

## Aberystwyth University

### *Algorithmic derivation of isochronicity conditions*

Lloyd, Noel G.; Pearson, Jane M.; Hill, Joe M.

*Published in:*

Nonlinear Analysis: Theory, Methods & Applications

*DOI:*

[10.1016/j.na.2006.04.018](https://doi.org/10.1016/j.na.2006.04.018)

*Publication date:*

2007

*Citation for published version (APA):*

Lloyd, N. G., Pearson, J. M., & Hill, J. M. (2007). Algorithmic derivation of isochronicity conditions. *Nonlinear Analysis: Theory, Methods & Applications*, 67(1), 52-69. <https://doi.org/10.1016/j.na.2006.04.018>

#### **General rights**

Copyright and moral rights for the publications made accessible in the Aberystwyth Research Portal (the Institutional Repository) are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Aberystwyth Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Aberystwyth Research Portal

#### **Take down policy**

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

tel: +44 1970 62 2400  
email: [is@aber.ac.uk](mailto:is@aber.ac.uk)

# Algorithmic derivation of isochronicity conditions

J.M. Hill, N.G. Lloyd, J.M. Pearson\*

*Institute of Mathematical and Physical Sciences, University of Wales Aberystwyth  
UK, SY23 3BZ*

---

## Abstract

This paper is concerned with obtaining the conditions under which the origin is an isochronous centre for certain planar polynomial systems. The necessity of these conditions is proved using a computer implementation of an algorithm first proposed by I.I. Pleshkan. Their sufficiency is then proved using various methods.

*Key words:* differential equations, isochronous centres  
*1991 MSC:* 34C07, 37G15

---

## 1 Introduction

We consider the planar differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

where  $P, Q$  are polynomials with real coefficients. We also use the complex form of (1)

$$i\dot{z} = Z(z, \bar{z}),$$

where  $z = x + iy$  and  $\bar{z}$  is the complex conjugate of  $z$ . Critical points of the system occur where  $P = Q = 0$ . A critical point is a *centre* if, within some neighbourhood, all orbits surrounding it are closed; a centre is *non-degenerate* if the linearised vector field has two non-zero eigenvalues. A centre is said to be *isochronous* if all the orbits surrounding it have the same *period* and it was shown in [20] that an isochronous centre is non-degenerate.

---

\* Corresponding author. Tel: +44 1970 622002; fax: +44 1970 611446.  
*Email address:* jmp@aber.ac.uk (J.M. Pearson).

The question of isochronicity was first discussed by Huygens in *Horologium Oscillatorium sive de motu pondulorum* (1673). Huygens showed that the cycloidal pendulum has a motion whose period (and hence frequency) is independent of the amplitude. Thus the cycloidal pendulum has isochronous oscillations, in contrast to the simple pendulum in which the period of oscillation increases with amplitude. The simplest form of the equation of the cycloidal pendulum is given in terms of arc length,  $s$ , as the independent variable

$$\ddot{s} + w^2 s = 0, \quad (2)$$

where  $w$  is a real constant. This is a special case of a class of second order differential equations

$$\ddot{s} + g(s) = 0, \quad (3)$$

which arise in the study of conservative mechanical systems. Urabe [29] proved that if  $g(s)$  is an odd function then (3) can only have an isochronous centre when it is system (2).

Many examples of isochronous systems exist in applications. Needham [21] discussed one such system that arises in the telecommunications industry. Atmospheric distortion can affect the transmission of high-speed digital signals and is removed by introducing tapped delay devices, which act as adaptive equalisers. The differential system modelling this situation has the property that small changes in the initial conditions can alter the amplitude, but not the frequency, of an oscillation.

The existence of isochronous centres for several classes of differential systems has been studied - see the survey paper [4] and references therein, for example. In particular, Loud [19] gave the results for quadratic systems, cubic systems with homogeneous nonlinearities were considered by Pleshkan [24] and results for other cubic systems can be found in [26], [3], [18] for example. Systems of higher degree have received less attention; there are results for quartic and quintic systems with homogeneous nonlinearities [1], [2] and for Liénard systems [7]. It was shown in [10] that some systems can have coexisting isochronous and non-isochronous centres. We shall present results for particular classes of cubic and quartic systems.

Any isochronous family of periodic orbits surrounds a unique non-degenerate critical point of centre type. So, without loss of generality, we can assume that the origin is the critical point of interest and transform system (1) to

$$\dot{x} = y + p(x, y), \quad \dot{y} = -x + q(x, y), \quad (4)$$

where  $p, q$  are polynomials without constant or linear terms. Every system can be transformed to a linear system at a non-singular point, but this is not true in general at singular (critical) points. A key property of isochronous systems, which was proved in [20], is that they can be transformed to a

linear system even at critical points. Centres of linear systems are clearly isochronous, which leads to the following result.

**Theorem 1** *A centre of system (4) is isochronous if and only if there exists an analytic change of coordinates of the form*

$$u(x, y) = x + o(|(x, y)|), \quad v(x, y) = y + o(|(x, y)|),$$

*which reduces the system to*

$$\dot{u} = v, \quad \dot{v} = -u.$$

**Proof.** See for example [20]. ■

The equivalent linear system in complex notation is  $i\dot{z} = z$ . Linearisation of differential systems forms the basis of several methods for the determination of the conditions under which a critical point is an isochronous centre. Pleshkan presented an algorithm in [24] which uses the complex form of (4). We describe this algorithm in the next section. Others have used the polar form of (4), see for example [9], or the real form as in [6]. Commuting systems have also been used [28] and in some instances this is a simpler approach than searching for a linearisation. However, not every polynomial system with an isochronous centre has a polynomial commutator. This would require that every centre in the plane was isochronous and Devlin [10] showed that isochronous and non-isochronous centres can coexist. Another approach is to consider the period function; if  $T(x)$  is the period of the solution through the point  $(x, 0)$  then  $T$  can be written as

$$T(x) = 2\pi + \alpha_1 x + \alpha_2 x^2 + \dots.$$

The origin is an isochronous centre if the period constants  $\alpha_i = 0$ , for all  $i$ . See [11] for an example of this approach, or an alternative method for calculating period constants as given in [16].

Although Pleshkan used his algorithm to find necessary and sufficient conditions for the origin to be an isochronous centre, we prefer to use the algorithm to show necessity only. We then prove sufficiency using various methods, some of which we describe in section 3. Using this approach avoids the need to find a basis for a set of very large multivariate polynomials and also serves as a check on the validity of the computer assisted calculation of the necessary conditions. In section 4 we demonstrate the use of our implementation of Pleshkan's algorithm by confirming the known results for the homogeneous cubic system

$$i\dot{z} = z + A_{30}z^3 + A_{21}z^2\bar{z} + A_{12}z\bar{z}^2 + A_{03}\bar{z}^3, \quad (5)$$

where the  $A_{ij}$  are complex coefficients. We have also used the algorithm to confirm the known results for the system

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3,\end{aligned}\tag{6}$$

where the  $a_i$ ,  $1 \leq i \leq 7$  are arbitrary real coefficients. The conditions for the origin to be an isochronous centre for (6), which is often referred to as the Kukles system, were established in [20] and [6].

We present two examples for which the conditions for isochronicity were previously unknown. In section 5, we consider

$$\begin{aligned}\dot{x} &= y(1 + x), \\ \dot{y} &= -x - a_1x^2 - a_2x^3 - a_3x^4 - a_4(x + a_5x^2)y - a_6y^2,\end{aligned}\tag{7}$$

where the  $a_i$ ,  $1 \leq i \leq 6$  are arbitrary real coefficients. System (7) was investigated by Cherkas in [5]. This example belongs to a class of systems of the form

$$\dot{x} = p_0(x) + p_1(x)y, \quad \dot{y} = q_0(x) + q_1(x)y + q_2(x)y^2.$$

These systems can be transformed to Liénard form using the transformation

$$(x, y, t) \rightarrow (x, (p_0(x) + p_1(x)y)\Phi(x), \tau),$$

where

$$\frac{dt}{d\tau} = \Phi(x) = \frac{1}{p_1(x)} \exp\left(-\int_0^x \frac{q_2(s)}{p_1(s)} ds\right).$$

Such a transformation can be used to show the sufficiency of certain centre conditions, see [8]. However, since it is a spatially-dependent transformation of time, it does not preserve isochronicity and cannot be used in this context.

Finally, in section 6, we consider the system

$$\begin{aligned}\dot{x} &= y(1 + kx), \\ \dot{y} &= -x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3.\end{aligned}\tag{8}$$

This is an extension of the Kukles system in that the term  $kxy$  is added to the  $\dot{x}$  equation, whilst the  $\dot{y}$  equation is unchanged. A specific example of system (8) can be used to model predator-prey populations with the inclusion of intratrophic predation, see [13]. We note that (7) and (8) have an invariant line parallel to the  $y$ -axis.

## 2 Pleshkan's Algorithm

In [24], Pleshkan presented an algorithm for deriving conditions under which a critical point of a polynomial system is an isochronous centre. His approach is to find the conditions under which the system can be linearised. We shall restate the algorithm, and we begin by writing (4) in complex form:

$$i\dot{z} = z + \sum_{r \geq 2} \Phi_r(z, \bar{z}) \quad (9)$$

where

$$\Phi_r(z, \bar{z}) = \sum_{j=0}^r c_{r-j,j} z^{r-j} \bar{z}^j,$$

$z = x + iy$ , the  $c_{r-j,j}$  are complex-valued and a bar denotes complex conjugation. We now seek a transformation

$$\omega = z + \sum_{r \geq 2} \Gamma_r(z, \bar{z}), \quad (10)$$

where

$$\Gamma_r(z, \bar{z}) = \sum_{j=0}^r d_{r-j,j} z^{r-j} \bar{z}^j$$

and the  $d_{r-j,j}$  are complex-valued, which reduces (9) to

$$i\dot{\omega} = \omega. \quad (11)$$

Substituting (10) into (11), we find

$$\sum_{k \geq 2} \left( z \frac{\partial \Gamma_k}{\partial z} - \bar{z} \frac{\partial \Gamma_k}{\partial \bar{z}} + \Phi_k \right) + \sum_{k \geq 2} \left( \sum_{r \geq 2} \frac{\partial \Gamma_k}{\partial z} \Phi_r - \frac{\partial \Gamma_k}{\partial \bar{z}} \bar{\Phi}_r \right) = \sum_{k \geq 2} \Gamma_k. \quad (12)$$

Considering terms of degree  $k$  in (12), we have

$$\Gamma_k - z \frac{\partial \Gamma_k}{\partial z} + \bar{z} \frac{\partial \Gamma_k}{\partial \bar{z}} = \Phi_k + \sum_{r=2}^{k-1} \left( \frac{\partial \Gamma_r}{\partial z} \Phi_{k-r+1} - \frac{\partial \Gamma_r}{\partial \bar{z}} \bar{\Phi}_{k-r+1} \right). \quad (13)$$

Equating the coefficients of  $z^{k-j} \bar{z}^j$  in (13), we have

$$\begin{aligned} \sum_{r=2}^{k-1} \sum_{p=0}^r (r-p) c_{k-j+p-r+1, j-p} d_{r-p,p} - \bar{c}_{k-p-j, j+p-r+1} d_{p,r-p} \\ + c_{k-j,j} = (2j+1-k) d_{k-j,j}, \end{aligned} \quad (14)$$

where terms with negative subscripts are zero. It follows from the above that if (9) is to have an isochronous centre at the origin, it is necessary and

sufficient that the equations given by (14) are consistent. The coefficients  $d_{j+1,j}$  can be chosen arbitrarily; we take

$$d_{j+1,j} = 0 \quad (j \geq 1).$$

Putting  $k = 2j + 1$  in (14), we obtain a set of conditions for the consistency of (14), namely

$$\Pi_j = 0 \quad (j \geq 1),$$

where

$$\Pi_j \equiv c_{j+1,j} + \sum_{r=2}^{2j} \sum_{p=0}^r (r-p)c_{j+p-r+2,j-p}d_{r-p,p} - \bar{c}_{j-p+1,j+p-r+1}d_{p,r-p}.$$

We call the  $\Pi_j$  the Pleshkan polynomials. Pleshkan showed that the conditions  $\Pi_j = 0$  ( $j \geq 1$ ) were necessary and sufficient for the origin to be an isochronous centre. There are infinitely many conditions but, by the Hilbert Basis Theorem, there is  $M$  such that  $\Pi_j = 0$  for  $j \leq M$  implies that  $\Pi_j = 0$  for all  $j$ . Thus calculating all the conditions for isochronicity is a terminating process, though  $M$  is not known *a priori*, and is equivalent to finding a basis for the Pleshkan polynomials for a given system. However the polynomials grow rapidly as  $j$  increases and finding the appropriate basis is non-trivial computationally. We prefer to calculate the first few polynomials, reducing each one modulo the preceding polynomials, until we have  $\Pi_m \neq 0$ , when  $\Pi_j = 0$  for  $j = 1, \dots, m-1$  except under certain conditions. The sufficiency of these conditions is then proved separately using a variety of methods. This approach has the added advantage of being an independent check on the computer assisted calculation of the Pleshkan polynomials and the subsequent derivation of the necessary conditions.

We have implemented the algorithm in REDUCE. This choice of computer algebra system was primarily due to the fact that the suite of programs developed by the Aberystwyth group is written in REDUCE. A copy of the code is available on request from the authors. It can also be found in [12]. Pleshkan proved that if the origin is a centre then the conditions  $\text{Re}(\Pi_j) = 0$  ( $j \geq 1$ ), where  $\text{Re}$  denotes the real part, are necessary and sufficient for the origin to be an isochronous centre. Introducing the centre conditions, if known, can simplify the calculations but often leads to several cases having to be considered. For example, the derivation of the conditions for the origin to be an isochronous centre for system (5) is relatively straightforward without introducing the centre conditions. For some cases it is possible to calculate the conditions for isochronicity where derivation of the centre conditions proves intractable. Next we describe some methods we use to prove the sufficiency of conditions obtained through the use of this algorithm.

### 3 Sufficiency

As noted previously, the vanishing of all the Pleshkan polynomials is necessary and sufficient for isochronicity, but in practice only a finite number can be calculated. In our approach, the sufficiency of the conditions we find must be proved separately. This is achieved using different methods, and we summarise some of these in this section. In many cases one of the first two theorems below is used. For some systems the transformation to the linear isochrone, which is known to exist, can be found and for others we make use of Urabe's result [29]. Explicit solutions of particular systems can sometimes be found, see [19].

**Theorem 2** *Suppose that system (4) is holomorphic, that is*

$$i\dot{z} = z + f(z)$$

*where  $f(z)$  is an analytic function without linear terms (equivalently, that  $p_x = q_y$ ,  $p_y = -q_x$ ). Then the origin is an isochronous centre.*

**Proof.** The origin is a centre, as the system is Hamiltonian. The period of a trajectory  $\gamma$  is given by

$$T(\gamma) = \left| \int_{\gamma} dt \right| = \left| \int_{\gamma} \frac{i}{z + f(z)} dz \right| = 2\pi \text{Residue} \left( \frac{1}{z + f(z)}, 0 \right) = 2\pi,$$

and so the origin is an isochronous centre. ■

**Theorem 3** *Let  $x = r \cos \theta$ ,  $y = r \sin \theta$ . Then system (4) has an isochronous centre at the origin if  $\dot{\theta} = -1$ .*

**Proof.** The result is immediate. ■

When choosing transformations to linearise systems, we must find a first integral of the right form. This is not a straightforward process, and the first integrals are not unique. First integrals can be found in numerous ways; for example using invariant functions and integrating factors. In this context, a function  $C$  is invariant with respect to system (4) if  $\dot{C} = C_x \dot{x} + C_y \dot{y} = LC$ , where the degree of the polynomial  $L$  (known as the cofactor of  $C$ ) is at most  $n - 1$ , where  $n$  is the degree of the system. The first stage is to search for invariant functions, products of which form an integrating factor, see [22] for examples of this technique. The integrating factor is then used to find a first integral, using the result of Lemma 4 below. A transformation is sought such that the first integral is of the form  $H(x, y) = X^2 + Y^2$ . Then system (4) becomes  $\dot{X} = Y$ ,  $\dot{Y} = -X$ .



**Lemma 4** Suppose that  $B(x, y)$  is an integrating factor of system (1). Let

$$H(x, y) = B(x, y)P(x, y)dy + f(x), \quad (15)$$

where  $f(x)$  is such that  $H_x(x, y) = -B(x, y)Q(x, y)$ . Then  $H(x, y)$  is a first integral of system (1).

**Proof.** By the definition of  $H(x, y)$  in (15), we have

$$\dot{H} = -BQP + \left( \int BPdy + f(x) \right)_y Q = -BQP + BPQ = 0,$$

and so  $H(x, y)$  is a first integral of system (1). ■

In some cases, one may be able to use more specialised results such as the following, which is attributed to Urabe. Consider the equation

$$\ddot{x} + g(x) = 0,$$

which is clearly equivalent to the first order planar system

$$\dot{x} = y, \quad \dot{y} = -g(x). \quad (16)$$

If  $xg(x) > 0$  for  $x \neq 0$ , then  $\frac{1}{2}y^2 + \int_0^x g(u)du = \text{constant}$  is a family of closed orbits surrounding the origin. Hence the origin is a centre of system (16).

**Theorem 5 (Urabe)** Define  $X$  by

$$\frac{1}{2}X^2 = \int_0^x g(u)du.$$

If the relationship above can be inverted, with  $x$  and  $X$  having the same sign, and we obtain  $x = \mu X + h(X)$ , where  $\mu$  is a constant and  $h(X)$  is an even function of  $X$ , then the origin is an isochronous centre of system (16).

**Proof.** See [29] or [25]. ■

More recently other methods for establishing sufficiency have been considered in [27]. However, these results are not required for the systems investigated in this paper.

## 4 Homogeneous Cubic Systems

We have tested our implementation of Pleshkan's algorithm on systems whose isochronicity conditions are already known. We have successfully

replicated the results for both quadratic and homogeneous cubic systems (see [19] and [24] respectively) amongst others. To illustrate our approach, we shall present the details of our investigation of the homogeneous cubic system (5). As the reduction of the Pleshkan polynomials is straightforward, in this example, we do not introduce the known conditions for the origin to be a centre. In [24], Pleshkan proved that the origin is an isochronous centre of system (5) if and only if one of the following holds:

- (1)  $A_{21} = A_{12} = A_{03} = 0$ ;
- (2)  $A_{21} = A_{03} = 0$ ,  $A_{12} = -\bar{A}_{30}$ ;
- (3)  $A_{21} = 0$ ,  $3\bar{A}_{30} + 7A_{12} = 0$ ,  $9A_{03}\bar{A}_{03} - 16A_{12}\bar{A}_{12} = 0$ ,  $9A_{03}^2\bar{A}_{12} - 16A_{12}^3 = 0$ .

We use our algorithm to compute the first six Pleshkan polynomials; we find  $\Pi_1 = A_{21}$ . Let  $A_{21} = 0$ , then

$$\Pi_2 = -3A_{03}\bar{A}_{03} - 4A_{12}\bar{A}_{12} - 4A_{30}A_{12}.$$

If  $A_{12} = 0$ , then  $\Pi_2 = A_{03}\bar{A}_{03}$  is a factor of  $\Pi_j$ ,  $j = 3, \dots, 6$ . With  $A_{21} = A_{12} = A_{03} = 0$  the system becomes

$$i\dot{z} = z + A_{30}z^3$$

and the origin is an isochronous centre by Theorem 2; this is condition (1) above.

Assume that  $A_{12}$  is non-zero and let

$$A_{30} = \frac{-3A_{03}\bar{A}_{03} - 4A_{12}\bar{A}_{12}}{4A_{12}}.$$

Then

$$\Pi_3 = \bar{A}_{03}(8A_{12}(-9A_{03}^2\bar{A}_{12} + 24A_{12}^3 + 8A_{12}^2\bar{A}_{30}) + 27A_{03}^3\bar{A}_{03}).$$

We first consider  $A_{03} = 0$ . We find that  $\Pi_4 = -A_{12}^2\bar{A}_{12}^2(A_{12} + \bar{A}_{30})$  is a factor of  $\Pi_6$  and that  $\Pi_5 = 0$ . When  $A_{21} = A_{03} = 0$ ,  $A_{12} = -\bar{A}_{30} = -a + ib$ , then, in polar coordinates, system (5) becomes

$$\dot{r} = 2r^3(a \sin(2\theta) + b \cos(2\theta)), \quad \dot{\theta} = -1$$

and the origin is an isochronous centre by Theorem 3; this is condition (2) above.

Assume now that  $A_{12}A_{03}$  is non-zero and let

$$\bar{A}_{03} = \frac{8A_{12}(9A_{03}^2\bar{A}_{12} - 24A_{12}^3 - 8A_{12}^2\bar{A}_{30})}{27A_{03}^3}.$$

Let  $R(A, B, c)$  denote the resultant of  $A$  and  $B$  with respect to  $c$ , and  $\#$  represent a (large) integer. Then  $\Pi_4 = \Pi_5 = 0$  only if

$$R(\Pi_4, \Pi_5, \bar{A}_{12}) = \#A_{12}^{12}A_{03}^{12}(3A_{12} + \bar{A}_{30})^4(7A_{12} + 3\bar{A}_{30})\phi_1 = 0,$$

where  $\phi_1$  is a homogeneous polynomial of degree six in  $A_{12}$  and  $\bar{A}_{30}$ . If  $\bar{A}_{30} = -3A_{12}$ , then  $\Pi_4 = A_{12}\bar{A}_{12}^2A_{03}^4 \neq 0$  and similarly if  $\phi_1 = 0$ , then we would require  $A_{12} = 0$  in order to achieve  $\Pi_4 = \Pi_5 = \Pi_6 = 0$ . When  $\bar{A}_{30} = -\frac{7}{3}A_{12}$ , we find that

$$\Pi_4 = A_{12}\chi(243A_{03}^2\bar{A}_{12} - 64A_{12}^3), \quad \Pi_5 = A_{12}\chi\phi_2$$

and

$$\Pi_6 = A_{12}\chi\phi_3,$$

where  $\chi = 9A_{03}^2\bar{A}_{12} - 16A_{12}^3$  and  $\phi_2, \phi_3$  are homogeneous polynomials of degree six in  $A_{03}, A_{12}$  and  $\bar{A}_{12}$ . If  $243A_{03}^2\bar{A}_{12} - 64A_{12}^3 = 0$ , then  $\phi_2 = \phi_3 = 0$  only if  $A_{12} = 0$ .

Let  $A_{12} = R_1e^{i\theta_1}$  and  $w = ze^{-\frac{1}{2}\theta_1}$ , then when  $A_{21} = 0, \bar{A}_{30} = -\frac{7}{3}A_{12}, 9A_{03}\bar{A}_{03} - 16A_{12}\bar{A}_{12} = 0, 9A_{03}^2\bar{A}_{12} - 16A_{12}^3 = 0$  the system becomes

$$i\dot{w} = w - \frac{7}{3}R_1w^3 + R_1w\bar{w}^2 + \frac{4}{3}R_1\bar{w}^3.$$

We note that such a transformation preserves isochronicity. The real form of the transformed system is

$$\dot{x} = y \left( 1 - 12R_1x^2 + \frac{8}{3}R_1y^2 \right), \quad \dot{y} = -x(1 + 4R_1y^2),$$

which admits  $C = 1 + 4R_1y^2 = 0$  as an invariant curve, with cofactor  $L = -8R_1xy$ . Hence we find  $B(x, y) = C^{-4}$  is an integrating factor and  $H(x, y) = (x^2 + y^2(1 + \frac{8}{3}R_1y^2)^2)C^{-3}$  is a first integral. Let  $X = xC^{-\frac{3}{2}}, Y = y(1 + \frac{8}{3}R_1y^2)C^{-\frac{3}{2}}$ , such that  $H(x, y) = X^2 + Y^2$ . Then  $\dot{X} = Y, \dot{Y} = -X$  and the system is transformed to the linear isochrone. Since this spatial transformation preserves isochronicity, the origin is an isochronous centre of the original system when condition (3) above holds.

The ability to replicate the known results for systems such as (5), (6) and others gives confidence in the integrity of our algorithm and reduction process. In the following two sections we determine conditions for the origin to be an isochronous centre for two systems for which such results were previously unknown.

## 5 The Cherkas System

We now derive necessary and sufficient conditions for the origin to be an isochronous centre for system (7) with at least one of the coefficients,  $a_i$  non-zero. System (7) was originally investigated by Cherkas [5], but it was later found that the centre conditions obtained were incomplete. The complete set of necessary and sufficient centre conditions is stated in [17] and [8].

**Theorem 6** *The origin is a centre for system (7) if and only if one of the following conditions hold:*

- (i)  $a_1 = a_5 - a_6, a_3 = a_5(a_2 + a_5 a_6), a_5 a_6(2a_5 - 2a_6 - 3) + a_2(2a_5 - a_6 - 2) = 0;$
- (ii)  $a_4 = 0;$
- (iii)  $a_5 = 0, a_6 = 1, a_1 = -1, a_3 = -a_2;$
- (iv)  $a_5 = 1, a_6 = 1, a_1 = 0, a_3 = \frac{2}{3}a_2;$
- (v)  $a_5 = \frac{1}{3}, a_6 = -\frac{2}{3}, a_1 = 1, a_3 = \frac{1}{9}a_2;$
- (vi)  $a_5 = \frac{2}{3}, a_6 = -\frac{4}{3}, a_1 = 2, a_3 = \frac{8}{9}(a_2 - 1);$
- (vii)  $a_5 = 0, a_6 = -3, a_1 = 3, a_3 = \frac{1}{3}a_2;$
- (viii)  $a_5 = 1, a_6 = -3, a_1 = 4, a_3 = 2a_2 - 8.$

**Proof.** See [8]. ■

We shall proceed by analysing each of the eight centre conditions separately, first obtaining necessary conditions and then turning to the question of sufficiency. We note that knowledge of the centre conditions is not essential, but their inclusion eases the computation of the Pleshkan polynomials and their subsequent reduction.

**Theorem 7** *The origin is an isochronous centre for system (7) if and only if one of the following conditions holds:*

- (1)  $a_1 = \frac{1}{2}, a_2 = a_3 = a_4 = 0, a_6 = -\frac{1}{2};$
- (2)  $a_1 = a_2 = a_3 = a_4 = 0, a_6 = -1;$
- (3)  $a_1 = a_2 = a_3 = a_4 = 0, a_6 = -\frac{1}{4};$
- (4)  $a_1 = \frac{1}{2}, a_2 = a_3 = a_4 = 0, a_6 = -2;$
- (5)  $a_1 = 2, a_2 = 1, a_3 = a_4 = 0, a_6 = -2;$
- (6)  $a_1 = 1, a_2 = \frac{1}{3}, a_3 = a_4 = 0, a_6 = -3;$
- (7)  $a_1 = \frac{3}{2}, a_2 = 1, a_3 = \frac{1}{4}, a_4 = 0, a_6 = -4;$
- (8)  $a_1 = \frac{3}{2}, a_2 = 1, a_3 = \frac{1}{4}, a_4 = 0, a_6 = -1;$
- (9)  $a_1 = 1, a_2 = \frac{1}{3}, a_3 = a_4 = 0, a_6 = -\frac{3}{4};$
- (10)  $a_1 = 2, a_2 = \frac{1}{9}(a_4^2 + 9), a_3 = a_5 = 0, a_6 = -2.$

**Proof.** The necessity and sufficiency of these conditions are proved in the following two sub-sections. ■

### 5.1 Necessary conditions for Isochronicity

**Lemma 8** *If any one of the conditions (iii) to (viii) of Theorem 6 holds, then the origin cannot be an isochronous centre.*

**Proof.** In each case, if the first Pleshkan polynomial is zero, then the second must be non-zero, hence the origin cannot be an isochronous centre. ■

**Lemma 9** *Suppose that condition (ii) of Theorem 6 holds. Then the origin is an isochronous centre for system (7) only if one of the conditions (1) to (9) of Theorem 7 holds.*

**Proof.** We calculate Pleshkan polynomials,  $\Pi_i$ , for system (7) with  $a_4 = 0$ . We find

$$\Pi_1 = 10a_1^2 + 10a_1a_6 + a_1 + 4a_6^2 + 5a_6 + 1 - 9a_2.$$

Let  $a_2 = \frac{1}{9}(10a_1^2 + 10a_1a_6 + a_1 + 4a_6^2 + 5a_6 + 1)$  then

$$\Pi_2 = \Phi - 27a_3(14a_1 + 12a_6 - 3),$$

where

$$\begin{aligned} \Phi = & 280a_1^4 + 770a_1^3a_6 - 70a_1^3 + 804a_1^2a_6^2 - 15a_1^2a_6 + 9a_1^2 + 362a_1a_6^3 \\ & + 12a_1a_6^2 - 108a_1a_6 - 28a_1 + 52a_6^4 - 35a_6^3 - 120a_6^2 - 35a_6 - 2. \end{aligned}$$

When  $14a_1 + 12a_6 = 3$  then  $\Pi_2 = \phi$ , a polynomial of degree 4 in  $a_6$ , and when  $\phi = 0$ ,  $\Pi_3 = \Pi_4 = 0$  is not possible. So we assume that  $(14a_1 + 12a_6 - 3)\phi \neq 0$  and let  $a_3 = \Phi/(27(14a_1 + 12a_6 - 3))$ . Now  $\Pi_3$  is a non-homogeneous polynomial of degree 8 in  $a_1, a_6$ ,  $\Pi_4$  is of degree 10 and  $\Pi_5$  is of degree 13. Let  $T_1 = R(\Pi_3, \Pi_4, a_1)$  and  $T_2 = R(\Pi_4, \Pi_5, a_1)$ . Then

$$T_1 = \#(a_6 + 1)^3(a_6 + 2)^3(a_6 + 3)(a_6 + 4)(2a_6 + 1)^6(4a_6 + 3)(4a_6 + 1)\phi^4\Delta_1, \quad (17)$$

$$T_2 = \#(a_6 + 1)^3(a_6 + 2)^3(a_6 + 3)(a_6 + 4)(2a_6 + 1)^6(4a_6 + 3)(4a_6 + 1)\phi^6\Delta_2,$$

where  $\Delta_1, \Delta_2$  are polynomials in  $a_6$  of degrees 48 and 64 respectively. Considering each linear factor of the right hand side of (17) we find that the first five Pleshkan polynomials are zero, when  $a_4 = 0$ , only if one of the conditions (1) to (9) of Theorem 7 holds. ■

**Lemma 10** *Suppose that condition (i) of Theorem 6 holds and  $a_4 \neq 0$ . Then the origin is an isochronous centre for system (7) only if condition (10) of Theorem 7 holds.*

**Proof.** We have  $a_1 = a_5 - a_6$ . Assume for the time being that  $(2 - 2a_5 + a_6) \neq 0$  and let  $a_2 = a_5a_6(2a_5 - 2a_6 - 3)(2 - 2a_5 + a_6)^{-1}$ . Now

$a_3 = a_5^2 a_6 (1 + a_6) (2a_5 - a_6 - 2)^{-1}$  and we compute that

$$\Pi_1 = \chi - (2a_5 - a_6 - 2)a_4^2,$$

where

$$\chi = 2 + 9a_6 + 18a_5^2 + 12a_6^2 - 20a_5^3 + 12a_5^2 a_6 + 4a_6^3.$$

Let  $a_4^2 = \chi(2a_5 - a_6 - 2)^{-1} > 0$ . We calculate that

$$\Pi_2 = -(2a_5 - 1)a_5^2 \Sigma_1$$

and

$$\Pi_3 = (2a_5 - 1)a_5^2 (\Sigma_2 + 315a_4 a_5 \Sigma_1 (2a_5 - a_6 - 2)i),$$

where

$$\Sigma_1 = 10a_5^2 - 10a_5 - 2a_6^2 - 3a_6$$

and

$$\begin{aligned} \Sigma_2 = & -26200a_5^5 + 28600a_5^4 a_6 + 45000a_5^4 + 9620a_5^3 a_6^2 - 34910a_5^3 a_6 \\ & - 11260a_5^3 - 5436a_5^2 a_6^3 - 8762a_5^2 a_6^2 + 24692a_5^2 a_6 - 7644a_5^2 \\ & - 972a_5 a_6^4 + 1524a_5 a_6^3 - 5299a_5 a_6^2 - 18496a_5 a_6 + 68a_5 \\ & - 1656a_6^4 - 5772a_6^3 - 4872a_6^2 + 114a_6 + 36. \end{aligned}$$

If  $a_5(2a_5 - 1) = 0$  then we must have  $a_4 = 0$ , which is contrary to assumptions. We calculate the resultant

$$R(\Sigma_1, \Sigma_2, a_6) = \#a_5(a_5 + 1)(a_5 - 1)^2(2a_5^2 - 2a_5 - 1)\Sigma_3,$$

where

$$\Sigma_3 = 21610a_5^4 - 4470a_5^3 - 5640a_5^2 + 2107a_5 - 153.$$

When  $a_5 = 1$ , we have  $a_6 = 0$  and if  $a_5 = -1$  then  $a_6 = -4$ . In both cases  $2a_5 - a_6 = 2$ , which is contrary to assumptions. When  $2a_5^2 - 2a_5 = 1$ , then  $\Sigma_1 = -(2a_6 + 5)(a_6 - 1)$ . If  $a_6 = 1$ , then  $a_4^2 < 0$  and if  $a_6 = -\frac{5}{2}$  then  $\Sigma_2 \neq 0$ . The factor of  $R(\Sigma_1, \Sigma_2, a_5)$  which corresponds to  $\Sigma_3$  is

$$\Sigma_4 = 17288a_6^4 + 4548a_6^3 - 18282a_6^2 - 34597a_6 - 11532.$$

Both  $\Sigma_3 = 0$  and  $\Sigma_4 = 0$  have two real roots; however when  $a_5$  and  $a_6$  take any of the four possible pairs of values then  $a_4^2 < 0$ .

We now return to the one excluded case. Let  $a_6 = 2(a_5 - 1)$ . To satisfy condition (i) of Theorem 6, we require  $-2a_5(a_5 - 1)(2a_5 - 1) = 0$ . Let  $a_5 = 0$ , then  $\Pi_1 = 9a_2 - a_4^2 - 9$  and  $\Pi_1$  is a factor of  $\Pi_j$ ,  $j = 2, \dots, 6$ . Hence

$$a_1 = 2, a_2 = \frac{1}{9}(a_4^2 + 9), a_3 = a_5 = 0, a_6 = -2,$$

is a necessary condition for the origin to be an isochronous centre for system (7) and this is condition (10) of Theorem 7. When  $a_5 = 1$ , then  $\Pi_1 = 9a_2 - a_4^2 - 12$ . If  $a_2 = \frac{1}{9}(a_4^2 + 12)$ , then  $\Pi_2 = -(2a_4^2 + 69) \neq 0$ , and so the origin cannot be an isochronous centre. Finally, if  $a_5 = \frac{1}{2}$ , then  $\Pi_1 = 9a_2 - a_4^2 - 9$ . Let  $a_2 = \frac{1}{9}(a_4^2 + 9)$ , then  $\Pi_2 = -4a_4^2$ , which is non-zero under current assumptions. ■

This completes the proof of the necessity of the conditions in Theorem 7.

## 5.2 Sufficient Conditions for Isochronicity

We now show, using various methods, that each of the necessary conditions listed in Theorem 7 is indeed also sufficient. We note that the sufficiency of all the conditions for the origin to be a centre in Theorem 6 can be proved under transformation to a system of Liénard form and subsequent symmetry arguments. However, such a transformation cannot be used to prove the sufficiency of the conditions for isochronicity, as it uses a space-dependent transformation of time which destroys isochronicity.

**Lemma 11** *If any one of conditions (1) to (10) stated in Theorem 7 holds, then the origin is an isochronous centre of system (7).*

**Proof.** System (7) reduces to a quadratic system when any one of the conditions (1) to (4) of Theorem 7 holds. The conditions for isochronicity in such cases were resolved by Loud in [19]. In particular, system (7) can be transformed to Loud's systems  $S_1$ ,  $S_2$ ,  $S_3$  and  $S_4$  respectively when conditions (1) to (4) hold.

We prove the sufficiency of the isochronicity conditions (5) to (9) of Theorem 7 by transforming the system to a second order differential equation and then applying Urabe's Result. We need to transform system (7) to an equation of the form  $\ddot{x} + g(x) = 0$ . When any one of the conditions (5) to (9) holds, system (7) can be written in the form

$$\dot{x} = y(x+1), \quad \dot{y} = f(x+1) + my^2,$$

where  $f$  is a function and  $m$  is a constant. Without loss of generality we can restrict our attention to  $x > -1$ . We have  $y = \dot{x}/(x+1)$ , hence

$$\ddot{x} = (x+1)f(x+1) + (m+1)\frac{(\dot{x})^2}{x+1}.$$

Let  $x + 1 = (w + 1)^{-\frac{1}{m}}$ . We take the positive root for  $w$ , so that  $x = 0$  if and only if  $w = 0$ . We have

$$\ddot{w} + m(w + 1)f\left((w + 1)^{-\frac{1}{m}}\right) = 0,$$

which is a system of the required form. According to Urabe's result, if we define

$$X^2 = 2m \int_0^w (z + 1)f\left((z + 1)^{-\frac{1}{m}}\right)dz,$$

where  $w$  has the same sign as  $X$ , then the origin is an isochronous centre of system (7) if and only if  $w = \mu X + h(X)$ , where  $\mu$  is a constant and  $h(X)$  is an even function of  $X$ . We present the details of this transformation for condition (5) of Theorem 7. In this case

$$f(x + 1) = (1 - (x + 1))(x + 1)^2 \quad \text{and} \quad m = 2.$$

Therefore, we define

$$\begin{aligned} X^2 &= 4 \int_0^w (z + 1) \left(1 - (z + 1)^{-\frac{1}{2}}\right) (z + 1)^{-1} dz \\ &= 4 \left( (w + 1)^{\frac{1}{2}} - 1 \right)^2. \end{aligned}$$

We must ensure that  $w$  has the sign of  $X$ , so

$$w = X + \frac{1}{4}X^2$$

and hence the origin is an isochronous centre of system (7) by Urabe's result. Similarly, for conditions (6) and (7), we find  $h(X) = 0$ , and for conditions (8) and (9) we have  $h(X) = -1 + \sqrt{X^2 + 1}$ . We find that  $\mu = 1$  in all cases. We conclude that the origin is an isochronous centre of system (7) when any one of the conditions (5) to (9) of Theorem 7 holds.

We now turn our attention to the one remaining condition in Theorem 7. We note that this occurs when centre condition (i) of Theorem 6 holds, whereas the other nine conditions occur when centre condition (ii) holds. When condition (10) of Theorem 7 holds, system (7) has an invariant curve  $C = 9(x + 1)^2 + a_4^2 x^2 + 3a_4 y = 0$ , with cofactor  $L = 2y - \frac{1}{3}a_4 x$ . The line  $x = -1$  is also invariant with respect to system (7). We find that these two invariant curves can be used to create an integrating factor,  $B(x, y) = 729(x + 1)C^{-3}$ . We note that  $C \neq 0$  in a neighbourhood of the origin. By Lemma 4, a first integral  $H(x, y) = \frac{1}{2}(81x^2(x + 1)^2 + 9(a_4 x^2 + 3y)^2)C^{-2}$ , can be constructed from this integrating factor. Let  $X = 9x(x + 1)(\sqrt{2}C)^{-1}$  and  $Y = 3(a_4 x^2 + 3y)(\sqrt{2}C)^{-1}$ , so that  $H(x, y) = X^2 + Y^2$ . Then system (7) becomes

$$\dot{X} = (x + 1)Y, \quad \dot{Y} = -(x + 1)X,$$



or in polar coordinates,

$$\dot{r} = 0, \quad \dot{\theta} = -(x + 1). \quad (18)$$

Simple manipulation gives

$$x + 1 = \frac{3 - \sqrt{2}a_4Y}{3 - \sqrt{2}a_4Y - 3\sqrt{2}X}.$$

We note that if  $3 - \sqrt{2}a_4Y - 3\sqrt{2}X = 0$ , then we must also have  $x = -1$ , which is not true in a neighbourhood of the origin. Therefore we may assume that the denominator is non-zero. The orbits of system (18) are given by  $r = \kappa$  (constant), which are the level curves of  $H(x, y)$ . Thus, on an orbit  $X^2 + Y^2 = \kappa^2$  and

$$\dot{\theta} = - \left( \frac{3 - \sqrt{2}a_4\kappa \sin \theta}{3 - \sqrt{2}a_4\kappa \sin \theta - 3\sqrt{2}\kappa \cos \theta} \right).$$

The period of this orbit is given by

$$T(\kappa) = \int_0^{2\pi} \frac{3 - \sqrt{2}a_4\kappa \sin \theta - 3\sqrt{2}\kappa \cos \theta}{3 - \sqrt{2}a_4\kappa \sin \theta} d\theta = 2\pi.$$

Thus, the origin is an isochronous centre of system (7). This completes the proof of Theorem 7. ■

### **Remark**

If the origin is an isochronous centre of system (7), then at most one other critical point of focus type exists, that is  $(-2, 0)$ . If this point is of focus type, it is an isochronous centre and the system is quadratic.

## **6 The Extended Kukles System**

We now turn our attention to system (8). We consider here the conditions for the origin to be an isochronous centre without prior knowledge of the centre conditions, which are not known in general.

**Theorem 12** *The origin is an isochronous centre for system (8) if one of the following conditions holds:*

- (1)  $a_1 = a_2 = 0, a_3 = k, a_4 = a_5 = a_6 = a_7 = 0;$
- (2)  $a_1 = a_2 = 0, a_3 = \frac{k}{4}, a_4 = a_5 = a_6 = a_7 = 0;$
- (3)  $a_1 = -\frac{k}{2}, a_2 = 0, a_3 = 2k, a_4 = a_5 = a_6 = a_7 = 0;$
- (4)  $a_1 = -\frac{k}{2}, a_2 = 0, a_3 = \frac{k}{2}, a_4 = a_5 = a_6 = a_7 = 0;$

- (5)  $a_1 = -k, a_2 = 0, a_3 = \frac{3k}{4}, a_4 = -\frac{k^2}{3}, a_5 = a_6 = a_7 = 0;$   
(6)  $a_1 = -k, a_2 = 0, a_3 = 3k, a_4 = -\frac{k^2}{3}, a_5 = a_6 = a_7 = 0;$   
(7)  $a_1 = -2k, a_3 = 2k, a_4 = -\frac{1}{9}(a_2^2 + 9k^2), a_5 = a_6 = a_7 = 0.$

**Proof.** System (8) with  $k = 0$  is the Kukles system, for which the conditions for the origin to be an isochronous centre are known. Therefore, without loss of generality, we can scale  $x, y$  by  $k \neq 0$  (noting that such a scaling does not destroy isochronicity). Hence we consider (8) with  $k = 1$ . Conditions (1) to (4) give the quadratic systems covered by Loud, so in each case the origin is an isochronous centre. Under condition (5) the scaled system becomes system (7) when condition (9) of Theorem 7 holds, for which sufficiency is proved using Urabe's result. Similarly, when condition (6) holds, system (8) is equivalent to system (7) when condition (6) of Theorem 7 holds. Again Urabe's result is used to prove sufficiency. Condition (7) leads to a special case of system (7) when condition (10) of Theorem 7 holds; in this case a first integral was obtained and all the orbits surrounding the origin were shown to have a period of  $2\pi$ , hence the origin is an isochronous centre. ■

We now consider whether or not the conditions contained in Theorem 12 are necessary, as well as sufficient, for the origin to be an isochronous centre for system (8). We calculate the first five Pleshkan polynomials for system (8) with  $k = 1$ . In this example we do not introduce conditions for the origin to be a centre, so the Pleshkan polynomials do not become real-valued upon reduction. We denote the real and imaginary parts of each polynomial by  $\text{Re}(\ast)$  and  $\text{Im}(\ast)$  respectively.

We find  $\text{Im}(\Pi_1) = a_2(a_1 + a_3) + a_5 + 3a_7$ . Let  $a_7 = -(a_2(a_1 + a_3) + a_5)/3$  then

$$\text{Re}(\Pi_1) = -9a_4 + \alpha,$$

where

$$\alpha = -10a_1^2 - 10a_1a_3 + a_1 - a_2^2 - 4a_3^2 + 5a_3 - 3a_6 - 1.$$

Let  $a_4 = \alpha/9$ . Then  $\text{Im}(\Pi_2) = 6a_2(a_3 - a_1 - 1)a_6 + \Phi$ , where

$$\begin{aligned} \Phi = & -35a_1^3a_2 - 78a_1^2a_2a_3 + 15a_1^2a_2 + 15a_1^2a_5 - 2a_1a_2^3 - 75a_1a_2a_3^2 \\ & + 54a_1a_2a_3 - 6a_1a_2 + 42a_1a_3a_5 + 30a_1a_5 - a_2^3a_3 - 2a_2^3 + 3a_2^2a_5 \\ & - 28a_2a_3^3 + 27a_2a_3^2 + 3a_2a_3 - 2a_2 + 15a_3^2a_5 + 42a_3a_5 - 3a_5. \end{aligned}$$

Assume for the time being that  $a_2(a_3 - a_1 - 1) \neq 0$  and let  $a_6 = \Phi/(6a_2(a_1 - a_3 + 1))$ . As no variable occurs linearly in the remaining Pleshkan polynomials we use resultant calculations to eliminate further variables. Some of the resultants we need to compute involve multivariate polynomials of high degree and cannot be obtained using the procedures available in Computer Algebra systems such as REDUCE and Maple. We use procedures we have developed for this purpose, see [23]; these exploit the

possibility of removing factors of a resultant as they arise during its calculation.

We calculate

$$\begin{aligned} R(\operatorname{Re}(\Pi_2), \operatorname{Im}(\Pi_3), a_5) &= \#a_2^6(a_1 - a_3 + 1)^4 S_1 T \\ R(\operatorname{Re}(\Pi_2), \operatorname{Re}(\Pi_3), a_5) &= \#a_2^6(a_1 - a_3 + 1)^6 S_2 T \\ R(\operatorname{Re}(\Pi_2), \operatorname{Im}(\Pi_4), a_5) &= \#a_2^6(a_1 - a_3 + 1)^6 S_3. \end{aligned}$$

Now  $T = a_2^4 + 2\beta a_2^2 + \gamma^2$ , where

$$\beta = 7a_1^2 + 10a_1 a_3 + 14a_1 + 7a_3^2 + 10a_3 + 1, \quad \gamma = 5a_1^2 + 14a_1 a_3 + 10a_1 + 5a_3^2 + 14a_3 - 1.$$

Clearly with  $a_2 \neq 0$ , we cannot have  $T = 0$  unless  $\beta < 0$ . For  $a_2^2$  to be a positive real root of  $T = 0$  we require  $a_1 + a_3 \geq 0$  or  $a_1 + a_3 \leq -2$ . However if  $a_1, a_3$  satisfy either of these conditions then  $\beta > 0$ . We conclude that  $T \neq 0$ , under current assumptions. We investigate the possibility that  $S_1 = S_2 = S_3 = 0$ . These are polynomials in  $a_1, a_2, a_3$  with  $a_2$  occurring to even powers only. We calculate

$$\begin{aligned} R(S_1, S_2, a_2^2) &= \#m^{29}(m+2)(4m+a_1+2)\gamma^4 \Omega_1 \Omega_2, \\ R(S_1, S_3, a_2^2) &= \#m^{35}(m+2)^2(4m+a_1+2)^{21} \gamma^4 \Omega_3 \Omega_4, \end{aligned}$$

where  $m = a_1 + a_3$  and  $\Omega_1$  is degree 69 in  $a_1$ , 93 in  $m$ ;  $\Omega_2$  is degree 32 in  $a_1$ , 46 in  $m$ ;  $\Omega_3$  is degree 94 in  $a_1$ , 127 in  $m$ ;  $\Omega_4$  is degree 40 in  $a_1$ , 58 in  $m$ . We cannot calculate the resultants of  $\Omega_1$  (or  $\Omega_2$ ),  $\Omega_3$  (or  $\Omega_4$ ) with respect to either  $m$  or  $a_1$  as the expressions generated exceed the space available on the Compaq Alpha XP1000 workstation, with 1Gb of memory, that we use. We have investigated the possibility of eliminating two variables simultaneously from  $S_1, S_2, S_3$  using an implementation of the Kapur-Saxena-Yang variant of the Dixon resultant due to Lewis [15]. This approach yields some simple factors and one, not necessarily irreducible, factor of degree 2069 in  $a_1$ . The simple factors confirm the results we have obtained, the other factor remains to be considered further. We conjecture that there are no conditions for the origin to be an isochronous centre, with  $\Omega_1 \Omega_2 = \Omega_3 \Omega_4 = 0$ , that are not covered elsewhere.

When  $(m+2)\gamma = 0$ , all possible isochronous centres have  $a_2 = 0$ . Consider  $m = a_1 + a_3 = 0$ ; either  $a_2 = 0$ , which is contrary to current assumptions, or

$$a_1 = -2, a_3 = 2, a_4 = \frac{-(a_2^2 + 9)}{9}, a_5 = a_6 = a_7 = 0 \quad (19)$$

is a necessary condition for the origin to be an isochronous centre. If  $4m + a_1 + 2 = 0$  then the origin can only be an isochronous centre when (19) holds.

Next we consider the situation with  $a_2 = 0$ . If  $a_5 = -3a_7$  then  $\text{Im}(\Pi_1) = 0$  and  $a_7$  is a factor of all the imaginary parts of the calculated Pleshkan polynomials. Consider first  $a_7 = 0$ , with  $a_4 = \alpha/9$ . We calculate the following resultants to eliminate  $a_6$ ;  $R_1 = R(\text{Re}(\Pi_2), \text{Re}(\Pi_3), a_6)$ ,  $R_2 = R(\text{Re}(\Pi_2), \text{Re}(\Pi_4), a_6)$  and  $R_3 = R(\text{Re}(\Pi_2), \text{Re}(\Pi_5), a_6)$  where  $R_1, R_2, R_3$  are polynomials of degrees 12, 20, 25 respectively in  $a_1, a_3$ . Further resultant calculations with respect to  $a_1$  give:

$$\begin{aligned} R(R_1, R_2, a_1) &= \#(4a_3 - 3)(a_3 - 3)(4a_3 - 1)(a_3 - 1)(a_3 - 2)(2a_3 - 1)\Omega_5\Omega_6, \\ R(R_1, R_3, a_1) &= \#(4a_3 - 3)(a_3 - 3)(4a_3 - 1)^3(a_3 - 1)(a_3 - 2)(2a_3 - 1)\Omega_7\Omega_8, \end{aligned}$$

where  $\Omega_5, \Omega_6, \Omega_7, \Omega_8$  are irreducible polynomials of degrees 72, 86, 96, 110 respectively in  $a_3$  and multiplicity of the linear factors is ignored. Clearly we cannot have  $\Omega_5\Omega_6 = \Omega_7\Omega_8 = 0$ .

Considering the linear factors of these resultants we find the following necessary conditions for the origin to be an isochronous centre:

- (i)  $a_1 = a_2 = 0, a_3 = 1, a_4 = a_5 = a_6 = a_7 = 0$ ;
- (ii)  $a_1 = a_2 = 0, a_3 = 1/4, a_4 = a_5 = a_6 = a_7 = 0$ ;
- (iii)  $a_1 = -1/2, a_2 = 0, a_3 = 2, a_4 = a_5 = a_6 = a_7 = 0$ ;
- (iv)  $a_1 = -1/2, a_2 = 0, a_3 = 1/2, a_4 = a_5 = a_6 = a_7 = 0$ ;
- (v)  $a_1 = -1, a_2 = 0, a_3 = 3/4, a_4 = -1/3, a_5 = a_6 = a_7 = 0$ ;
- (vi)  $a_1 = -1, a_2 = 0, a_3 = 3, a_4 = -1/3, a_5 = a_6 = a_7 = 0$ ;
- (vii)  $a_1 = -2, a_2 = 0, a_3 = 2, a_4 = -1, a_5 = a_6 = a_7 = 0$ .

When  $a_2 = 0$ , but  $a_7 \neq 0$  we have

$$a_5 = -3a_7, \quad a_6 = -5a_1 - 8a_1a_3 - 3a_1 - 3a_3^2 - 3a_3 - 3a_4, \quad \text{Re}(\Pi_1) = \gamma$$

and  $\text{Re}(\Pi_2), \text{Im}(\Pi_3), \text{Re}(\Pi_3), \text{Im}(\Pi_4)$  are polynomials in  $a_1, a_3, a_4, a_7$ . We calculate

$$\begin{aligned} R(\text{Re}(\Pi_2), \text{Im}(\Pi_3), a_4) &= \#a_7^2U_1, \\ R(\text{Re}(\Pi_2), \text{Re}(\Pi_3), a_4) &= \#U_2, \\ R(\text{Re}(\Pi_2), \text{Im}(\Pi_4), a_4) &= \#a_7^2U_3, \end{aligned}$$

where  $U_1, U_2, U_3$  are polynomials in  $a_1, a_3, a_7$ . Further calculation gives

$$\begin{aligned} R(U_1, U_2, a_7) &= \#(a_1 - a_3 + 1)^4V_1^2V_2^2, \\ R(U_1, U_3, a_7) &= \#(a_1 - a_3 + 1)^4V_3^2V_4^2. \end{aligned}$$

When  $a_1 - a_3 = -1$  we find that  $U_1 = U_2 = 0$  if and only if  $a_1 = -\frac{1}{2}, a_3 = \frac{1}{2}$ . Finally

$$\begin{aligned} R(\gamma, V_1, a_3) &= \#(2a_1 + 1)^4W_1, & R(\gamma, V_2, a_3) &= \#(2a_1 + 1)^2W_2, \\ R(\gamma, V_3, a_3) &= \#(2a_1 + 1)^4W_3, & R(\gamma, V_4, a_3) &= \#(2a_1 + 1)^3W_4, \end{aligned}$$

where the  $W_i$  are distinct irreducible polynomials of degree 14 in  $a_1$  alone. When  $a_1 = -\frac{1}{2}$  we must have  $a_3 = \frac{1}{2}$  and  $\text{Re}(\Pi_2) = a_4^2 + a_7^2$ , which is non-zero under current assumptions. We conclude that there are no isochronous centres with  $a_2 = 0$ ,  $a_7 \neq 0$ .

When  $a_2 \neq 0$ , but  $a_3 - a_1 = 1$  and  $a_4 = \alpha/9$ , we have  $\text{Im}(\Pi_2) = \delta + a_5\psi$ , where

$$\delta = -4a_2(72a_1^3 + 72a_1^2 + a_1a_2^2 + 18a_1 + a_2^2) \quad \text{and} \quad \psi = 24a_1^2 + 48a_1 + 18 + a_2^2.$$

We find that if  $\delta = \psi = 0$  there are no isochronous centres for real values of the coefficients  $a_i$ . Similarly, when  $\psi \neq 0$  and  $a_5 = -\delta/\psi$  there are no isochronous centres under current hypotheses.

In terms of the coefficients before scaling, conditions (i) to (vi) and (19) are precisely conditions (1) to (7) of Theorem 12. We note that when  $k = 0$ , the origin is an isochronous centre for system (8) if and only if condition (7) of Theorem 12 holds. This gives us the following result.

**Theorem 13** *If  $\Omega_1\Omega_2 \neq 0$  or  $\Omega_3\Omega_4 \neq 0$  then the origin is an isochronous centre for system (8) if and only if one of the conditions of Theorem 12 holds.*

**Proof.** We have shown above that the conditions of Theorem 12 are necessary, sufficiency follows from Theorem 12. ■

**Theorem 14** *If  $a_7 = 0$  then the origin is an isochronous centre for system (8) if and only if one of the conditions of Theorem 12 holds.*

**Proof.** The proof follows from Theorems 12 and 13. ■

### **Remark**

We have obtained all the necessary and sufficient conditions for the origin to be an isochronous centre for the sub-class with  $a_7 = 0$ . Moreover when  $a_7 \neq 0$ , for all the known centre conditions which are given in [14], there are no isochronous centres. This leads us to conjecture that the origin is an isochronous centre for system (8) if and only if one of the conditions of Theorem 12 holds. Then, if it is an isochronous centre, at most one other critical point of focus type can exist, that is  $(-\frac{2}{k}, 0)$ . If this point is of focus type, it is an isochronous centre and the system is quadratic.

## 7 Concluding Remarks

We are confident in the validity of our computer implementation of Pleshkan's algorithm. This conviction stems from the agreement with known results and also the independent proofs of sufficiency.

We have investigated differential systems of the form

$$\dot{x} = y(1 + x), \quad \dot{y} = -x + q(x, y), \quad (20)$$

where  $q$  is a polynomial without constant or linear terms. The conditions for the origin to be an isochronous centre when  $q$  is a quadratic polynomial were established by Loud [19]. For system (8), where  $q$  is a polynomial of degree 3, we find that the origin is an isochronous centre if and only if the origin is a centre (not necessarily an isochronous centre) for the quadratic part of (8). Similarly, for (7) (where  $q$  is of degree 4) the origin must be a centre for the quadratic part, and for the quadratic plus cubic parts, for the origin to be an isochronous centre. This is certainly not the case for systems in general, see [18] for example. We pose the question - is it true for systems of the form (20) whatever the degree of  $q$ ? That is, if  $q$  is of degree  $n$  is it a necessary condition for isochronicity that the origin be a centre for the sub-systems of degree 2 to degree  $n - 1$ ?

Assuming this is the case we consider the quartic system with

$$q = a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3 \\ + a_8x^4 + a_9x^3y + a_{10}x^2y^2 + a_{11}xy^3 + a_{12}y^4,$$

with at least one of the degree 4 terms present. Now the origin is a centre for the quadratic part of (20) if  $a_2(a_1 + a_3) = 0$ . We find that, when  $a_1 = -a_3$ , and the origin is a centre for the cubic part of (20), the origin cannot be an isochronous centre for (20). Furthermore, when  $a_2 = 0$ , with the origin as a centre for the cubic part of (20), the origin can only be an isochronous centre if  $a_5 = a_7 = 0$  also.

Finally we suggest that, if the origin is an isochronous centre for systems of the form (20), there are no critical points at infinity other than those on the invariant line  $x = -1$ .

## References

- [1] J Chavarriga, J Giné and I A García, Isochronous centers of a linear center perturbed by fourth degree homogeneous polynomial, *Bull. Sci. Math.* **123** (1999), pp 77-96.

- [2] J Chavarriga, J Giné and I A García, Isochronous centers of a linear center perturbed by fifth degree homogeneous polynomials, *J. Comput. Appl. Math.* **126** (2000), pp 351-368.
- [3] J Chavarriga, I A García and J Giné, Isochronicity into a family of time-reversible cubic vector fields, *App. Maths and Comp.* **121** (2001), pp 129-145.
- [4] J Chavarriga and M Sabatini, A Survey of Isochronous Centers, *Qualitative Theory of Dynamical Systems* **1** (1999), pp 1-70.
- [5] L A Cherkas, Conditions for a Liénard equation to have a centre, *Differentsial'nye Uravneniya* **12** (1976), pp 201-206.
- [6] C J Christopher and J Devlin, Isochronous centers in planar polynomial systems, *SIAM J. Math. Anal.* **28** (1) (1997), pp 162-177.
- [7] C Christopher and J Devlin, On the Classification of Liénard Systems with Amplitude-Independent Periods, *J. Diff. Equations* **200** (2004), pp 1-17.
- [8] C J Christopher, N G Lloyd and J M Pearson, On Cherkas's method for centre conditions, *Nonlin. World* **2** (1995), pp 459-469.
- [9] A Cima, A Gasull, V Mañosa and F Mañosas, Algebraic properties of the Liapunov and period constants, *Rocky. Mount. J. Math.* **27** (1997), pp 471-501.
- [10] J Devlin, Coexisting isochronous and nonisochronous centres, *Bull. Lond. Math. Soc.* **28** (1996), pp 495-500.
- [11] A Gasull and V Mañosa, An explicit expression of the first Liapunov and period constants with applications, *J. Math. Anal. Appl.* **211** (1997), pp 190-212.
- [12] J M Hill, Isochronicity and Coexisting Fine Foci, PhD Thesis, The University College of Wales, Aberystwyth (2004).
- [13] J M Hill, N G Lloyd and J M Pearson, Properties of a predator-prey model with intratrophic predation, preprint.
- [14] J M Hill, N G Lloyd and J M Pearson, The extended Kukles system, preprint.
- [15] R H Lewis, Fermat, Computer Algebra System, [www.bway.net/~lewis](http://www.bway.net/~lewis) .
- [16] Y Liu and W Huang, A new method to determine isochronous center conditions for polynomial differential systems, *Bull. Sci. Math.* **127** (2003), pp 133-148.
- [17] N G Lloyd and J M Pearson, Limit Cycles and Centres: An Example, *Diff. Equations, Dyn. Systems and Control Science*, ed. Elworthy, K D, Everitt, W N & Lee, E B (Marcel Dekker, New York 1993), pp 143-158.
- [18] N G Lloyd, C J Christopher, J Devlin, J M Pearson and N Yasmin, Quadratic-like cubic systems, *Diff. Equations and Dyn. Systems* **5** (3/4) (1997), pp 329-345.

- [19] W S Loud, Behaviour of the period of solutions of certain plane autonomous systems near centers, *Contributions to Diff. Equations* **3** (1) (1964), pp 21-36.
- [20] P Mardešić, C Rousseau and B Toni, Linearization of Isochronous Centers, *J. Diff. Equations* **121** (1995), pp 67-108.
- [21] D J Needham, A centre theorem for two-dimensional complex holomorphic systems and its generalizations, *Proc. Roy. Soc. London Ser. A* **450** (1995), pp 225-232.
- [22] J M Pearson, N G Lloyd and C J Christopher, Algorithmic derivation of centre conditions, *SIAM Review* **38** (4) (1996), pp 619-636.
- [23] J M Pearson and N G Lloyd, Space saving calculation of the resultant of multivariate polynomials, Institute of Mathematical and Physical Sciences, University of Wales, Aberystwyth, internal report.
- [24] I I Pleshkan, A new method of investigating the isochronicity of a system of two differential equations, *Differential Equations* **5** (1969), pp 796-802.
- [25] M Robnik and V G Romanovski, On Urabe's criteria of isochronicity, *J. of Phys. A: Math. Gen.* **32** (1999), pp 1279-1283.
- [26] V G Romanovski, Y H Wu and C Christopher, Linearizability of three families of cubic systems, *J. Phys. A: Math. Gen.* **36** (2003), pp 145-152.
- [27] M Sabatini, On the period function of Liénard systems, *J. Diff. Equations* **152** (2) (1999), pp 467-487.
- [28] M Sabatini, Quadratic isochronous centers commute, *Applicaciones Mathematicae* **26**, (3) (1999), pp 357-362.
- [29] M Urabe, Potential forces which yield periodic motions of a fixed period, *J. Math. Mech.* **10** (1961), pp 569-578.