{-1}, \tag{36}$$

while (see also Eq. (22))

$$\delta = \frac{a_0^2}{4R} - \frac{2}{a_0}\sqrt{\frac{2\Delta\gamma}{m}}. \tag{37}$$

Let us now compare the asymptotic solution (36) with the approximate solution constructed above. First, from the second and third equations (15), we derive the equation

$$\frac{R_2}{R_1} = \frac{s^2\bigl[6s^2+s+1+2\xi(1+3s^2)\bigr]}{s(s^2+s+6)+2\xi(3+s^2)},$$

which, in view of (33) and (35), implies

$$\xi \simeq \frac{9a_2-4a_0}{4(a_0-3a_2)}. \tag{38}$$

At the same time, the first equation (15) can be rewritten as

$$\frac{1}{a_0\delta}\sqrt{\frac{2\Delta\gamma}{m}} \simeq \frac{s^2\eta(s)(1+\mu a_2/a_0)}{s(1+s)+2\xi(1+s^2)}, \tag{39}$$

where the first formula (34) was already taken into account.

The substitution of (35) and (38) into Eq.(39) results in the equation

$$\frac{a_2}{a_0} = \left(\frac{2}{a_0\delta}\sqrt{\frac{2\Delta\gamma}{m}}+1\right)\left(\frac{3}{a_0\delta}\sqrt{\frac{2\Delta\gamma}{m}}+3\right)^{-1},$$

which, after eliminating δ by means of Eq. (37), reduces to Eq. (36).

Thus, in the case of a nearly circular area of contact the obtained approximate solution agrees with the asymptotic solution up to the second-order terms. This, in particular, means that the adopted approximation for the contact pressure (5) is asymptotically exact for nearly circular contacts. On the other hand, the asymptotic Ansatz developed in [18], which was established for an arbitrary smooth gap function close to a paraboloid of revolution, may suggest a more elaborate approximation (incorporating more free parameters) in order to extend the analytical approach to the range of smaller values of the ratio ρ.