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Fuzzy Interpolation with Generalized Representative Values

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Abstract

Fuzzy interpolative reasoning offers the potential to model problems using sparse rule bases, as opposed to dense rule bases deployed in traditional fuzzy systems. It thus supports the simplification of complex fuzzy models in terms of rule number and facilitates inferences when limited knowledge is available. This paper presents an interpolative reasoning method by means of scale and move transformations. It can be used to interpolate fuzzy rules involving arbitrarily complex polygonal fuzzy sets. In particular, the paper introduces the general definition on representative values (RVs) employed by fuzzy interpolation and presents three useful implementations of this definition. This provides a degree of freedom to choose appropriate RVs to meet different requirements. The interpolation mechanism associated with the general RV definition is outlined and a comparative study of the interpolation results over different RV implementations is given.

1 Introduction

Fuzzy rule interpolation helps reduce the complexity of fuzzy models and supports inference in systems that employ sparse rule sets [7]. With interpolation, fuzzy rules which may be approximated from their neighbouring rules can be omitted from the rule base. This leads to the complexity reduction of fuzzy models. When given observations have no overlap with the antecedent values of rules, classical fuzzy inference methods have no rule to fire, but interpolative reasoning methods can still obtain certain conclusions. Despite these significant advantages, earlier work in fuzzy interpolative reasoning does not guarantee the convexity of the derived fuzzy sets [9][13], which is often a crucial requirement of fuzzy reasoning to attain more easily interpretable practical results.

In order to eliminate the non-convexity drawback, there has been considerable work reported in the literature. For instance, Vas, Kalmar and Kóczy have proposed an algorithm [10] that reduces the problem of non-convex conclusions. Qiao, Mizumoto and Yan [8] have published an improved method which uses similarity transfer reasoning to guarantee the attainment of convex results. Hsiao, Chen and Lee [4] have introduced a new interpolative method which exploits the slopes of the fuzzy sets. General fuzzy interpolation and extrapolation techniques [1], and a modified α-cut based method [2], have also been proposed. In addition, Bouchon, Marsala and Rifqi have created an interpolative method by exploiting the concept of graduality [3], and Yam and Kóczy [11][12] have proposed a fuzzy interpolative method based on Cartesian representation.

Nevertheless, some of the existing methods may include complex computation. It becomes more difficult when they are extended to multiple variables interpolation. Some others may only apply to simple fuzzy membership functions limited to triangular or trapezoidal. Others may not be able to obtain unique as well as normal and convex fuzzy (NCF) results. This paper, based on the initial work carried out by the authors [5][6], introduces a general RV definition (which includes the RV definitions used in previous work, of course) and summaries the computation steps for this general RV definition. Like the work in [5][6], the enhanced interpolation method avoids the problems mentioned above and ensures unique, normal and convex results. It also provides the flexibility to employing different RVs to suit different application requirements.

The rest of the paper is organized as follows. Section 2 introduces the general representative value definition for arbitrarily complex polygonal fuzzy sets. Section 3 describes scale and move transformations, and presents the computation procedure by using the general RV definition. Section 4 compares the interpolation results obtained by employing different RV definitions. Finally, Section 5 concludes the paper and points out further work.

2 General Representative Value

To facilitate the discussion of the transformation based interpolation method, the *representative value* (RV) of the polygonal fuzzy sets involved must be defined first. This value captures the overall location of the fuzzy set, and will be used as the guide to perform transformations as presented in the next section. In general, given a fuzzy set, different RVs may be defined. Whilst different RVs may lead to different interpolation results (although the transformations are to be applied in the same way), they offer a degree of freedom to suit different application requirements.

The RVs, employed in previous work [5][6] are expected to be specific cases and hence outlined below. Considering a triangular fuzzy set A, denoted as (a_0, a_1, a_2), as shown in Fig. 1, the typical RV for triangular sets is defined by

$$Rep(A) = \frac{a_0 + a_1 + a_2}{3}. \tag{1}$$

This happens to be the x value of the center of

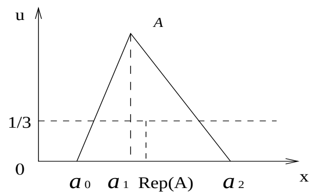

Figure 1: The classical RV of a triangular fuzzy set

gravity of such a fuzzy set [5].

To be compatible to this definition, the RV for a trapezoidal fuzzy set $A = (a_0, a_1, a_2, a_3)$ (as shown in Fig. 2) is defined by [6]:

$$Rep(A) = \frac{1}{3}(a_0 + \frac{a_1 + a_2}{2} + a_3). \tag{2}$$

Indeed, this definition subsumes the RV of a triangular set as its specific case. This is because when a_1 and a_2 in a trapezoid are collapsed into a single value a_1, the set degenerates into a triangular.

It becomes more complicated to deal with more complex fuzzy sets such as hexagonal fuzzy sets (as shown in Fig. 3). The simplest solution is to use the average of all the odd points as the RV of that fuzzy set:

$$Rep(A) = \frac{a_0 + a_1 + a2 + a_3 + a4 + a_5}{6}. \tag{3}$$

An alternative RV [6] can be defined by

$$Rep(A) = \frac{(a_0 + a_5) + (1 - \frac{\alpha}{2})(a_1 + a_4) + \frac{1}{2}(a_2 + a_3)}{5 - \alpha}, \tag{4}$$

where α is the membership value of both a_1 and a_4. This definition assigns different pairs of points with different weights. That is, a weighted average is taken as the representative value.

Another alternative RV definition for hexagonal fuzzy sets is compatible to the less complex fuzzy sets including triangular, trapezoidal and pentagonal. For example, if a_1 and a_4 happen to be on the lines between a_0, a_2 and a_3, a_5, respectively, such a hexagonal fuzzy set becomes a trapezoidal set, the definition is thus equivalent to (2). Such a compatible definition can be written as:

$$\begin{aligned} Rep(A) = \frac{1}{3}[a_0 + (1 - \frac{\alpha}{2})(a_1 - a_1') + \frac{1}{2}(a_2 + a_3) \\ +(1 - \frac{\alpha}{2})(a_4 - a_4') + a_5], \end{aligned} \tag{5}$$

where $a_1' = \alpha a_2 + (1 - \alpha)a_0$ and $a_4' = \alpha a_3 + (1 - \alpha)a_5$, see Fig 3.

Now, considering a generalized RV definition for an arbitrary polygonal fuzzy set with n odd points, $A = (a_0, \ldots, a_{n-1})$, as shown in (4). Note that the two top points (of the membership value 1) do not have to be different (e.g., a trapezoidal having the same top value is just a triangular). Although this figure explicitly assumes that evenly paired odd points are on each α-cut level, this does not affect the generality of the

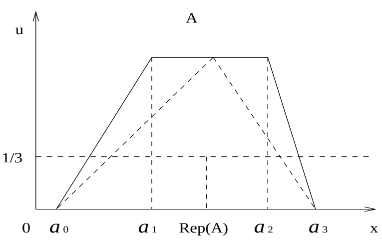

Figure 2: The RV of a trapezoidal fuzzy set

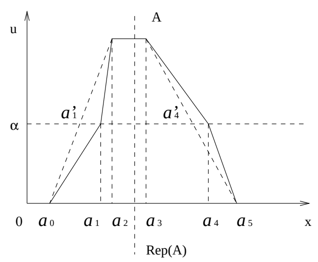

Figure 3: The RV of a hexagonal fuzzy set

fuzzy set as artificially odd points can be created to consist of evenly paired odd points. Clearly, a general fuzzy membership function with n odd points has $\lfloor \frac{n}{2} \rfloor$ *supports* (horizontal intervals between a pair of odd points which have the same membership value) and $2(\lceil \frac{n}{2} \rceil - 1)$ *slopes* (non-horizontal intervals between two consecutive odd points). A general RV definition of such an ar-

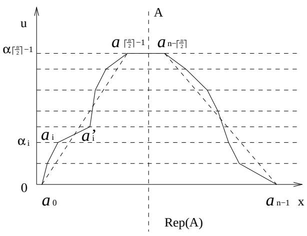

Figure 4: The RV of an arbitrarily complex fuzzy set

bitrary polygonal fuzzy set can be written as:

$$Rep(A) = \sum_{i=0}^{n-1} w_i a_i, \quad (6)$$

where w_i is the weight assigned to point a_i.

The above general definition has little practical value if no knowledge is available for the weighting scheme used. Specifying it helps avoid this problem. The simplest case (which is called the *average RV* hereafter) is that all points take the same weight value, i.e., $w_i = \frac{1}{n}$. The RV is therefore written as:

$$Rep(A) = \frac{1}{n} \sum_{i=0}^{n-1} a_i. \quad (7)$$

Note that (1) is a particular implementation of this definition.

To maintain maximal generality, it is useful to have a *compatible RV* definition. A fuzzy set represented using more odd points is the same fuzzy set that can be represented by using less odd points, then their RVs ought to be identical. One such definition can be written as:

$$Rep(A) = \frac{1}{3}[(a_0 + a_{n-1}) + \frac{1}{2}(a_{\lceil \frac{n}{2} \rceil - 1} + a_{n - \lceil \frac{n}{2} \rceil}) \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 2} (1 - \frac{\alpha_i}{2})(a_i - a'_i + a_{n-1-i} - a'_{n-1-i})], \quad (8)$$

where $a'_i = \alpha_i a_{\lceil \frac{n}{2} \rceil - 1} + (1 - \alpha_i) a_0$ and $a'_{n-1-i} = \alpha_i a_{n - \lceil \frac{n}{2} \rceil} + (1 - \alpha_i) a_{n-1}$ (see Fig 4). Note that this definition subsumes cases (1), (2) and (5).

An alternative specification of the general RV definition (named the *weighted average RV*) assumes that the weights increase upwardly from the bottom support to the top support. This weight assignment strategy is inspired by the assumption that different odd points may have different weights, and by observation that the weights should reflect the significance of the fuzzy membership values. For instance, assuming the weights increase upwardly from $\frac{1}{2}$ to 1, the weight w_i thus can be calculated by $w_i = \frac{1+\alpha_i}{2}$ (where α_i is the fuzzy membership value of a_i, $i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 1\}$), and then be normalized by the total of w_i, $i = \{0, \ldots, n-1\}$. The RV is thus defined by

$$Rep(A) = \frac{\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \frac{1+\alpha_i}{2}(a_i + a_{n-1-i})}{\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \frac{1+\alpha_i}{2}}. \quad (9)$$

The general RV definition can be simplified if the lengths of the $\lfloor \frac{n}{2} \rfloor$ supports $S_0, \ldots, S_{\lfloor \frac{n}{2} \rfloor - 1}$ (the index in ascending order from the bottom to the top) are known. As $a_{n-1-i} = a_i + S_i$, $i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, the general form of (6) thus can be re-written as:

$$\begin{aligned} Rep(A) &= a_0(w_0 + w_{n-1}) + S_0 w_{n-1} + \ldots \\ &\quad + a_{\lceil \frac{n}{2} \rceil - 1}(w_{\lceil \frac{n}{2} \rceil - 1} + w_{n - \lceil \frac{n}{2} \rceil}) \\ &\quad + S_{\lceil \frac{n}{2} \rceil - 1} w_{n - \lceil \frac{n}{2} \rceil} \\ &= \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} a_i(w_i + w_{n-1-i}) + C, \end{aligned} \quad (10)$$

where $C = S_0 w_{n-1} + \ldots + S_{\lceil \frac{n}{2} \rceil - 1} w_{n - \lceil \frac{n}{2} \rceil}$ is a constant.

In summary, the general RV definition (6) subsumes all the RV representations defined previously in [5][6]. It provides a range of possible choices to suit different problems. This generalized definition is a linear combination of all the odd points a fuzzy set involves. Note that, non-linear combinations of such points, such as the one including the product of two or more points' values, is not valid as the interpolation is itself linear.

3 Transformation Based Interpolation

3.1 Construct the Intermediate Rule

To be concise, the simplest case is herein used to illustrate the underlying techniques for fuzzy interpolation. Given two adjacent rules as follows

$$If\ X\ is\ A_1\ then\ Y\ is\ B_1,$$
$$If\ X\ is\ A_2\ then\ Y\ is\ B_2,$$

which are denoted as $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ respectively, together with an observation A^* which is located between fuzzy sets A_1 and A_2, an interpolation is performed to achieve the fuzzy result

B^*. In another form this simplest case can be represented through the *modus ponens* interpretation (11), and shown in Fig. 5.

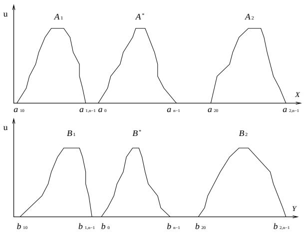

Figure 5: Interpolation with arbitrarily polygonal fuzzy membership functions

$$\begin{array}{l} \text{observation: } X \text{ is } A^* \\ \text{rules: if } X \text{ is } A_1 \text{, then } Y \text{ is } B_1 \\ \qquad\quad \text{if } X \text{ is } A_2 \text{, then } Y \text{ is } B_2 \\ \hline \text{conclusion: } Y \text{ is } B^* ? \end{array} \tag{11}$$

Here, $A_i = (a_{i0}, \dots, a_{i,n-1})$, $B_i = (b_{i0}, \dots, b_{i,n-1})$, $i = \{1, 2\}$, and $A^* = (a_0, \dots, a_{n-1})$, $B^* = (b_0, \dots, b_{n-1})$.

The transformation based interpolation method begins with constructing a new fuzzy set A' which has the same RV as that of A^*. To support this work, the distance between A_1 and A_2 is herein defined by

$$d(A_1, A_2) = d(Rep(A_1), Rep(A_2)). \tag{12}$$

An interpolative ratio λ_{Rep} ($0 \le \lambda_{Rep} \le 1$) is introduced to represent the important impact of A_2 upon the construction of A':

$$\begin{aligned} \lambda_{Rep} &= \frac{d(A_1, A^*)}{d(A_1, A_2)} \\ &= \frac{d(Rep(A_1), Rep(A^*))}{d(Rep(A_1), Rep(A_2))}. \end{aligned} \tag{13}$$

That is to say, if $\lambda_{Rep} = 0$, A_2 plays no part in constructing A', while if $\lambda_{Rep} = 1$, A_2 plays a full role in determining A'. Then by using the simplest linear interpolation, a'_i, $i = \{0, \dots, n-1\}$, of A' are calculated as follows:

$$a'_i = (1 - \lambda_{Rep}) a_{1i} + \lambda_{Rep} a_{2i}, \tag{14}$$

which are collectively abbreviated to

$$A' = (1 - \lambda_{Rep}) A_1 + \lambda_{Rep} A_2. \tag{15}$$

Now, A' has the same representative value as A^*.

proof 1 *As* $Rep(A') = \sum_{i=0}^{n-1} w_i a'_i$, *from (14) and (13), it follows that*

$$\begin{aligned} & Rep(A') \\ &= \sum_{i=0}^{n-1} w_i [(1 - \lambda_{Rep}) a_{1i} + \lambda_{Rep} a_{2i}] \\ &= (1 - \lambda_{Rep}) \sum_{i=0}^{n-1} w_i a_{1i} + \lambda_{Rep} \sum_{i=0}^{n-1} w_i a_{2i} \\ &= (1 - \lambda_{Rep}) Rep(A_1) + \lambda_{Rep} Rep(A_2) \\ &= Rep(A^*) \end{aligned} \tag{16}$$

It is worth noting that A' is a convex fuzzy set as the following holds given $a_{1i} \le a_{1,i+1}$, $a_{2i} \le a_{2,i+1}$, where $i = \{0, \dots, n-2\}$, and $0 \le \lambda_{Rep} \le 1$:

$$\begin{aligned} & a'_{i+1} - a'_i \\ &= (1 - \lambda_{Rep})(a_{1,i+1} - a_{1i}) + \lambda_{Rep}(a_{2,i+1} - a_{2i}) \ge 0. \end{aligned}$$

Similarly, the consequent fuzzy set B' can be obtained by

$$B' = (1 - \lambda_{Rep}) B_1 + \lambda_{Rep} B_2. \tag{17}$$

In so doing, the newly derived rule $A' \Rightarrow B'$ involves the use of only normal and convex fuzzy sets.

As $A' \Rightarrow B'$ is derived from $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, it is feasible to perform fuzzy reasoning with this new rule without further reference to its originals. The interpolative reasoning problem is therefore changed from (11) to the new *modus ponens* interpretation:

$$\begin{array}{l} \text{observation: } X \text{ is } A^* \\ \text{rule: if } X \text{ is } A' \text{, then } Y \text{ is } B' \\ \hline \text{conclusion: } Y \text{ is } B^* ? \end{array} \tag{18}$$

This interpretation retains the same results as (11) in dealing with the extreme cases: If $A^* = A_1$, then from (13) $\lambda_{Rep} = 0$, and according to (15) and (17), $A' = A_1$ and $B' = B_1$, so the conclusion $B^* = B_1$. Similarly, if $A^* = A_2$, then $B^* = B_2$.

Other than the extreme cases, *similarity* measures are used to support the application of this new *modus ponens*. In particular, (18) can be interpreted as

The more similar X *to* A', *the more similar* Y *to* B'. (19)

Suppose that a certain degree of similarity between A' and A^* is established, it is intuitive to require that the consequent parts B' and B^* attain the same similarity degree. The question is now how to obtain an operator which can represent the similarity degree between A' and A^*, and to allow transforming B' to B^* with the desired degree of similarity. To this end, the following two component transformations are proposed.

3.2 Scale Transformation for Generalized RVs

Consider applying scale transformation to an arbitrary polygonal fuzzy membership function $A = (a_0, \dots, a_{n-1})$ (as shown in Fig. 6) to generate $A' = (a'_0, \dots, a'_{n-1})$ such that they have the same RV, and $a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i)$, where s_i are scale rates and $i = \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$. In order to achieve this,

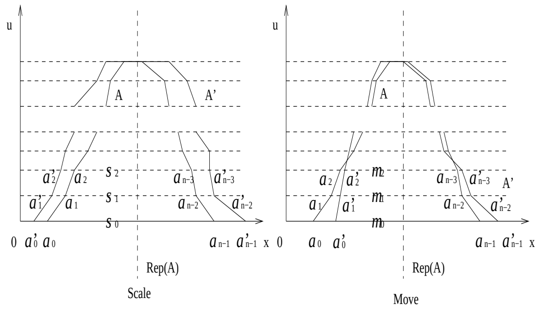

Figure 6: Scale and move transformations

$\lfloor \frac{n}{2} \rfloor$ equations $a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i)$, $i = \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$, are imposed to obtain the supports with desired lengths, and $(\lceil \frac{n}{2} \rceil - 1)$ equations $\frac{a'_{i+1} - a'_i}{a'_{n-1-i} - a'_{n-2-i}} = \frac{a_{i+1} - a_i}{a_{n-1-i} - a_{n-2-i}}$, $i = \{0, \dots, \lceil \frac{n}{2} \rceil - 2\}$ are imposed to equalise the ratios between the left $(\lceil \frac{n}{2} \rceil - 1)$ slopes' lengths and the right $(\lceil \frac{n}{2} \rceil - 1)$ slopes' lengths of A' to those counterparts of the original fuzzy set A. The equation $\sum_{i=0}^{n-1} w_i a'_i = \sum_{i=0}^{n-1} w_i a_i$ which ensures the representative values to remain the same before and after the transformation is added to make up of $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil - 1 + 1 = n$ equations. All these n equations are collectively written as:

$$
\begin{cases}
a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i) = S_i \\
(i = \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\}) \\
\frac{a'_{i+1} - a'_i}{a'_{n-1-i} - a'_{n-2-i}} = \frac{a_{i+1} - a_i}{a_{n-1-i} - a_{n-2-i}} = R_i \\
(i = \{0, \dots, \lceil \frac{n}{2} \rceil - 2\}) \\
\sum_{i=0}^{n-1} w_i a'_i = \sum_{i=0}^{n-1} w_i a_i
\end{cases} \quad (20)
$$

where S_i is the i-th support length of the resultant fuzzy set and R_i is the ratio between the left i-th slope length and the right i-th slope length. Solving these n equations simultaneously results in an unique and convex fuzzy set A' given that the resultant set has the support lengths in a descending order from the bottom to the top. This can be proved as follows.

proof 2 *As $R_i \geq 0$ $(i = \{0, \dots, \lceil \frac{n}{2} \rceil - 2\})$ and $S_i \geq S_{i+1}$ $(i = \{0, \dots, \lfloor \frac{n}{2} \rfloor - 2\})$, from (20), the conclusions below can be drawn:*

$$
\begin{cases}
a'_{i+1} - a'_i = \frac{R_i}{1+R_i}(S_i - S_{i+1}) \geq 0 \\
i = \{0, \dots, \lceil \frac{n}{2} \rceil - 2\} \\
a'_{n-\lceil \frac{n}{2} \rceil} - a'_{\lceil \frac{n}{2} \rceil - 1} = S_{\lceil \frac{n}{2} \rceil - 1} \geq 0 \\
a'_{i+1} - a'_i = \frac{1}{1+R_{n-i-2}}(S_{n-i-2} - S_{n-i-1}) \geq 0 \\
i = \{n - \lceil \frac{n}{2} \rceil, \dots, n-2\}
\end{cases}
$$

From this proof, it is clear that given a fuzzy set A and the support scale rates s_i, the use of a different RV will not affect the geometrical shape of the resultant fuzzy set. Instead, it only affects the position of the transformed fuzzy set.

However, arbitrarily choosing the i-th support scale rate when the $(i-1)$-th scale rate is fixed may lead the i-th support to becoming wider than the $(i-1)$-th support, i.e., $S_i > S_{i-1}$. To avoid this, the i-th *scale ratio* $\mathbb{S}_i$, which represents the actual increase of the ratios between the i-th supports and the $i-1$th supports, before and after the transformation, normalised over the maximal possible such increase (in the sense it does not lead to non-convexity), is introduced to restrict s_i with respect to s_{i-1}:

$$
\mathbb{S}_i = \begin{cases}
\frac{\frac{s_i(a_{n-i-1} - a_i)}{s_{i-1}(a_{n-i} - a_{i-1})} - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}}{1 - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}} & if\ s_i \geq s_{i-1} \geq 0 \\
\frac{\frac{s_i(a_{n-i-1} - a_i)}{s_{i-1}(a_{n-i} - a_{i-1})} - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}}{\frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}} & if\ s_{i-1} \geq s_i \geq 0
\end{cases} \quad (21)
$$

If $\mathbb{S}_i \in [0, 1]$ (when $s_i \geq s_{i-1} \geq 0$) or $\mathbb{S}_i \in [-1, 0]$ (when $s_{i-1} \geq s_i \geq 0$), then $S_{i-1} \geq S_i$. This can be shown as follows:

proof 3 *When $s_i \geq s_{i-1} \geq 0$, assume $S_i > S_{i-1}$, i.e, $s_i(a_{n-i-1} - a_i) > s_{i-1}(a_{n-i} - a_{i-1})$,*

$$\therefore \frac{s_i(a_{n-i-1} - a_i)}{s_{i-1}(a_{n-i} - a_{i-1})} > 1.$$

Also,

$$\because 1 \geq \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}} \geq 0,$$
$$\therefore \mathbb{S}_i > 1.$$

This conflicts with $\mathbb{S}_i \in [0, 1]$. The assumption is therefore wrong. So $S_{i-1} \geq S_i$.
When $s_{i-1} \geq s_i \geq 0$,

$$\because a_{n-i} - a_{i-1} \geq a_{n-i-1} - a_i,$$
$$\therefore s_{i-1}(a_{n-i} - a_{i-1}) \geq s_i(a_{n-i-1} - a_i),$$
$$\therefore S_{i-1} \geq S_i.$$

In summary, if given s_i $(i = \{0, \dots, \lfloor \frac{n}{2} \rfloor - 1\})$ such that $\mathbb{S}_i \in [0, 1]$ or $\mathbb{S}_i \in [-1, 0]$ (depending on whether $s_i >= s_{i-1}$ or not), $i = \{1, \dots, \lfloor \frac{n}{2} \rfloor - 1\}$, the scale transformation guarantees to generate an NCF fuzzy set.

Conversely, if two convex sets $A = (a_0, \dots, a_{n-1})$ and $A' = (a'_0, \dots, a'_{n-1})$ which have the same RV are given, the scale rate of the bottom support, s_0, and the scale ratio of the i-th support, $\mathbb{S}_i$ ($\mathbb{S}_i$, $i = \{1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$) can be calculated by:

$$s_0 = \frac{a'_{n-1} - a'_0}{a_{n-1} - a_0} \tag{22}$$

$$\mathbb{S}_i = \begin{cases} \dfrac{\frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}} - \frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}}{1-\frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}} \in [0,1] \\ (if\ \frac{a'_{n-i-1}-a'_i}{a_{n-i-1}-a_i} \geq \frac{a'_{n-i}-a'_{i-1}}{a_{n-i}-a_{i-1}} \geq 0) \\ \dfrac{\frac{a'_{n-i-1}-a'_i}{a'_{n-i}-a'_{i-1}} - \frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}}{\frac{a_{n-i-1}-a_i}{a_{n-i}-a_{i-1}}} \in [-1,0] \\ (if\ \frac{a'_{n-i}-a'_{i-1}}{a_{n-i}-a_{i-1}} \geq \frac{a'_{n-i-1}-a'_i}{a_{n-i-1}-a_i} \geq 0) \end{cases} \tag{23}$$

Given that A and A' both are convex, the ranges of $\mathbb{S}_i$ as indicated above can be proved as follows.

proof 4 *When* $\frac{a'_{n-i-1}-a'_i}{a_{n-i-1}-a_i} \geq \frac{a'_{n-i}-a'_{i-1}}{a_{n-i}-a_{i-1}} \geq 0$,

$$\because 1 \geq \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}} \geq \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}} \geq 0,$$
$$\therefore 1 \geq \mathbb{S}_i \geq 0.$$

When $\frac{a'_{n-i}-a'_{i-1}}{a_{n-i}-a_{i-1}} \geq \frac{a'_{n-i-1}-a'_i}{a_{n-i-1}-a_i} \geq 0$,

$$\because 1 \geq \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}} \geq \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}} \geq 0,$$
$$\therefore 0 \geq \mathbb{S}_i \geq -1.$$

3.3 Move Transformation for Generalized RVs

Now, consider the move transformation (also shown in Fig. 6) applied to an arbitrary polygonal fuzzy membership function $A = (a_0, \ldots, a_{n-1})$ to generate $A' = (a'_0, \ldots, a'_{n-1})$ such that they have the same representative value and the same lengths of supports, and $a'_i = a_i + l_i$, $i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$. In order to achieve this, the move transformation is decomposed to $(\lceil \frac{n}{2} \rceil - 1)$ sub-moves. The i-th sub-move ($i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$) moves the i-th support (index from the bottom to the top beginning with 0) to the desired place. It moves all the odd points on and above the i-th support, whilst keeping unaltered for those points under this support. To measure the degree of the i-th sub-move, the first maximal possible move distance (in the sense that the sub-move does not lead to the upper non-convexity) should be worked out first. To simplify the description of the sub-move procedure, only the right direction move (from a_i's point of view) is considered in the discussion hereafter. The left direction simply mirrors this operation.

If the i-th point is supposed to move to the right direction, the maximal position $a_i^{(i)*}$ can be calculated as follows when $\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1}(w_j + w_{n-1-j}) > 0$:

$$a_i^{(i)*} = \frac{\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1} a_j(w_j + w_{n-1-j}) - A}{\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1}(w_j + w_{n-1-j})} \tag{24}$$

where $A = \sum_{\substack{w_k + w_{n-1-k} < 0 \\ i<k<\lceil \frac{n}{2} \rceil}} [(S_{k-1} - S_k) \sum_{m=k}^{\lceil \frac{n}{2} \rceil - 1}(w_m + w_{n-1-m})]$ and S_k is the length of the k-th support (either before or after move transformation as they are the same). If however $\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1}(w_j + w_{n-1-j}) < 0$, the maximal position $a_i^{(i)*}$ is calculated similarly to (24) except that the condition $w_k + w_{n-1-k} < 0$ in term A is changed to $w_k + w_{n-1-k} > 0$.

proof 5 *As the sub-move does not change the RV and supports' lengths, according to (10), it can be assumed that*

$$\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} a'_i(w_i + w_{n-1-i}) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} a_i(w_i + w_{n-1-i}) = D$$

Also, as the i-th sub-move does not move the points under the i-th support, it can be assumed that

$$\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1} a'_j(w_j + w_{n-1-j}) = \sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1} a_j(w_j + w_{n-1-j}) = E$$

Consider moving point $a_i^{(i-1)}$ (a_i's new position after the $(i-1)$-th sub-move) to the right direction and $\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1}(w_j + w_{n-1-j}) > 0$,

$$a'_i(w_i + w_{n-1-i}) = E - \sum_{j=i+1}^{\lceil \frac{n}{2} \rceil - 1} a'_j(w_j + w_{n-1-j})$$

$$\leq \begin{cases} E - \sum_{j=i+1}^{\lceil \frac{n}{2} \rceil - 2} a'_j(w_j + w_{n-1-j}) \\ -a'_{\lceil \frac{n}{2} \rceil - 2}(w_{\lceil \frac{n}{2} \rceil - 1} + w_{n-\lceil \frac{n}{2} \rceil}) \\ (if\ w_{\lceil \frac{n}{2} \rceil - 1} + w_{n-\lceil \frac{n}{2} \rceil} > 0) \\ E - \sum_{j=i+1}^{\lceil \frac{n}{2} \rceil - 2} a'_j(w_j + w_{n-1-j}) \\ -a'_{\lceil \frac{n}{2} \rceil - 2}(w_{\lceil \frac{n}{2} \rceil - 1} + w_{n-\lceil \frac{n}{2} \rceil}) \\ -(S_{\lceil \frac{n}{2} \rceil - 2} - S_{\lceil \frac{n}{2} \rceil - 1})(w_{\lceil \frac{n}{2} \rceil - 1} + w_{n-\lceil \frac{n}{2} \rceil}) \\ (if\ w_{\lceil \frac{n}{2} \rceil - 1} + w_{n-\lceil \frac{n}{2} \rceil} < 0) \end{cases}$$

where $S_{\lceil \frac{n}{2} \rceil - 2}$ and $S_{\lceil \frac{n}{2} \rceil - 1}$ are the lengths of the $(\lceil \frac{n}{2} \rceil - 2)$-th and $(\lceil \frac{n}{2} \rceil - 1)$-th supports, respectively. That is to say, if $w_{\lceil \frac{n}{2} \rceil - 1} + w_{n-\lceil \frac{n}{2} \rceil} > 0$, in order to get the maximal value of $a'_i(w_i +$

w_{n-1-i}), $a'_{\lceil \frac{n}{2} \rceil -1}$ is assigned the same value as that of $a'_{\lceil \frac{n}{2} \rceil -2}$. This leads to the top left slope to being vertical. Similarly, if however $w_{\lceil \frac{n}{2} \rceil -1} + w_{n-\lceil \frac{n}{2} \rceil} < 0$, $a'_{\lceil \frac{n}{2} \rceil -1} = a'_{\lceil \frac{n}{2} \rceil -2} + S_{\lceil \frac{n}{2} \rceil -2} - S_{\lceil \frac{n}{2} \rceil -1}$ the top right slope will be vertical. Applying this procedure from the top down to the i-th support leads to the following

$$a'_i(w_i + w_{n-1-i}) \leq E - a'_i \sum_{j=i+1}^{\lceil \frac{n}{2} \rceil -1} (w_j + w_{n-1-j})$$

$$- \sum_{\substack{w_k + w_{n-1-k} < 0 \\ i < k < \lceil \frac{n}{2} \rceil}} [(S_{k-1} - S_k) \sum_{m=k}^{\lceil \frac{n}{2} \rceil -1} (w_m + w_{n-1-m})],$$

which can therefore be rearranged to being expressed as (24). The proof for the case with $\sum_{j=i}^{\lceil \frac{n}{2} \rceil -1} (w_j + w_{n-1-j}) < 0$ is omitted as it simply follows the discussion. Note that it is meaningless for $\sum_{j=i}^{\lceil \frac{n}{2} \rceil -1} (w_j + w_{n-1-j}) = 0$. With such a weight vector, the RV cannot represent the overall location of any given fuzzy set. This is because the RV of a fuzzy set always remains the same when the fuzzy set is merely moved without changing the geometrical shape.

From the proof, the other extreme points $a_j^{(i)*}$ ($j = \{i+1, \dots, \lceil \frac{n}{2} \rceil - 1\}$) which are on the left side of the fuzzy set in the i-th sub-move can be calculated by:

$$a_j^{(i)*} = \begin{cases} a_{j-1}^{(i)*} \quad if\ w_j + w_{n-1-j} > 0 \\ a_{j-1}^{(i)*} + S_{j-1} - S_j \quad if\ w_j + w_{n-1-j} < 0 \end{cases} \tag{25}$$

It can be shown that all the extreme points determine an NCF fuzzy set $A^{(i)*}$ (as illustrated in Fig. 7) which must have at least a vertical slope between any two consecutive α-cuts above the i-th support. This fuzzy set has the same RV as $A^{(i-1)}$. That is:

$$\sum_{j=0}^{\lceil \frac{n}{2} \rceil -1} a_j^{(i)*}(w_j + w_{n-1-j}) = \sum_{j=0}^{\lceil \frac{n}{2} \rceil -1} a_j^{(i-1)}(w_j + w_{n-1-j}) \tag{26}$$

The proof is ignored here as it is obvious from the calculation of the extreme point $a_i^{(i)*}$.

The move to the left direction from the viewpoint of a_i is omitted as it mirrors the right direction move.

From this, the first maximal move distance can be calculated. However, the i-th sub-move does not only need to consider the possible upper non-convexity, but also to pay attention to the possible lower non-convexity. Otherwise it may still lead to non-convexity as shown in Fig. 7. To avoid this, the second maximal move distance is calculated as $a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}$. It is intuitive to select the minimal of these two maximal move distances as the actual maximal move distance for use which will not lead to either upper or lower non-convexity. The move ratio $\mathbb{M}_i$, which is used to measure the degree of such a sub-move, is thus calculated by:

$$\mathbb{M}_i = \begin{cases} \frac{l_i - (a_i^{(i-1)} - a_i)}{min\{a_i^{(i)*} - a_i^{(i-1)}, a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}\}} \\ (if\ l_i \geq (a_i^{(i-1)} - a_i)) \\ \frac{l_i - (a_i^{(i-1)} - a_i)}{min\{a_i^{(i-1)} - a_i^{(i)*}, a_i^{(i-1)} - a_{i-1}^{(i-1)}\}} \\ (if\ l_i \leq (a_i^{(i-1)} - a_i)) \end{cases} \tag{27}$$

where the notation $a_i^{(i-1)}$ represents a_i's new position after the $(i-1)$-th sub-move. Initially, $a_i^{(-1)} = a_i$.

If $\mathbb{M}_i \in [0, 1]$ when $l_i \geq (a_i^{(i-1)} - a_i)$, or $\mathbb{M}_i \in [-1, 0]$ when $l_i \leq (a_i^{(i-1)} - a_i)$, the sub-move is carried out as follows: The odd points under the i-th support are not changed:

$$a_j^{(i)} = a_j^{(i-1)}, \quad j = \{0, \dots, i-1, n-i, \dots, n-1\}$$

while the other points $a_i^{(i-1)}, a_{i+1}^{(i-1)}, \dots, a_{n-1-i}^{(i-1)}$ are being moved. At the beginning, when $i = 0$, all odd points are being moved of course. If moving to the right direction from the viewpoint of $a_i^{(i-1)}$, i.e., $\mathbb{M}_i \in [0, 1]$, the moving distances of $a_j^{(i-1)}$ ($j = \{i, i+1, \dots, \lceil \frac{n}{2} \rceil - 1\}$) which are on the left side of the fuzzy set $A^{(i-1)}$ are calculated by multiplying $\mathbb{M}'_i$ with the distances between the extreme positions $a_j^{(i)*}$ and themselves. In so doing, $a_j^{(i-1)}$ will move the same proportion of distances to their respective extreme positions. Thus, $a_j^{(i)}$ can be computed by:

$$a_j^{(i)} = a_j^{(i-1)} + \mathbb{M}'_i(a_j^{(i)*} - a_j^{(i-1)}), \tag{28}$$

where

$$\mathbb{M}'_i = \mathbb{M}_i \frac{min\{a_i^{(i)*} - a_i^{(i-1)}, a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}\}}{a_i^{(i)*} - a_i^{(i-1)}}. \tag{29}$$

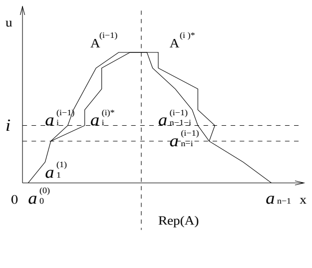

Figure 7: The extreme move positions in the i-th sub-move

It represents the *applied move ratio* for the i-th sub-move. If $\mathbb{M}_i \in [0,1]$, $\mathbb{M}'_i \in [0,\mathbb{M}_i]$. The adoption of applied move ratio $\mathbb{M}'_i$ avoids the potential lower non-convexity. Such a move strategy leads to an NCF set $A^{(i)} = \{a_0^{(i)},\ldots,a_{n-1}^{(i)}\}$ which has the same representative value as A and has the new point $a_i^{(i)}$ on the desired position, i.e., $Rep(A^{(i)}) = Rep(A)$ and $a_i^{(i)} = a_i + l_i$.

proof 6 *Consider the i-th point during the i-th sub-move* ($i = \{0,\ldots,\lceil\frac{n}{2}\rceil-2\}$)*, substituting (27) and (29) to (28) leads to* $a_i^{(i)} = a_i + l_i$*, which is the desired position for* a_i *to be moved to. As the i-th support length is fixed,* a_{n-1-i} *is also moved to the desired position via this sub-move. Initially, the* 0*-th sub-move moves* a_0 *and* a_{n-1} *to the correct positions, and the first sub-move moves* a_1 *and* a_{n-2} *to the correct positions while keeping* a_0 *and* a_{n-1} *unchanged. Following this by induction, the i-th sub-move moves* $a_0,\ldots,a_i,a_{n-1-i},\ldots,a_{n-1}$ *to the correct positions.*

The distances between $a_{j+1}^{(i)}$ *and* $a_j^{(i)}$ ($j = \{i,i+1,\ldots,\lceil\frac{n}{2}\rceil-2\}$) *are calculated as follows according to (28):*

$$a_{j+1}^{(i)}-a_j^{(i)} = (a_{j+1}^{(i-1)}-a_j^{(i-1)})(1-\mathbb{M}'_i)+\mathbb{M}'(a_{j+1}^{(i)*}-a_j^{(i)*}).$$

Initially, when $i=0$, $a_{j+1}^{(i-1)}-a_j^{(i-1)} = a_{j+1}^{(-1)} - a_j^{(-1)} = a_{j+1}-a_j \geq 0$ *and* $a_{j+1}^{(i)*}-a_j^{(i)*} = a_{j+1}^{(0)*} - a_j^{(0)*} \geq 0$ ($j = \{0,1,\ldots,\lceil\frac{n}{2}\rceil-2\}$) *as* A *and* $A^{(0)*}$ *are convex. This leads to* $a_{j+1}^{(0)} - a_j^{(0)} \geq 0$, $j = \{0,1,\ldots,\lceil\frac{n}{2}\rceil-2\}$*, which in turn leads to* $a_{j+1}^{(1)} - a_j^{(1)} \geq 0$, $j = \{1,\ldots,\lceil\frac{n}{2}\rceil-2\}$*. Also, as this sub-move causes moves to the right direction,* $a_1^{(1)} \geq a_0^{(0)} = a_0^{(1)}$*. So* $a_{j+1}^{(1)}-a_j^{(1)} \geq 0$, $j = \{0,\ldots,\lceil\frac{n}{2}\rceil-2\}$*. By induction, it follows that*

$$a_{j+1}^{i} - a_j^{i} \geq 0, \quad j = \{0,\ldots,\lceil\frac{n}{2}\rceil-2\ \}.$$

The new positions of a_j ($j = \{n-\lceil\frac{n}{2}\rceil,\ldots,n-1-i\}$) *which are on the right side of* A *can be calculated similarly:*

$$a_j^{(i)} = a_j^{(i-1)} + \mathbb{M}'_i(a_{n-1-j}^{(i)*} - a_{n-1-j}^{(i-1)}). \quad (30)$$

Thus, the distances between $a_{j+1}^{(i)}$ *and* $a_j^{(i)}$ ($j = \{n-\lceil\frac{n}{2}\rceil,\ldots,n-2-i\}$) *are calculated by:*

$$a_{j+1}^{(i)} - a_j^{(i)} = a_{j+1}^{(i-1)} - a_j^{(i-1)}$$
$$+\mathbb{M}'_i(a_{n-2-j}^{(i)*} - a_{n-2-j}^{(i-1)} - a_{n-1-j}^{(i)*} + a_{n-1-j}^{(i-1)}).$$

From (25),

$$a_{n-1-j}^{(i)*} = \begin{cases} a_{n-2-j}^{(i)*} \quad (if\ w_{n-1-j}+w_j > 0) \\ a_{n-2-j}^{(i)*} + S_{n-2-j} - S_{n-1-j} \\ (if\ w_{n-1-j}+w_j < 0) \end{cases}$$

$$\therefore a_{n-2-j}^{(i)*} - a_{n-1-j}^{(i)*} \geq S_{n-1-j} - S_{n-2-j}$$
$$\therefore a_{j+1}^{(i)} - a_j^{(i)} \geq a_{j+1}^{(i-1)} - a_j^{(i-1)} + \mathbb{M}'_i(a_j^{(i-1)} - a_{j+1}^{(i-1)})$$
$$= (a_{j+1}^{(i-1)} - a_j^{(i-1)})(1-\mathbb{M}'_i) \geq 0$$

Initially, $(a_{j+1}^{(i-1)} - a_j^{(i-1)})(1-\mathbb{M}'_i) = (a_{j+1}^{(-1)} - a_j^{(-1)})(1-\mathbb{M}'_i) = (a_{j+1}-a_j)(1-\mathbb{M}'_i) \geq 0$ ($j = \{n-\lceil\frac{n}{2}\rceil,\ldots,n-2\}$)*. This leads to* $a_{j+1}^{(0)} - a_j^{(0)} \geq 0$ ($j = \{n-\lceil\frac{n}{2}\rceil,\ldots,n-2\}$)*, which in turn leads to* $a_{j+1}^{(1)} - a_j^{(1)} \geq 0$ ($j = \{n-\lceil\frac{n}{2}\rceil,\ldots,n-3\}$)*. Also, the adoption of applied move ratio ensures* $a_{n-1}^{(1)} = a_{n-1}^{(0)} \geq a_{n-2}^{(1)}$*, so* $a_{j+1}^{(1)} - a_j^{(1)} \geq 0$ ($j = \{n-\lceil\frac{n}{2}\rceil,\ldots,n-2\}$)*. Again, by induction,*

$$a_{j+1}^{(i)} - a_j^{(i)} \geq 0 \quad j = \{n-\lceil\frac{n}{2}\rceil,\ldots,n-2\}.$$

Also, as $a_{n-\lceil\frac{n}{2}\rceil}^{(i)} - a_{\lceil\frac{n}{2}\rceil-1}^{(i)} = S_{\lceil\frac{n}{2}\rceil-1} \geq 0$*. Thus, it can be summarized that*

$$a_{j+1}^{(i)} - a_j^{(i)} \geq 0 \quad j = \{0,\ldots,n-2\},$$

i.e., $A^{(i)}$ *is an NCF set.*

The representative value of A *after the i-th sub-move,* $Rep(A^{(i)})$*, is the same as its original* $Rep(A)$*. This is because the following holds according to (28), (30) and (26):*

$$\sum_{j=0}^{\lceil\frac{n}{2}\rceil-1} a_j^{(i)}(w_j + w_{n-1-j})$$
$$= \sum_{j=0}^{\lceil\frac{n}{2}\rceil-1} a_j^{(i-1)}(w_j + w_{n-1-j})$$
$$= \ldots$$
$$= \sum_{j=0}^{\lceil\frac{n}{2}\rceil-1} a_j(w_j + w_{n-1-j})$$

The proofs of these properties including moving to the desired position, preservation of both RV and convexity for moving to the left direction (i.e., $\mathbb{M}_i \in [-1,0]$*) are omitted as they simply mirror the derivations as given above.*

In summary, if given move ratios $\mathbb{M}_i \in [-1,1]$, ($i = \{0,\ldots,\lceil\frac{n}{2}\rceil-2\}$), the ($\lceil\frac{n}{2}\rceil-1$) sub-moves transform the given NCF set $A = (a_0,\ldots,a_{n-1})$ to a new NCF set $A' = (a'_0,\ldots,a'_{n-1})$ with the same lengths of supports and the same RV.

In the converse case, where two convex fuzzy sets $A = (a_0,\ldots,a_{n-1})$ and $A' = (a'_0,\ldots,a'_{n-1})$ which have the same representative value are given, the move ratio $\mathbb{M}_i$, $i = \{0,1,\ldots,\lceil\frac{n}{2}\rceil-2\}$, are computed by:

$$\mathbb{M}_i = \begin{cases} \frac{a'_i - a_i^{(i-1)}}{min\{a_i^{(i)*} - a_i^{(i-1)}, a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}\}} \\ (if\ a'_i \geq a_i^{(i-1)}) \\ \\ \frac{a'_i - a_i^{(i-1)}}{min\{a_i^{(i-1)} - a_i^{(i)*}, a_i^{(i-1)} - a_{i-1}^{(i-1)}\}} \\ (if\ a'_i \leq a_i^{(i-1)}) \end{cases} \tag{31}$$

where $a_i^{(i-1)}$ is the a_i's new position after the $(i-1)$-th sub-move. Initially, when $i = 0$, $a_i^{(-1)} = a_i$. This sub-move (bottom sub-move) will not lead to underneath non-convexity as there are no odd points below, whilst the other sub-moves need to consider situations where non-convexity arises both upper and lower. When $i = 0$, $a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}$ and $a_i^{(i-1)} - a_{i-1}^{(i-1)}$ are not defined. In order to keep the expression the same for (31), both of them take value 1 to present the bottom case.

Given that $A = (a_0, \ldots, a_{n-1})$ and $A' = (a'_0, \ldots, a'_{n-1})$ are both convex, the ranges of $\mathbb{M}_i$ (i.e., $\mathbb{M}_i \in [0, 1]$ when $a'_i \geq a_i^{(i-1)}$ or $\mathbb{M}_i \in [-1, 0]$ when $a'_i \leq a_i^{(i-1)}$), $i = \{0, 1, \ldots, \lceil \frac{n}{2} \rceil - 2\}$, are obvious and hence no proof is needed.

3.4 Summary

As indicated earlier, it is intuitive to maintain the similarity degree between the consequent parts $B' = (b'_0, \ldots, b'_{n-1})$ and $B^* = (b^*_0, \ldots, b^*_{n-1})$ to be the same as that between the antecedent parts $A' = (a'_0, \ldots, a'_{n-1})$ and $A^* = (a^*_0, \ldots, a^*_{n-1})$, in performing interpolative reasoning. Now that the proposed scale and move transformations allow the similarity degree between two fuzzy sets to be measured by the *scale rate*, *scale ratios* and *move ratios*, the desired conclusion B^* can be obtained as follows:

1. Calculate scale rates s_i $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ of the i-th support from A' to A^* as follows:

$$s_i = \frac{a^*_{n-1-i} - a^*_i}{a'_{n-1-i} - a'_i}. \tag{32}$$

2. Calculate scale rate s_0 of the bottom support (or just get from the first step) and scale ratios $\mathbb{S}_i$ $(i = \{1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ of the i-th support from A' to A^* according to (22) and (23):

$$s_0 = \frac{a^*_{n-1} - a^*_0}{a'_{n-1} - a'_0} \tag{33}$$

$$\mathbb{S}_i = \begin{cases} \frac{\frac{a^*_{n-i-1} - a^*_i}{a^*_{n-i} - a^*_{i-1}} - \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}}}{1 - \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}}} \\ (if\ \frac{a^*_{n-i-1} - a^*_i}{a^*_{n-i} - a^*_{i-1}} \geq \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}} \geq 0) \\ \\ \frac{\frac{a^*_{n-i-1} - a^*_i}{a^*_{n-i} - a^*_{i-1}} - \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}}}{\frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}}} \\ (if\ \frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}} \geq \frac{a^*_{n-i-1} - a^*_i}{a^*_{n-i} - a^*_{i-1}} \geq 0) \end{cases} \tag{34}$$

As A' and A^* are both convex, $\mathbb{S}_i \in [0, 1]$ (when $s_i \geq s_{i-1}$) or $\mathbb{S}_i \in [-1, 0]$ (when $s_{i-1} \geq s_i$) holds.

3. Apply scale transformation to A' with scale rates s_i $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ calculated in the first step to obtain A'' by simultaneously solving n linear equations. As $\mathbb{S}_i \in [-1, 1]$ $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$, it enables A'' to have all its support lengths arranged in descending order from the bottom to the top. This, together with the scale transformation, guarantees A'' to be a unique, normal and convex fuzzy set, which has the same representative value as A^* and has the same $\lfloor \frac{n}{2} \rfloor$ support lengths as those of A^*.

4. Assign scale rate s'_0 of the bottom support of B' to the value of s_0 (i.e., $s'_0 = s_0$) as it does not give rise to non-convexity. The scale ratios $\mathbb{S}'_i$, $(i = \{1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ of the i-th support of B' are in the form

$$\mathbb{S}'_i = \begin{cases} \frac{\frac{s'_i(b'_{n-i-1} - b'_i)}{s'_{i-1}(b'_{n-i} - b'_{i-1})} - \frac{b'_{n-i-1} - b'_i}{b'_{n-i} - b'_{i-1}}}{1 - \frac{b'_{n-i-1} - b'_i}{'*_{n-i} - b'_{i-1}}} \\ (if\ s_i \geq s_{i-1} \geq 0) \\ \\ \frac{\frac{s'_i(b'_{n-i-1} - b'_i)}{s'_{i-1}(b'_{n-i} - b'_{i-1})} - \frac{b'_{n-i-1} - b'_i}{b'_{n-i} - b'_{i-1}}}{\frac{b'_{n-i-1} - b'_i}{b'_{n-i} - b'_{i-1}}} \\ (if\ s_{i-1} \geq s_i \geq 0) \end{cases} \tag{35}$$

They are required to equal $\mathbb{S}_i$ ($i = \{1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$) as calculated in step 2. Solving this along with the initial status $(s'_0 = s_0)$ leads to the following scale rates

s_i' $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$:

$$s_i' = \begin{cases} s_i & (if\ i = 0) \\ \dfrac{s_{i-1}'(s_i - s_{i-1})(\frac{b_{n-i}' - b_{i-1}'}{b_{n-i-1}' - b_i'} - 1)}{s_{i-1}(\frac{a_{n-i}' - a_{i-1}'}{a_{n-i-1}' - a_i'} - 1)} + s_{i-1}' \\ (if\ s_i \geq s_{i-1} \geq 0) \\ \dfrac{s_{i-1}' s_i}{s_{i-1}} \\ (if\ s_{i-1} \geq s_i \geq 0) \end{cases} \quad (36)$$

5. Apply scale transformation to B' using s_i' $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ as calculated in step 4 to obtain $B'' = (b_0'', \ldots, b_{n-1}'')$, by simultaneously solving the n linear equations. As B' is convex and $\mathbb{S}_i' = \mathbb{S}_i \in [-1, 1]$, it enables B'' to have descending support lengths from the bottom to the top. This, together with the scale transformation, ensures B'' to be a unique, normal and convex fuzzy set.

6. Decompose the move transformation to $(\lceil \frac{n}{2} \rceil - 1)$ sub-moves. For $i = 0, 1, \ldots, \lceil \frac{n}{2} \rceil - 2$,

   (a) Calculate the i-th sub-move ratio $\mathbb{M}_i$ from $A^{(i-1)}$ to A^* according to (31), where $A^{(i-1)}$ is the fuzzy set obtained after the $(i-1)$-th sub-move. Initially, $A^{(-1)} = A''$. As $A^{(i-1)}$ and A^* are both convex, $\mathbb{M}_i \in [-1, 1]$.

   (b) Apply move transformation to $A^{(i-1)}$ using $\mathbb{M}_i$ to obtain $A^{(i)} = \{a_0^{(i)}, a_1^{(i)}, \ldots, a_n^{(i)}\}$. As $\mathbb{M}_i \in [-1, 1]$ and $A^{(i-1)}$ is convex, $A^{(i)}$ is convex and has the same RV as A''.

   (c) Apply move transformation to $B^{(i-1)}$ using $\mathbb{M}_i$ to obtain $B^{(i)} = \{b_0^{(i)}, b_1^{(i)}, \ldots, b_n^{(i)}\}$. Again, it is convex and has the same RV as B''.

7. When the for loop of step 6 terminates, the procedure returns that $A^{(\lceil \frac{n}{2} \rceil - 2)} = A^*$ and $B^{(\lceil \frac{n}{2} \rceil - 2)}$, which is the resultant fuzzy set B^*.

Clearly, B' and B^* will retain the same similarity degree as that between the antecedent parts A' and A^*.

The interpolation of two rules involving multiple antecedent variables is extend-able by averaging the scale rate, scale ratios and move ratios. Interested readers are referred to [5] for detailed discussion.

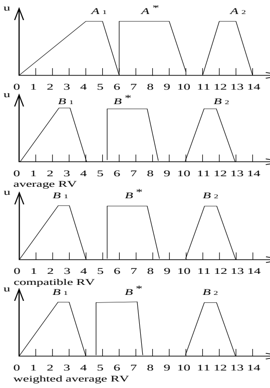

Figure 8: Example 1

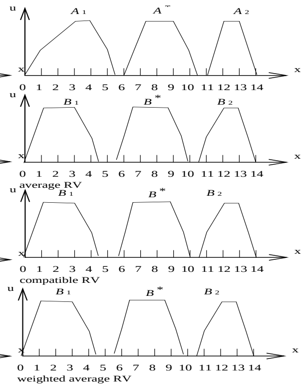

Figure 9: Example 2

4 EXPERIMENTAL RESULTS

In this section, the use of the average RV, compatible RV and weighted average RV to conduct fuzzy interpolation is demonstrated and the results are compared. Note that (9) is used to calculate weighted RV. For simplicity, both examples discussed below concern the interpolation between two adjacent rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$.

Example 1. This example shows the use of the proposed approach in dealing with trapezoidal fuzzy sets. All the conditions are given in Table 1 and Fig. 8, which also include the results of interpolation. The interpolations by using three different RV representations are carried out, according to the steps summarized in Section 3.4, resulting in three unique and NCF fuzzy sets respectively. It is interesting to note that these three results almost have the same geometrical shape although their positions are slightly different. This is because all the calculations are the same except that of the RV definition. This empirically shows that although different RVs may be chosen for use given a specific problem, their influence on the final interpolative outcomes is not drastic. This helps ensure the stability of the inference method employed.

Table 1: Results for example 1, with $A^* = (6, 6, 9, 10)$

Attribute Values	Results	
$A_1 = (0, 4, 5, 6)$	RV	B^*
$A_2 = (11, 12, 13, 14)$	average	(5.12, 5.12, 7.48, 8.18)
$B_1 = (0, 2, 3, 4)$	compatible	(5.23, 5.23 ,7.61, 8.32)
$B_2 = (10, 11, 12, 13)$	w_average	(4.73, 4.73, 7.02, 7.70)

Example 2. This example shows an interpolation of rules concerning hexagonal fuzzy sets, and it also demonstrates the effect of interpolation involving different shapes of fuzzy sets. All the attribute values and results with respect

to the observation $A^* = (6, 6.5, 7, 9, 10, 10.5)$ are shown in Table 2 and Fig. 9. Note that in this example, the two intermediate points a_1 and a_4 of each fuzzy set involved have a membership value of 0.5. Three unique and NCF fuzzy sets are obtained using three RVs respectively. Similar to example 1, the resultant fuzzy sets possess almost same geometrical shape but of slightly deferent positions.

Table 2: Results for example 2, with $A^* = (6, 6.5, 7, 9, 10, 10.5)$

Attribute Values	Results	
$A_1 = (0, 1, 3, 4, 5, 5.5)$	RV	B^*
$A_2 = (11, 11.5, 12, 13, 13.5, 14)$	average	(5.64, 5.98, 6.29, 8.63, 9.46, 9.93)
$B_1 = (0, 0.5, 1, 3, 4, 4.5)$	w_average	(5.69, 6.03, 6.36, 8.69, 9.53, 10.00)
$B_2 = (10.5, 11, 12, 13, 13.5, 14)$	compatible	(5.55, 5.88,6.19, 8.52, 9.34, 9.81)

5 CONCLUSIONS

This paper has proposed a generalized, scale and move transformation-based, interpolative reasoning method which can handle interpolation of arbitrarily complex polygonal fuzzy sets. It has introduced a generalized representation of representative values of fuzzy sets and provided three useful specifications. This helps ensure the uniqueness, normality, convexity of interpolated fuzzy sets, as well as provide a degree of freedom to choose different RVs to meet particular application requirements.

There is still room to improve the present work. In particular, the analysis of the interpolative method's sensitivity to changes in the shape of membership functions is worth considering, and more comparisons to other approaches are desirable. In addition, this research only uses two rules to conduct interpolation, but interpolation involving more rules may be utilized in fuzzy modelling. An extension of the proposed method to cope with such a problem is currently begin carried out. Finally, this work does not look into the extrapolation problem, further effort to estimate this issue seems useful.

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