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# Short Papers

## Modeling Random Fuzzy Renewal Reward Processes

Qiang Shen, Ruiqing Zhao, and Wansheng Tang

**Abstract**—This short paper discusses the modeling of random fuzzy renewal reward processes in which the interarrival times and rewards are represented by nonnegative random fuzzy variables. Based on random fuzzy theory, a random fuzzy variable denotes a measurable function from a credibility space to a collection of random variables. Under this setting, the long-run expected reward per unit time is addressed and the theorem on random fuzzy renewal rewards is established. The utility of this research is demonstrated with a realistic application case.

**Index Terms**—Credibility measure, fuzzy variables, interarrival time, random fuzzy variables, renewal processes, stochastic processes.

### I. INTRODUCTION

Renewal reward processes play an important role in process theory. A large number of models, such as dispatching a train and queuing in a bank, are special cases of these processes. Thus, modeling such processes has a significant value in real-world applications. Having recognized this, stochastic renewal reward processes have been well developed [22]. Indeed, based on probability theory, much research has been reported to perform such modeling (e.g., [1], [2], [5], [22], [26], and [27]). In representing stochastic renewal processes, an underlying assumption is that the interarrival times and rewards are deemed to be random variables. However, in many cases, consideration of randomness alone will not help to evaluate a process in a satisfactory manner. This is because fuzziness and randomness in one process are often mixed up, and it is not easy to distinguish between them from a practical viewpoint. They are usually required to be considered simultaneously.

There are two approaches to deal with these kinds of phenomena. One is the fuzzy random theory first introduced by Kwakernaak [7], [8]. Roughly speaking, a fuzzy random variable is a mathematical descriptor for a fuzzy random phenomenon and defined as a measurable function from a probability space to a collection of fuzzy sets. Based on fuzzy random theory, the modeling of typical fuzzy random processes has been considered by several authors. Hwang [6] investigated a renewal process in which the interarrival times were considered independent and identically distributed (i.i.d.) fuzzy random variables and a theorem on the fuzzy rate of fuzzy random renewal processes was provided. Popova and Wu [21] considered a renewal reward process with fuzzy random interarrival times and rewards. Also, the long-run average fuzzy reward per unit time was stated in [21]. Zhao and Tang [30] addressed some important properties of fuzzy random renewal processes, which are generated by a sequence of i.i.d. fuzzy random

interarrival times. This includes an extended version of the renewal equation, Blackwell's renewal theorem, and Smith's key renewal theorem, all in fuzzy random form. Li *et al.* [10] introduced the concept of a fuzzy random delayed renewal process in which the interarrival times between two events are characterized as fuzzy random variables. They also defined a fuzzy random equilibrium renewal process as a special case of the fuzzy random delayed renewal process [10]. By dealing with interarrival times as exponentially distributed fuzzy random variables, Li *et al.* [9] considered a fuzzy random homogeneous Poisson process and a fuzzy random compound Poisson process, and established several important properties of these two types of processes.

The other approach is to use the random fuzzy theory developed by Liu [12]. Briefly, a random fuzzy variable is a measurable function from a possibility space to a collection of random variables. In this approach, the expected value operator of random fuzzy variables was introduced in [18]. Based on this theory, Shen *et al.* [24] conducted an investigation into the modeling of alternating renewal processes. This resulted in a theorem on the limit value of the average chance of a given random fuzzy event, in terms of "a system being on at time  $t$ ." These renewal processes were represented by sequences of positive random fuzzy vectors. Zhao *et al.* [31] considered a random fuzzy process in which the interarrival times were modeled as i.i.d. random fuzzy variables, and described certain properties of the random fuzzy renewal processes, including the random fuzzy elementary renewal theorem and Blackwell's theorem. As a continuation of the significant initial work of [31], this short paper considers the modeling of another important type of random fuzzy process—the random fuzzy renewal reward process.

The short paper is organized as follows. For completeness, Section II briefly outlines the background of fuzzy variables and random fuzzy variables within the framework of the credibility theory [12], in terms of basic concepts and properties relevant to the present short paper. Section III describes the random fuzzy renewal reward processes and presents the random fuzzy renewal reward theorem. Section IV shows a worked example, illustrating the effectiveness of the proposed theorem. Section V concludes the short paper.

### II. FUZZY VARIABLES AND RANDOM FUZZY VARIABLES

To cope with uncertain and imprecise process modeling, the possibility measure [28] or the necessity measure [4], [23] is commonly used. In fact, the main difference between these two measures is that they consider the same question from different angles. The possibility measure assesses a given vague concept in terms of affirmation, and the necessity measure does so in terms of disaffirmation. However, their use may lead to situations where the former overrates the possibility for a vague concept to be correctly captured while the latter underrates such a possibility. To have a balanced approach, and based on the basic intuitions behind the possibility and necessity measures, a self-dual measure, called credibility, has been introduced [16], [17].

Given a universe  $\Theta$ , let  $\mathcal{P}(\Theta)$  denote the power set of  $\Theta$ . The credibility measure of a set  $A$ ,  $\text{Cr}\{A\}$ , can be defined [16] such that

$$\text{Cr}\{A\} = \frac{1}{2} (\text{Pos}\{A\} + \text{Nec}\{A\}) \quad (1)$$

where  $\text{Pos}$  and  $\text{Nec}$  represent the possibility and necessity measures, respectively. The triplet  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$  is called a credibility space. The rest of this section introduces the notions and notations closely relevant

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to the development of the work to be presented in the next section, based on the concepts of credibility space and credibility measure.

**Definition 1:** A fuzzy variable is defined as a function from the credibility space  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$  to the real line  $\mathfrak{R}$ .

**Remark 1 ([13]):** Let  $\xi$  be a fuzzy variable on the credibility space  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$ . Then its membership function can be derived from the credibility measure such that

$$\mu(x) = \min(2\text{Cr}\{\xi = x\}, 1), \quad x \in \mathfrak{R}. \quad (2)$$

**Definition 2 ([12]):** Let  $\xi$  be a fuzzy variable and  $\alpha \in (0, 1]$ . Then

$$\xi'_\alpha = \inf\{x | \mu(x) \geq \alpha\} \quad \text{and} \quad \xi''_\alpha = \sup\{x | \mu(x) \geq \alpha\} \quad (3)$$

are called the  $\alpha$ -pessimistic value and the  $\alpha$ -optimistic value of  $\xi$ , respectively.

**Proposition 1 ([18]):** Let  $\xi$  and  $\eta$  be two fuzzy variables. Then:

- 1) for any  $\alpha \in (0, 1]$ ,  $(\xi + \eta)'_\alpha = \xi'_\alpha + \eta'_\alpha$ ;
- 2) for any  $\alpha \in (0, 1]$ ,  $(\xi + \eta)''_\alpha = \xi''_\alpha + \eta''_\alpha$ .

Furthermore, if  $\xi$  and  $\eta$  are nonnegative (i.e.,  $\text{Cr}\{\xi < 0\} = 0$  and  $\text{Cr}\{\eta < 0\} = 0$ ), then:

- 3) for any  $\alpha \in (0, 1]$ ,  $(\xi \cdot \eta)'_\alpha = \xi'_\alpha \cdot \eta'_\alpha$ ;
- 4) for any  $\alpha \in (0, 1]$ ,  $(\xi \cdot \eta)''_\alpha = \xi''_\alpha \cdot \eta''_\alpha$ .

**Definition 3 ([16], [17]):** Let  $\xi$  be a fuzzy variable. The expected value  $E[\xi]$  of  $\xi$  is defined as

$$E[\xi] = \int_0^\infty \text{Cr}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\} dr \quad (4)$$

provided that at least one of the two integrals is finite (avoiding the case  $\infty - \infty$ ). Especially, if  $\xi$  is a nonnegative fuzzy variable, then  $E[\xi] = \int_0^\infty \text{Cr}\{\xi \geq r\} dr$ .

**Proposition 2 ([18]):** Let  $\xi$  be a fuzzy variable with finite expected value  $E[\xi]$ . Then

$$E[\xi] = \frac{1}{2} \int_0^1 [\xi'_\alpha + \xi''_\alpha] d\alpha \quad (5)$$

where  $\xi'_\alpha$  and  $\xi''_\alpha$  are the  $\alpha$ -pessimistic value and the  $\alpha$ -optimistic value of  $\xi$ , respectively.

**Definition 4 ([19]):** The fuzzy variables  $\xi_1, \xi_2, \dots, \xi_n$  are said to be independent if

$$\text{Cr}\{\xi_i \in B_i, i = 1, 2, \dots, n\} = \inf_{1 \leq i \leq n} \text{Cr}\{\xi_i \in B_i\} \quad (6)$$

for any sets  $B_1, B_2, \dots, B_n$  of  $\mathfrak{R}$ .

**Definition 5 ([15]):** The fuzzy variables  $\xi_1, \xi_2, \dots, \xi_n$  are said to be identically distributed if

$$\text{Cr}\{\xi_i \in B\} = \text{Cr}\{\xi_j \in B\}, \quad i, j = 1, 2, \dots, n \quad (7)$$

for any set  $B$  of  $\mathfrak{R}$ .

**Proposition 3 ([14]):** Suppose  $(\Theta_i, \mathcal{P}(\Theta_i), \text{Cr}_i), i = 1, 2, \dots, n$ , are credibility spaces. Let  $\Theta = \Theta_1 \times \Theta_2 \times \dots \times \Theta_n = \prod_{i=1}^n \Theta_i$  and  $\text{Cr}\{A\} = \sup_{(\theta_1, \theta_2, \dots, \theta_n) \in A} \inf_i \text{Cr}_i\{\theta_i\}$ ,  $A \in \mathcal{P}(\Theta)$ . Then the set function  $\text{Cr}$  is a credibility measure on  $\mathcal{P}(\Theta)$  and  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$  is a credibility space (called the product credibility space of  $(\Theta_i, \mathcal{P}(\Theta_i), \text{Cr}_i), i = 1, 2, \dots, n$ ).

**Proposition 4 ([15]):** Let  $(\Theta_i, \mathcal{P}(\Theta_i), \text{Cr}_i), i = 1, 2, \dots$ , be an arbitrary sequence of credibility spaces and  $\Theta = \prod_{i=1}^\infty \Theta_i$ . Define  $\text{Cr}$  on  $\mathcal{P}(\Theta)$  such that  $\text{Cr}\{A\} = \sup_{(\theta_1, \theta_2, \dots) \in A} \inf_i \text{Cr}_i\{\theta_i\}$ ,  $A \in \mathcal{P}(\Theta)$ . Then the set function  $\text{Cr}$  is a fuzzy measure on  $\mathcal{P}(\Theta)$  and  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$  is a credibility space (called the infinite product credibility space of  $(\Theta_i, \mathcal{P}(\Theta_i), \text{Cr}_i), i = 1, 2, \dots$ ).

Let  $(\Omega, \mathcal{A}, \text{Pr})$  be a probability space and  $\mathcal{F}$  be a collection of random variables defined on probability space  $(\Omega, \mathcal{A}, \text{Pr})$ . Then the following definitions can be introduced and propositions established.

**Definition 6 ([12]):** A random fuzzy variable is a measurable function from a credibility space  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$  to a collection of random variables  $\mathcal{F}$ .

**Example 1:** Consider an example of a random variable as given in [12]. Let  $\xi$  be the lifetime of a system. Usually, the probability distribution of  $\xi$  is assumed to be known completely except for the values of certain parameters. For instance, it might be known that the lifetime  $\xi$  is an exponentially distributed random variable with an unknown mean  $\lambda$

$$\phi(x) = \begin{cases} \frac{1}{\lambda} e^{-x/\lambda}, & \text{if } 0 \leq x < \infty \\ 0, & \text{otherwise.} \end{cases}$$

In statistics, an interval estimate or point estimate of the value of  $\lambda$  is provided by experimental data. In many practical situations, however, such data are often unavailable. If the value of  $\lambda$  is represented as a fuzzy variable, defined on the credibility space  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$ , then  $\xi$  is a random fuzzy variable that can be described by

$$\xi(\lambda(\theta)) \sim \mathcal{E}\mathcal{X}\mathcal{P}(\lambda(\theta)) \quad (8)$$

where  $\lambda$  is a fuzzy variable on  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$  and the symbol  $\mathcal{E}\mathcal{X}\mathcal{P}$  is an abbreviation of the exponential distribution.

**Proposition 5 ([15]):** Let  $\xi$  be a random fuzzy variable defined on the credibility space  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$ . Then, we have the following.

- 1) The probability  $\text{Pr}\{\xi(\theta) \in B\}$  is a fuzzy variable for any Borel set  $B$  of  $\mathfrak{R}$ .
- 2) The expected value  $E[\xi(\theta)]$  is a fuzzy variable provided that  $E[\xi(\theta)]$  is finite for each  $\theta \in \Theta$ .

**Definition 7:** A random fuzzy variable  $\xi$  defined on the credibility space  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$  is said to be positive if and only if  $\text{Pr}\{\xi(\theta) \leq 0\} = 0$  for each  $\theta \in \Theta$  with  $\text{Cr}\{\theta\} > 0$ .

**Definition 8 ([18]):** Let  $\xi$  be a random fuzzy variable defined on the credibility space  $(\Theta, \mathcal{P}(\Theta), \text{Cr})$ . The expected value  $E[\xi]$  is defined by

$$E[\xi] = \int_0^\infty \text{Cr}\{\theta \in \Theta | E[\xi(\theta)] \geq r\} dr - \int_{-\infty}^0 \text{Cr}\{\theta \in \Theta | E[\xi(\theta)] \leq r\} dr \quad (9)$$

provided that at least one of the two integrals is finite. Especially, if  $\xi$  is a nonnegative random fuzzy variable, then  $E[\xi] = \int_0^\infty \text{Cr}\{\theta \in \Theta | E[\xi(\theta)] \geq r\} dr$ .

**Remark 2 ([12]):** If the random fuzzy variable  $\xi$  degenerates to a random variable, then the expected value operator becomes

$$E[\xi] = \int_0^\infty \text{Pr}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Pr}\{\xi \leq r\} dr \quad (10)$$

which is just the conventional mathematical expectation of random variable  $\xi$ . If the random fuzzy variable  $\xi$  degenerates to a fuzzy variable, then the expected value operator becomes

$$E[\xi] = \int_0^\infty \text{Cr}\{\xi \geq r\} dr - \int_{-\infty}^0 \text{Cr}\{\xi \leq r\} dr \quad (11)$$

which is just the expected value of fuzzy variable  $\xi$ .

**Definition 9 ([11]):** The random fuzzy variables  $\xi_1, \xi_2, \dots, \xi_n$  are independent if:

- 1)  $\xi_1(\theta), \xi_2(\theta), \dots, \xi_n(\theta)$  are independent random variables for each  $\theta$ ;

2)  $E[\xi_1(\cdot)], E[\xi_2(\cdot)], \dots, E[\xi_n(\cdot)]$  are independent fuzzy variables.

*Definition 10 ([11]):* The random fuzzy variables  $\xi$  and  $\eta$  are identically distributed if

$$\sup_{Cr\{A\} \geq \alpha} \inf_{\theta \in A} \{\Pr\{\xi(\theta) \in B\}\} = \sup_{Cr\{A\} \geq \alpha} \inf_{\theta \in A} \{\Pr\{\eta(\theta) \in B\}\} \quad (12)$$

for any  $\alpha \in (0, 1]$  and Borel set  $B$  of real numbers.

*Definition 11 ([31]):* The random fuzzy variables  $\xi_i, i \in I$  are said to be i.i.d. if and only if  $\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n}$  are i.i.d. random fuzzy variables for all finite collections  $\{i_1, i_2, \dots, i_n\}$  of  $I$ , where  $I$  is an index set.

### III. RANDOM FUZZY RENEWAL REWARD PROCESSES

Let  $\xi_i$  denote the interarrival times between the  $(i-1)$ th and  $i$ th events,  $i = 1, 2, \dots$ , respectively. Suppose  $\xi_i$ 's are random fuzzy variables, each defined on their respective credibility spaces  $(\Theta_i, \mathcal{P}(\Theta_i), Cr_i)$ . Let  $S_0 = 0$  and

$$S_n = \xi_1 + \xi_2 + \dots + \xi_n \quad \forall n \geq 1. \quad (13)$$

Also, let  $N(t)$  denote the total number of events that have occurred by time  $t$ . Then

$$N(t) = \max_{n \geq 0} \{n \mid 0 < S_n \leq t\} \quad (14)$$

which is hereafter referred to as a random fuzzy renewal variable.

Between the  $(i-1)$ th and  $i$ th events,  $i = 1, 2, \dots$ , a random fuzzy reward  $\eta_i$ , defined on the credibility space  $(\Theta'_i, \mathcal{P}(\Theta'_i), Cr'_i)$ , is interpreted as the reward earned at the end of the  $i$ th renewal cycle. Here, the credibility spaces  $(\Theta_i, \mathcal{P}(\Theta_i), Cr_i)$  and  $(\Theta'_i, \mathcal{P}(\Theta'_i), Cr'_i)$  are allowed to be different.

In general, suppose:

- the random fuzzy variables  $\xi_1, \xi_2, \dots$  are i.i.d. positive random fuzzy variables;
- the random fuzzy variables  $\eta_1, \eta_2, \dots$  are i.i.d. positive random fuzzy variables;
- the sequences  $\{\xi_i\}$  and  $\{\eta_i\}$  are mutually independent;
- the image sets  $\mathcal{F}_i$  of  $\xi_i$  are totally ordered with respect to stochastic ordering, i.e., for any  $\theta_1, \theta_2 \in \Theta_i$  and  $r \in \mathfrak{R}$ , either

$$\Pr\{\xi_i(\theta_1) \leq r\} \leq \Pr\{\xi_i(\theta_2) \leq r\} \\ \text{(denoted by } \xi_i(\theta_2) \leq_d \xi_i(\theta_1) \text{)}$$

or

$$\Pr\{\xi_i(\theta_1) \leq r\} \geq \Pr\{\xi_i(\theta_2) \leq r\} \\ \text{(denoted by } \xi_i(\theta_1) \leq_d \xi_i(\theta_2) \text{)}$$

where  $i = 1, 2, \dots$

Let  $C(t)$  denote the total reward earned by time  $t$ . Then

$$C(t) = \sum_{i=1}^{N(t)} \eta_i \quad (15)$$

where  $N(t)$  is the random fuzzy renewal variable defined by (14). Thus,  $C(t)$  is a random fuzzy variable defined on credibility space  $(\Theta, \mathcal{P}(\Theta), Cr)$ , where  $(\Theta, \mathcal{P}(\Theta), Cr)$  is an infinite product credibility

space defined by

$$\Theta = \prod_{i=1}^{\infty} (\Theta_i, \Theta'_i) \quad (16)$$

and

$$Cr\{A\} = \sup_{((\theta_1, \theta'_1), (\theta_2, \theta'_2), \dots) \in A} \inf_i \min(Cr_i\{\theta_i\}, Cr'_i\{\theta'_i\})$$

for any  $A \in \mathcal{P}(\Theta)$ .

For any fixed  $\theta$ ,  $E[\xi_i(\theta)]$ ,  $E[\eta_i(\theta)]$ , and  $E[C(t)(\theta)]$  represent the expected values of the random variables  $\xi_i(\theta)$ ,  $\eta_i(\theta)$ ,  $E[N(t)(\theta)]$ , and  $C(t)(\theta)$ , respectively. When  $\theta$  is varied over all of  $\Theta$ ,  $E[\xi_i(\theta)]$ ,  $E[\eta_i(\theta)]$ ,  $E[N(t)(\theta)]$ , and  $E[C(t)(\theta)]$ , as functions of  $\theta \in \Theta$ , are fuzzy variables. The  $\alpha$ -pessimistic and  $\alpha$ -optimistic values of the fuzzy variables  $E[\xi_i(\theta)]$ ,  $E[\eta_i(\theta)]$ ,  $E[N(t)(\theta)]$ , and  $E[C(t)(\theta)]$  will then play an important role in random fuzzy renewal processes. Such values are obtained by

$$E[\xi_i(\theta)]'_\alpha = \inf \{x \mid \mu_{E[\xi_i(\theta)]}(x) \geq \alpha\} \quad (17)$$

$$E[\xi_i(\theta)]''_\alpha = \sup \{x \mid \mu_{E[\xi_i(\theta)]}(x) \geq \alpha\} \quad (18)$$

$$E[\eta_i(\theta)]'_\alpha = \inf \{x \mid \mu_{E[\eta_i(\theta)]}(x) \geq \alpha\} \quad (19)$$

$$E[\eta_i(\theta)]''_\alpha = \sup \{x \mid \mu_{E[\eta_i(\theta)]}(x) \geq \alpha\} \quad (20)$$

$$E[N(t)(\theta)]'_\alpha = \inf \{x \mid \mu_{E[N(t)(\theta)]}(x) \geq \alpha\} \quad (21)$$

$$E[N(t)(\theta)]''_\alpha = \sup \{x \mid \mu_{E[N(t)(\theta)]}(x) \geq \alpha\} \quad (22)$$

$$E[C(t)(\theta)]'_\alpha = \inf \{x \mid \mu_{E[C(t)(\theta)]}(x) \geq \alpha\} \quad (23)$$

$$E[C(t)(\theta)]''_\alpha = \sup \{x \mid \mu_{E[C(t)(\theta)]}(x) \geq \alpha\}. \quad (24)$$

*Theorem 1:* Let  $(\xi_1, \eta_1), (\xi_2, \eta_2), \dots$  be a sequence of pairs of i.i.d. positive random fuzzy interarrival times defined on the infinite product credibility space  $(\Theta, \mathcal{P}(\Theta), Cr)$ ,  $N(t)$  the random fuzzy renewal variable as given by (14), and  $C(t)$  the total reward as defined by (15). Then, for any  $\alpha \in (0, 1]$

$$E[C(t)(\theta)]'_\alpha = E[N(t)(\theta)]'_\alpha \cdot E[\eta_1(\theta)]'_\alpha \quad (25)$$

$$E[C(t)(\theta)]''_\alpha = E[N(t)(\theta)]''_\alpha \cdot E[\eta_1(\theta)]''_\alpha \quad (26)$$

where  $E[C(t)(\theta)]'_\alpha, E[C(t)(\theta)]''_\alpha, E[N(t)(\theta)]'_\alpha, E[N(t)(\theta)]''_\alpha, E[\eta_1(\theta)]'_\alpha$ , and  $E[\eta_1(\theta)]''_\alpha$  are the  $\alpha$ -pessimistic and  $\alpha$ -optimistic values of  $E[C(t)(\theta)]$ ,  $E[N(t)(\theta)]$ , and  $E[\eta_1(\theta)]$ , respectively.

*Proof:* For each  $\theta \in \Theta$ ,

$$E[C(t)(\theta)] = E \left[ \sum_{i=1}^{N(t)(\theta)} \eta_i(\theta) \right]. \quad (27)$$

From the independence of the random fuzzy sequences  $\{\xi_i\}$  and  $\{\eta_i\}$ , it follows that the random variable  $N(t)(\theta)$  is independent of the random sequence  $\{\eta_i(\theta)\}$ . Thus

$$\Pr \left\{ \sum_{i=1}^{N(t)(\theta)} \eta_i(\theta) \geq r \right\} \\ = \sum_{k=1}^{\infty} \Pr \{N(t)(\theta) = k\} \Pr \{\eta_1(\theta) + \dots + \eta_k(\theta) \geq r\}. \quad (28)$$

Hence,

$$\begin{aligned}
& E \left[ \sum_{i=1}^{N(t)(\theta)} \eta_i(\theta) \right] \\
&= \int_0^{+\infty} \Pr \left\{ \sum_{i=1}^{N(t)(\theta)} \eta_i(\theta) \geq r \right\} dr \\
&= \int_0^{+\infty} \sum_{k=1}^{\infty} \Pr \{N(t)(\theta) = k\} \Pr \{\eta_1(\theta) + \dots + \eta_k(\theta) \geq r\} dr \\
&= \sum_{k=1}^{\infty} \Pr \{N(t)(\theta) = k\} \int_0^{+\infty} \Pr \{\eta_1(\theta) + \dots + \eta_k(\theta) \geq r\} dr \\
&= \sum_{k=1}^{\infty} \Pr \{N(t)(\theta) = k\} (E[\eta_1(\theta)] + \dots + E[\eta_k(\theta)]). \quad (29)
\end{aligned}$$

Applying Proposition 1 to the above leads to

$$\begin{aligned}
& E \left[ \sum_{i=1}^{N(t)(\theta)} \eta_i(\theta) \right]_{\alpha}' \\
&= \left( \sum_{k=1}^{\infty} \Pr \{N(t)(\theta) = k\} (E[\eta_1(\theta)] + \dots + E[\eta_k(\theta)]) \right)_{\alpha}' \\
&= \sum_{k=1}^{\infty} (\Pr \{N(t)(\theta) = k\} (E[\eta_1(\theta)] + \dots + E[\eta_k(\theta)]))'_{\alpha} \\
&= \sum_{k=1}^{\infty} (\Pr \{N(t)(\theta) = k\})'_{\alpha} (E[\eta_1(\theta)] + \dots + E[\eta_k(\theta)])'_{\alpha} \\
&= \sum_{k=1}^{\infty} (\Pr \{N(t)(\theta) = k\})'_{\alpha} (E[\eta_1(\theta)]'_{\alpha} + \dots + E[\eta_k(\theta)]'_{\alpha}) \\
&= \sum_{k=1}^{\infty} (\Pr \{N(t)(\theta) = k\})'_{\alpha} \cdot k E[\eta_1(\theta)]'_{\alpha} \\
&= \left( \sum_{k=1}^{\infty} k \Pr \{N(t)(\theta) = k\} \right)'_{\alpha} \cdot E[\eta_1(\theta)]'_{\alpha} \\
&= E[N(t)(\theta)]'_{\alpha} \cdot E[\eta_1(\theta)]'_{\alpha}. \quad (30)
\end{aligned}$$

Hence, the result (25) holds. Assertion (26) can be established through a similar proof procedure. Thus, the theorem is proven.

**Theorem 2:** Let  $(\xi_1, \eta_1), (\xi_2, \eta_2), \dots$  be a sequence of pairs of i.i.d. nonnegative random fuzzy interarrival times defined on the infinite product credibility space  $(\Theta, \mathcal{P}(\Theta), Cr)$ ,  $N(t)$  the random fuzzy renewal variable as given by (14), and  $C(t)$  the total reward as defined by (15). Then

$$\lim_{t \rightarrow \infty} \frac{E[C(t)(\theta)]'_{\alpha}}{t} = \frac{E[\eta_1(\theta)]'_{\alpha}}{E[\xi_1(\theta)]'_{\alpha}} \quad (31)$$

$$\lim_{t \rightarrow \infty} \frac{E[C(t)(\theta)]''_{\alpha}}{t} = \frac{E[\eta_1(\theta)]''_{\alpha}}{E[\xi_1(\theta)]''_{\alpha}} \quad (32)$$

provided that the  $\alpha$ -pessimistic value  $E[\xi_1(\theta)]'_{\alpha}$  and the  $\alpha$ -optimistic value  $E[\xi_1(\theta)]''_{\alpha}$  of the fuzzy variables  $E[\xi_1(\theta)]$  are continuous at point  $\alpha, \alpha \in (0, 1]$ .

*Proof:* The following results have been proved in [31]:

$$\lim_{t \rightarrow \infty} \frac{E[N(t)(\theta)]'_{\alpha_0}}{t} = \frac{1}{E[\xi_1(\theta)]''_{\alpha_0}} \quad (33)$$

and

$$\lim_{t \rightarrow \infty} \frac{E[N(t)(\theta)]''_{\alpha_0}}{t} = \frac{1}{E[\xi_1(\theta)]'_{\alpha_0}}. \quad (34)$$

Applying Theorem 1 to these results leads to

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{E[C(t)(\theta)]'_{\alpha}}{t} &= \lim_{t \rightarrow \infty} \frac{E[N(t)(\theta)]'_{\alpha} \cdot E[\eta_1(\theta)]'_{\alpha}}{t} \\
&= \frac{E[\eta_1(\theta)]'_{\alpha}}{E[\xi_1(\theta)]''_{\alpha}} \quad (35)
\end{aligned}$$

and

$$\begin{aligned}
\lim_{t \rightarrow \infty} \frac{E[C(t)(\theta)]''_{\alpha}}{t} &= \lim_{t \rightarrow \infty} \frac{E[N(t)(\theta)]''_{\alpha} \cdot E[\eta_1(\theta)]''_{\alpha}}{t} \\
&= \frac{E[\eta_1(\theta)]''_{\alpha}}{E[\xi_1(\theta)]'_{\alpha}}. \quad (36)
\end{aligned}$$

The theorem is therefore proven.

Now, let  $\xi$  be one of the fuzzy variables with the  $\alpha$ -pessimistic value  $E[\xi_1(\theta)]'_{\alpha}$  and the  $\alpha$ -optimistic value  $E[\xi_1(\theta)]''_{\alpha}$ , and  $\eta$  be one of the fuzzy variables with the  $\alpha$ -pessimistic value  $E[\eta_1(\theta)]'_{\alpha}$  and the  $\alpha$ -optimistic value  $E[\eta_1(\theta)]''_{\alpha}$ , then the following theorem can be established.

**Theorem 3 (Random fuzzy renewal reward theorem):** Let  $(\xi_1, \eta_1), (\xi_2, \eta_2), \dots$  be a sequence of pairs of i.i.d. positive random fuzzy interarrival times defined on the infinite product credibility space  $(\Theta, \mathcal{P}(\Theta), Cr)$ ,  $N(t)$  the random fuzzy renewal variable as given by (14), and  $C(t)$  the total reward as defined by (15). If  $E[\eta/\xi]$  is finite, then

$$\lim_{t \rightarrow \infty} \frac{E[C(t)]}{t} = E \left[ \frac{\eta}{\xi} \right]. \quad (37)$$

*Proof:* It follows from Definition 8 and Proposition 2 that

$$\frac{E[C(t)]}{t} = \frac{1}{2} \int_0^1 \left( \frac{E[C(t)(\theta)]'_{\alpha}}{t} + \frac{E[C(t)(\theta)]''_{\alpha}}{t} \right) d\alpha \quad (38)$$

and

$$E \left[ \frac{\eta}{\xi} \right] = \frac{1}{2} \int_0^1 \left( \frac{E[\eta_1(\theta)]'_{\alpha}}{E[\xi_1(\theta)]'_{\alpha}} + \frac{E[\eta_1(\theta)]''_{\alpha}}{E[\xi_1(\theta)]''_{\alpha}} \right) d\alpha. \quad (39)$$

Hence, it suffices to prove that

$$\lim_{t \rightarrow \infty} \int_0^1 \frac{E[C(t)(\theta)]'_{\alpha}}{t} d\alpha = \int_0^1 \frac{E[\eta_1(\theta)]'_{\alpha}}{E[\xi_1(\theta)]'_{\alpha}} d\alpha \quad (40)$$

and

$$\lim_{t \rightarrow \infty} \int_0^1 \frac{E[C(t)(\theta)]''_{\alpha}}{t} d\alpha = \int_0^1 \frac{E[\eta_1(\theta)]''_{\alpha}}{E[\xi_1(\theta)]''_{\alpha}} d\alpha. \quad (41)$$

The following shows the proof for assertion (40), while assertion (41) can be proven similarly.

First of all, it follows from Theorem 2 that

$$\lim_{t \rightarrow \infty} \frac{E[C(t)(\theta)]'_{\alpha}}{t} = \frac{E[\eta_1(\theta)]'_{\alpha}}{E[\xi_1(\theta)]'_{\alpha}} \quad (42)$$

provided that  $E[\xi_1(\theta)]'_{\alpha}$  is continuous at point  $\alpha, \alpha \in (0, 1]$ .

What is now needed is to prove that there exists one integrable function  $g_1(\alpha, t)$  such that

$$\frac{E[C(t)(\theta)]'_{\alpha}}{t} \leq g_1(\alpha, t). \quad (43)$$

In order to do this, define a sequence of i.i.d. random variables  $\{\xi_i(\theta'')\}$ ,  $\theta'' \in A_i$ , such that

$$\xi_i(\theta) \leq_d \xi_i(\theta'') \quad (44)$$

for any  $\theta \in A_i$ , where  $A_i$  is the  $\alpha$ -level set of  $E[\xi_i(\theta)]$ ,  $i = 1, 2, \dots$ .

Let  $S_n(\theta'') = \sum_{i=1}^n \xi_i(\theta'')$ , and thus,  $\{S_n(\theta''), n \geq 1\}$  is a stochastic renewal process with the renewal variable

$$N(t)(\theta'') = \max_{n \geq 0} \{n \mid S_n(\theta'') \leq t\}. \quad (45)$$

In this case, it is obvious that

$$E[N(t)(\theta'')] = E[N(t)(\theta)]'_\alpha. \quad (46)$$

Define a new stochastic renewal process  $\{\bar{S}_n(\theta''), n \geq 1\}$  by letting

$$\bar{S}_n(\theta'') = \sum_{i=1}^n \bar{\xi}_i(\theta'') \quad (47)$$

and

$$\bar{N}(t)(\theta'') = \max_{n \geq 0} \{n \mid \bar{S}_n(\theta'') \leq t\} \quad (48)$$

where

$$\bar{\xi}_i(\theta'') = \begin{cases} \xi_i(\theta''), & \text{if } \xi_i(\theta'') \leq M \\ M, & \text{if } \xi_i(\theta'') > M \end{cases} \quad (49)$$

and  $M < t$ . Since

$$\bar{S}_{\bar{N}(t)(\theta'')+1}(\theta'') \leq t + M \quad (50)$$

it follows from Wald's equation in stochastic sense (see [22]) that

$$(E[\bar{N}(t)(\theta'')] + 1) \cdot E[\bar{\xi}_1(\theta'')] \leq t + M. \quad (51)$$

Thus

$$\frac{E[\bar{N}(t)(\theta'')] + 1}{t} \leq \frac{t + M}{tE[\bar{\xi}_1(\theta'')] } < \frac{2}{E[\bar{\xi}_1(\theta'')]}. \quad (52)$$

Since  $\bar{S}_n(\theta'') \leq S_n(\theta'')$ ,

$$\bar{N}(t)(\theta'') \geq N(t)(\theta'') \quad \text{and} \quad E[\bar{N}(t)(\theta'')] \geq E[N(t)(\theta'')]. \quad (53)$$

By (46), (52), and (53), the following holds:

$$\begin{aligned} \frac{E[C(t)(\theta)]'_\alpha}{t} &= \frac{E[N(t)(\theta)]'_\alpha \cdot E[\eta_1(\theta)]'_\alpha}{t} \\ &= \frac{E[N(t)(\theta'')] \cdot E[\eta_1(\theta)]'_\alpha}{t} \\ &< \frac{(E[N(t)(\theta'')] + 1) E[\eta_1(\theta)]'_\alpha}{t} \\ &< \frac{2E[\eta_1(\theta)]'_\alpha}{E[\bar{\xi}_1(\theta'')]}. \end{aligned}$$

Moreover, it follows from (49) that

$$E[\bar{\xi}_1(\theta'')] \uparrow E[\xi_1(\theta'')], \quad \text{as } M \rightarrow \infty. \quad (54)$$

That is,

$$\frac{E[\eta_1(\theta)]'_\alpha}{E[\bar{\xi}_1(\theta'')] } \downarrow \frac{E[\eta_1(\theta)]'_\alpha}{E[\xi_1(\theta)]'_\alpha}, \quad \text{as } M \rightarrow \infty. \quad (55)$$

It follows from (44) that  $E[\xi_1(\theta'')] = E[\xi_1(\theta)]'_\alpha$ . Thereby,

$$\frac{E[\eta_1(\theta)]'_\alpha}{E[\bar{\xi}_1(\theta'')] } \downarrow \frac{E[\eta_1(\theta)]'_\alpha}{E[\xi_1(\theta)]'_\alpha}, \quad \text{as } M \rightarrow \infty. \quad (56)$$

By the assumption that  $E[\eta/\xi]$  is finite, it follows that  $E[\eta_1(\theta)]'_\alpha/E[\xi_1(\theta)]'_\alpha$  is integrable. Thus, from the monotonic convergence theorem (see [3]),  $E[\eta_1(\theta)]'_\alpha/E[\bar{\xi}_1(\theta'')]$  is integrable and

$$\int_0^1 \frac{2E[\eta_1(\theta)]'_\alpha}{E[\bar{\xi}_1(\theta'')] } d\alpha \downarrow \int_0^1 \frac{2E[\eta_1(\theta)]'_\alpha}{E[\xi_1(\theta)]'_\alpha} d\alpha \quad (57)$$

as  $M \rightarrow \infty$ . Therefore, there exists  $g_1(\alpha, t) = 2E[\eta_1(\theta)]'_\alpha/E[\bar{\xi}_1(\theta'')]$  such that

$$\frac{E[C(t)(\theta)]'_\alpha}{t} \leq g_1(\alpha, t). \quad (58)$$

Finally, since  $E[N(t)(\theta)]'_\alpha$  is an almost surely continuous function of  $\alpha$ ,  $\alpha \in (0, 1]$ , the dominated convergence theorem (again, see [3]) implies that (40) holds.

As indicated previously, assertion (41) can be similarly proven, and hence, the theorem is proven.

*Remark 3:* If the random fuzzy variables  $\xi_i$  and  $\eta_i$  degenerate to random variables, then the result in Theorem 3 degenerates to the form

$$\lim_{t \rightarrow \infty} \frac{E[C(t)]}{t} = \frac{E[\eta_1]}{E[\xi_1]} \quad (59)$$

which is just the conventional result in the stochastic case [22]. Thus, the previous theorem is an extension of the corresponding stochastic theorem.

*Remark 4:* If the random fuzzy variables  $\xi_i$  and  $\eta_i$  degenerate to fuzzy variables, then the result in Theorem 3 degenerates to the form

$$\lim_{t \rightarrow \infty} \frac{E[C(t)]}{t} = E\left[\frac{\eta_1}{\xi_1}\right] \quad (60)$$

which is just the conventional result in the fuzzy case [29]. Thus, the previous theorem is an extension of the corresponding fuzzy theorem.

It is interesting to note that as with conventional stochastic renewal reward processes [22], if a renew cycle is completed every time a renewal occurs, then Theorem 3 states that the long-run expected return is just the expected return earned during a cycle, divided by the expected time of that cycle.

#### IV. APPLICATION EXAMPLE

To illustrate the utility of random fuzzy renewal award process modeling, this section presents a simple, worked example. The main purpose of this exercise is to show how the essential mathematical concepts introduced in the last section may be applicable to addressing a realistic problem. Detailed computations regarding this example are omitted, while comprehensive real-world applications of the proposed modeling approach are beyond the scope of this short paper and remain as active research.

Consider an insurance company dealing with its claims. The inter-arrival times between the  $(n - 1)$ th and  $n$ th claims  $\xi_n$  can be modeled by i.i.d. positive random fuzzy variables with the following exponential distribution

$$\begin{cases} 1 - e^{-x/u_n}, & \text{if } 0 \leq x < \infty \\ 0, & \text{otherwise,} \end{cases}$$

where  $u_n$  are i.i.d. fuzzy variables  $n = 1, 2, \dots$ . Also, the claim sizes  $\eta_n$  can be represented as i.i.d. positive random fuzzy variables with the following exponential distribution:

$$\begin{cases} 1 - e^{-x/v_n}, & \text{if } 0 \leq x < \infty \\ 0, & \text{otherwise} \end{cases}$$

where  $v_n$  are i.i.d. fuzzy variables. Moreover,  $\{\xi_n\}$  and  $\{\eta_n\}$  may be assumed to be mutually independent.

Let  $N(t)$  denote the total number of claims that have occurred by time  $t$ , where  $N(t) = \max_{n \geq 0} \{n \mid 0 < S_n \leq t\}$ . Then,  $C(t) = \sum_{i=1}^{N(t)} \eta_i$  represents the total amount of the claims accumulated up to time  $t$ , and the process  $\{C(t), t \geq 0\}$  is a random fuzzy renewal reward process. Thus, it follows from Theorem 3 that the average expected claim size in the long run can be calculated as

$$\lim_{t \rightarrow \infty} \frac{E[C(t)]}{t} = E\left[\frac{v_1}{u_1}\right].$$

As a particular instance of this application problem, for computational simplicity, assume that all values of the fuzzy variables are represented in triangular form. Let the number of days  $u_1$  be (20, 25, 30), and the claim amount  $v_1$  be (80, 100, 120) in British pounds sterling. By Definition 2

$$\begin{aligned} v'_{1\alpha} &= 80 + 20\alpha & v''_{1\alpha} &= 120 - 20\alpha \\ u'_{1\alpha} &= 20 + 5\alpha & u''_{1\alpha} &= 30 - 5\alpha. \end{aligned}$$

Then

$$\begin{aligned} E\left[\frac{v_1}{u_1}\right] &= \frac{1}{2} \int_0^1 \left( \frac{v'_{1\alpha}}{u'_{1\alpha}} + \frac{v''_{1\alpha}}{u''_{1\alpha}} \right) d\alpha \\ &= 20(\ln 3 - \ln 2) - 4. \end{aligned}$$

The average expected claim size in the long run can therefore be computed as

$$\lim_{t \rightarrow \infty} \frac{E[C(t)]}{t} = 20(\ln 3 - \ln 2) - 4 \approx 4.11.$$

In other words, the proposed modeling approach leads to the result that under the above-assumed distributions, the insurance company may expect to receive claims at the rate of £4.11 per day.

## V. CONCLUSION

This short paper has proposed a method to describe the renewal reward processes involving random fuzzy interarrival times and rewards. The approach takes advantage of i.i.d. nonnegative random fuzzy variables via considering the  $\alpha$ -pessimistic and  $\alpha$ -optimistic values of such variables. In doing so, important properties of random fuzzy renewal reward processes, including the theorem on random fuzzy elementary renewal rewards, are established. This short paper generalizes the mechanism for modeling conventional stochastic renewal processes.

The results of this research are currently being applied to address the problem of risk and reliability analysis, in the area of terrorism risk management. This is rooted in the observation that the loss brought by a terrorist attack may be represented by a random fuzzy variable. Therefore, the average loss per time unit may be evaluated by applying the theorem on random fuzzy renewal rewards, as developed in the short paper. This ongoing investigation will help better reveal the potential, and also the limitations, of the proposed approach. Finally, it is worth noting that the ideas presented here can be extended similarly to dealing with other renewal processes such as alternating

renewal processes [24], age-dependent branching processes, delayed renewal processes, and stationary point processes. Jointly, such extensions would enable the modeling of a variety of real-world application problems.

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