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Asymptotics of Perturbations to the Wave Equation

Matthias Hieber and Ian Wood

Dedicated to Jerry Goldstein on the occasion of his 60th birthday

1 Introduction

Determining the asymptotic behaviour of solutions to linear PDEs is often a delicate matter. Even if the solution is given by a C_0 -semigroup T acting on a suitable Banach space, it can be difficult to calculate the growth bound of T . It is a well-established procedure to calculate or to estimate the spectrum of the generator A and to try to relate the location of the spectrum in the complex plane to the asymptotic behaviour of the solution. It is however well-known that the spectral bound and the growth bound of T do not coincide in general (see the counterexamples given in [Are95], [EN00], [vN96], [Ren94], and [Zab75]). Therefore, one is interested in the question of when the growth bound $\omega(T)$ and the spectral bound $s(A)$ do coincide. When this is the case, we say that the principle of linear stability holds.

The principle of linear stability holds whenever we have a suitable spectral mapping theorem for the semigroup. This is the case for a wide variety of semigroups. In [EN00] it is shown that the spectral bound and the growth bound coincide for eventually norm-continuous semigroups. This includes analytic semigroups which allows us to deal with parabolic PDEs. For hyperbolic equations in one dimension affirmative results are given in [NSRL86] and [Ren93]. For higher dimensions however, there are counterexamples where the equality of bounds does not hold. In [Ren94] such a counterexample is given which is just a first order perturbation of the wave equation in two dimensions.

Inspired by this example, we try to find conditions on the perturbation guaranteeing equality of the bounds. The well-posedness of this kind of problem is treated in the monograph by Goldstein [Gol85]. We show that for a class of self-adjoint perturbations the equality of bounds which exists for the wave equation is preserved. Finally we show that Renardy's construction of a counterexample can be extended to higher order equations. Here, we will make use of the theory of cosine functions which was partly developed by Goldstein (cf. [Gol74] and [Gol85, section 2.8]).

Further results on the stability of the semigroup have recently been gained by using Fourier multiplier properties of the resolvent. For details see [Hie01], [LR00], [LS00] and [Wei97].

2 Renardy's Example

We consider a first order perturbation of the wave equation,

$$\partial_t^2 u = \partial_x^2 u + \partial_y^2 u + e^{iy} \partial_x u, \quad (x, y) \in \mathbb{R}^2$$

where u is 2π -periodic in x and y .

We rewrite the problem in $H := H_{per}^1(\Omega) \times L_{per}^2(\Omega)$ in the following way.

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{A} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $\Omega = (-\pi, \pi) \times (-\pi, \pi)$,

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta + e^{iy} \partial_x & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{D}(\mathcal{A}) = H_{per}^2(\Omega) \times H_{per}^1(\Omega).$$

The Hilbert space H is equipped with the norm

$$\|(u, v)\|_H := \sqrt{\|u\|_{H^1(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2}.$$

In [Ren94], Renardy proves the following result.

Theorem 2.1 *On $H_{per}^1(\Omega) \times L_{per}^2(\Omega)$ the operator*

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta + e^{iy} \partial_x & 0 \end{pmatrix} \quad \text{with} \quad \mathcal{D}(\mathcal{A}) = H_{per}^2(\Omega) \times H_{per}^1(\Omega)$$

generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ and we have $s(\mathcal{A}) = 0$, but $\omega(T) \geq \frac{1}{2}$.

3 Self-Adjoint Perturbations

In this section, we consider perturbations of the wave equation on a bounded domain in \mathbb{R}^n with zero boundary conditions and on $\Omega = (-\pi, \pi)^2$ with periodic boundary conditions. The perturbation is chosen such that the perturbed operator remains self-adjoint. Using the theory of self-adjoint operators on Hilbert spaces, we will see that the equality $s(\mathcal{A}) = \omega(T)$ still holds.

We consider operators on H of the form

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta + if(y) \partial_x & 0 \end{pmatrix},$$

where f is a bounded real-valued function and

- for the case of Dirichlet boundary conditions we have

$$H = H_0^1(\Omega) \times L^2(\Omega), \quad \mathcal{D}(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$$

and

- for the periodic case

$$H = H_{per}^1(\Omega) \times L_{per}^2(\Omega) \text{ and } \mathcal{D}(\mathcal{A}) = H_{per}^2(\Omega) \times H_{per}^1(\Omega).$$

In both cases we are dealing with bounded perturbations of the wave operator, so using [Gol85, Theorem 7.8], we see that the operator \mathcal{A} generates a semigroup on H . For further information on the semigroup generated by \mathcal{A} we refer to [Gol69].

We now determine the spectral bound.

For $\lambda^2 \notin \sigma(\Delta + if(y)\partial_x)$, the resolvent of \mathcal{A} is given by

$$R(\lambda, \mathcal{A}) = \begin{pmatrix} \lambda & I \\ \Delta + if(y)\partial_x & \lambda \end{pmatrix} R(\lambda^2, \Delta + if(y)\partial_x) \quad (3.1)$$

If we now make the further assumption that

- $\|f\|_\infty \leq d(\Omega)^{-1}$ where $d(\Omega)$ is the diameter of Ω in the case of Dirichlet boundary conditions,
- or $\|f\|_\infty \leq 1$ in the case of periodic boundary conditions,

then by a straightforward calculation, we see that $\Delta + if(y)\partial_x$ is a negative self-adjoint operator. Therefore its spectrum lies in $]-\infty, 0]$. Then from (3.1), $\sigma(\mathcal{A}) \subseteq i\mathbb{R}$ and $s(\mathcal{A}) = 0$.

Our next task is to determine the growth bound of the semigroup.

Since H is a Hilbert space, by the Gearhart-Prüss-Theorem (see [ABHN01, Theorem 5.2.1] or, for a proof using Fourier multipliers see [Hie01]), we have that $s_0(\mathcal{A}) = \omega(T)$ where $s_0(\mathcal{A})$ denotes the pseudo-spectral bound.

From (3.1), we obtain

$$\begin{aligned} \|R(\lambda, \mathcal{A})\|_{\mathcal{L}(H)}^2 &\leq \left\| \lambda (\lambda^2 - (\Delta + if(y)\partial_x))^{-1} \right\|_{\mathcal{L}(H_0^1(\Omega))}^2 \\ &\quad + \left\| (\lambda^2 - (\Delta + if(y)\partial_x))^{-1} \right\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))}^2 \\ &\quad + \left\| (\Delta + if(y)\partial_x) (\lambda^2 - (\Delta + if(y)\partial_x))^{-1} \right\|_{\mathcal{L}(H_0^1(\Omega), L^2(\Omega))}^2 \\ &\quad + \left\| \lambda (\lambda^2 - (\Delta + if(y)\partial_x))^{-1} \right\|_{\mathcal{L}(L^2(\Omega))}^2. \end{aligned}$$

In order to prove $s_0(\mathcal{A}) = 0$, we have to show that all four terms are uniformly bounded for $Re\lambda > \epsilon$ for all $\epsilon > 0$.

We start with two lemmas.

Lemma 3.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. On $H_0^1(\Omega)$ the norms*

$$\|\cdot\|_{H^1} \text{ and } \left\| (-\Delta - if(y)\partial_x + 1)^{\frac{1}{2}} \cdot \right\|_{L^2}$$

are equivalent whenever $M := \|f\|_\infty < (d(\Omega))^{-1}$.

In the periodic case we have a similar statement.

Lemma 3.2 *Let $\Omega = (-\pi, \pi)^2 \subseteq \mathbb{R}^2$.
On $H_{per}^1(\Omega)$ the norms*

$$\|\cdot\|_{H^1} \text{ and } \left\| (-\Delta - if(y)\partial_x + 1)^{\frac{1}{2}} \cdot \right\|_{L^2}$$

are equivalent whenever $M := \|f\|_{\infty} \leq 1$.

In both cases the proof consists of simple calculations. For Lemma 3.1 we make use of Poincaré's inequality which is why the diameter of Ω comes in.

From now on, the proof for the periodic case works in exactly the same way as for the Dirichlet case. We will therefore only give the proof for the case where Ω is a bounded domain and we have zero boundary conditions.

Proposition 3.3 *For all $\epsilon > 0$,*

$$\left\| \lambda(\lambda^2 - (\Delta + if(y)\partial_x))^{-1} \right\|_{\mathcal{L}(L^2(\Omega))}$$

is uniformly bounded on $S := \{\lambda : \operatorname{Re}\lambda > \epsilon\}$.

Proof:

The idea is the following.

For any $\theta \in (0, \frac{\pi}{2})$ we can get an estimate on the resolvent in the sector $\sum_{\theta+\frac{\pi}{2}}$ using the fact that $\Delta + if(y)\partial_x$ generates a bounded analytic semigroup. However, as $\theta \rightarrow \frac{\pi}{2}$, the constants tend to infinity. Therefore, we need another estimate on the resolvent for those λ^2 that are outside the sector $\sum_{\theta+\frac{\pi}{2}}$. Here we can use that $\Delta + if(y)\partial_x$ is self-adjoint, so the norm of the resolvent at λ^2 can be estimated by $(\operatorname{Im}\lambda^2)^{-1}$.

□

We now use the spectral decomposition of the operator $-(\Delta + if(y)\partial_x)$. Let $\{f_j, j \in \mathbb{N}\}$ be the set of orthonormal eigenfunctions of $-\Delta - if(y)\partial_x$ with the corresponding eigenvalues $\lambda_j \geq 0$. Then we have for $u \in H^2(\Omega) \cap H_0^1(\Omega)$

$$(-\Delta - if(y)\partial_x)u = \sum_{n=1}^{\infty} \lambda_n \langle u, f_n \rangle_{L^2} f_n.$$

Proposition 3.4 *For all $\epsilon > 0$, the terms*

$$\left\| \lambda(\lambda^2 - (\Delta + if(y)\partial_x))^{-1} \right\|_{\mathcal{L}(H_0^1(\Omega))} \quad , \quad \left\| (\lambda^2 - (\Delta + if(y)\partial_x))^{-1} \right\|_{\mathcal{L}(L^2(\Omega), H_0^1(\Omega))}$$

and

$$\left\| (\Delta + if(y)\partial_x)(\lambda^2 - (\Delta + if(y)\partial_x))^{-1} \right\|_{\mathcal{L}(H_0^1(\Omega), L^2(\Omega))}$$

are uniformly bounded on $S := \{\lambda : \operatorname{Re}\lambda > \epsilon\}$.

Proof: The key to the proof is given by Lemma 3.1 and the following estimates. For $\lambda = a + ib$ with $a > \epsilon > 0$ we have

$$\sup_{\lambda \in S, \lambda_n \geq 0} \left| \frac{\lambda}{\lambda^2 + \lambda_n} \right| \leq \frac{1}{\epsilon}, \quad \sup_{\lambda \in S, \lambda_n \geq 0} \left| \frac{1}{\lambda^2 + \lambda_n} \right| \leq \frac{1}{\epsilon^2}$$

and

$$\sup_{\lambda \in S, \lambda_n \geq 0} \left| \frac{\sqrt{\lambda_n}}{\lambda^2 + \lambda_n} \right| \leq \frac{1}{2\epsilon}. \quad (3.2)$$

We only consider the last term, the others can then be estimated in a similar way.

Let $u \in H_0^1(\Omega)$. Then

$$\begin{aligned} & \|(\Delta + if(y)\partial_x)(\lambda^2 - (\Delta + if(y)\partial_x))^{-1}u\|_{L^2} \\ &= \left\| \sum_n \frac{\lambda_n}{\lambda^2 + \lambda_n} \langle u, f_n \rangle_{L^2} f_n \right\|_{L^2} \\ &\leq \sup_{\lambda \in S, n \in \mathbb{N}} \left| \frac{\sqrt{\lambda_n}}{\lambda^2 + \lambda_n} \right| \left\| \sum_n \sqrt{\lambda_n} \langle u, f_n \rangle_{L^2} f_n \right\|_{L^2} \\ &\leq C \left\| \sum_n \sqrt{1 + \lambda_n} \langle u, f_n \rangle_{L^2} f_n \right\|_{L^2} \quad (\text{using estimate (3.2)}) \\ &= C \left\| (-\Delta - if(y)\partial_x + 1)^{1/2} u \right\|_{L^2} \\ &\leq C \|u\|_{H^1} \quad (\text{by Lemma 3.1}). \end{aligned}$$

□

Collecting all our results, we obtain

Theorem 3.5 *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $H = H_0^1(\Omega) \times L^2(\Omega)$. Then the operator*

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta + if(y)\partial_x & 0 \end{pmatrix} \quad \text{with} \quad \mathcal{D}(\mathcal{A}) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$$

and

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad \text{with} \quad \|f\|_\infty < (d(\Omega))^{-1}$$

generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on H and

$$s(\mathcal{A}) = \omega(T) = 0.$$

For the periodic case, we obtain

Theorem 3.6 Let $\Omega = (-\pi, \pi)^2 \subseteq \mathbb{R}^2$.
On $H = H_{per}^1(\Omega) \times L_{per}^2(\Omega)$ the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta + if(y)\partial_x & 0 \end{pmatrix} \quad \text{with} \quad \mathcal{D}(\mathcal{A}) = H_{per}^2(\Omega) \times H_{per}^1(\Omega)$$

and

$$f : (-\pi, \pi) \rightarrow \mathbb{R} \quad \text{with} \quad \|f\|_\infty \leq 1$$

generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ and

$$s(\mathcal{A}) = \omega(T) = 0.$$

Choosing $f(y) = \sin y$, we see that in Renardy's example, it is the term $\cos y \partial_x$ which destroys the equality of growth and spectral bound.

Our results suggest that symmetric lower order perturbations of the Laplacian might guarantee that the bounds remain equal. This could be the case even if the bounds do not stay equal to zero.

4 A Higher Order Equation

We now show that, by following Renardy's example, we can construct an operator for a fourth order differential equation where the growth bound of the generated semigroup and the spectral bound of the generator differ.

We consider the equation

$$\partial_t^2 u = -\partial_x^4 u - \partial_y^4 u - ie^{iy} \partial_x^2 u \quad , \quad (x, y) \in \mathbb{R}^2$$

where u is 2π -periodic in both x and y .

Again, we rewrite the problem in the following way.

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \tilde{\mathcal{A}} \begin{pmatrix} u \\ v \end{pmatrix}$$

where $\Omega = (-\pi, \pi) \times (-\pi, \pi)$,

$$\tilde{\mathcal{A}} = \begin{pmatrix} 0 & I \\ -\partial_x^4 - \partial_y^4 - ie^{iy} \partial_x^2 & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{D}(\tilde{\mathcal{A}}) = H_{per}^4(\Omega) \times H_{per}^2(\Omega).$$

The underlying Hilbert space is $H := H_{per}^2(\Omega) \times L_{per}^2(\Omega)$ with norm

$$\|(u, v)\|_H := \sqrt{\|u\|_{H^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2}.$$

4.1 Generation of a Semigroup

We first want to show that $\tilde{\mathcal{A}}$ generates a strongly continuous semigroup. Let

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -\partial_x^4 - \partial_y^4 & 0 \end{pmatrix} \text{ and } \mathcal{C} = \begin{pmatrix} 0 & 0 \\ -ie^{iy}\partial_x^2 & 0 \end{pmatrix}$$

Then \mathcal{C} is bounded on H , so it is enough to show that \mathcal{A} with $\mathcal{D}(\mathcal{A}) = H_{per}^4(\Omega) \times H_{per}^2(\Omega)$ generates a strongly continuous semigroup on H .

We introduce the following operators.

- Define A by $Au = -\partial_x^4 u - \partial_y^4 u$, $\mathcal{D}(A) = H_{per}^4(\Omega)$.
- Let $f(x, y) = \sum_{m,n} f_{m,n} e^{imx} e^{iny} \in L_{per}^2(\Omega)$.
We define $Cos, Sin : \mathbb{R}^+ \rightarrow \mathcal{L}(L_{per}^2(\Omega))$ by

$$\begin{aligned} Cos(t)f(x, y) &:= \sum_{m,n} f_{m,n} \cos(t\sqrt{m^4 + n^4}) e^{imx} e^{iny}, \\ Sin(t)f(x, y) &:= tf_{0,0} + \sum_{m^2+n^2 \neq 0} \frac{1}{\sqrt{m^4 + n^4}} f_{m,n} \sin(t\sqrt{m^4 + n^4}) e^{imx} e^{iny}. \end{aligned}$$

Proposition 4.1 $(T(t))_{t \geq 0} = \begin{pmatrix} Cos(t) & Sin(t) \\ ASin(t) & Cos(t) \end{pmatrix}_{t \geq 0}$ is a strongly continuous semigroup on H with generator \mathcal{A} .

Proof: The proof is done in the following steps.

- I. $(T(t))_{t \geq 0}$ is strongly continuous.
- II. $(T(t))_{t \geq 0}$ is exponentially bounded.
- III. The generator of the semigroup is \mathcal{A} .

I. $(T(t))_{t \geq 0}$ is strongly continuous.

Let $f \in H_{per}^2(\Omega)$. For the first entry in the matrix we obtain

$$\begin{aligned} &\|Cos(t)f - Cos(s)f\|_{H^2}^2 \\ &= \left\| \sum_{m,n} \left(\cos(t\sqrt{m^4 + n^4}) - \cos(s\sqrt{m^4 + n^4}) \right) f_{m,n} e^{imx} e^{iny} \right\|_{H^2}^2 \\ &\leq C \sum_{m,n} \left| \cos(t\sqrt{m^4 + n^4}) - \cos(s\sqrt{m^4 + n^4}) \right|^2 (1 + m^4 + n^4) |f_{m,n}|^2. \end{aligned}$$

Given $\epsilon > 0$, choose N_0 sufficiently large such that

$$4C \sum_{|m|+|n| > N_0} (1 + m^4 + n^4) |f_{m,n}|^2 < \frac{\epsilon^2}{2}.$$

For m, n with $|m| + |n| \leq N_0$, by continuity of the cosine function, there exists $\delta_{m,n}$ such that $|t - s| < \delta_{m,n}$ implies

$$\left| \cos(t\sqrt{m^4 + n^4}) - \cos(s\sqrt{m^4 + n^4}) \right|^2 < \frac{\epsilon^2}{8C(1 + m^4 + n^4)|f_{m,n}|^2 N_0^2}.$$

Let $\delta := \min \delta_{m,n}$. Then for $|t - s| < \delta$,

$$\| \text{Cos}(t)f - \text{Cos}(s)f \|_{H^2}^2 \leq \sum_{|m|+|n| \leq N_0} \frac{\epsilon^2}{8N_0^2} + \frac{\epsilon^2}{2} \leq \epsilon^2.$$

The other entries of the matrix can be treated in the same way. This proves that $(T(t))_{t \geq 0}$ is strongly continuous.

II. $(T(t))_{t \geq 0}$ is exponentially bounded. It is easy to check that all terms are exponentially bounded.

III. The generator of the semigroup is \mathcal{A} .

By [ABHN01, Theorem 3.1.7]¹ it remains to show that

$$R(\lambda, \mathcal{A}) = \int_0^\infty e^{-\lambda t} \begin{pmatrix} \text{Cos}(t) & \text{Sin}(t) \\ \text{ASin}(t) & \text{Cos}(t) \end{pmatrix} dt \quad (4.1)$$

for $\text{Re}\lambda > \omega$, where ω is the growth bound of the semigroup.

By explicitly calculating the right hand side of equation (4.1), we see that the equation holds for $\text{Re}\lambda > 0$. Thus, \mathcal{A} is the generator of the semigroup $(T(t))_{t \geq 0}$, which ends the proof of the proposition. □

We have therefore shown that the perturbed operator $\tilde{\mathcal{A}}$ also generates a strongly continuous semigroup.

4.2 Calculating the Growth and the Spectral Bound

Proposition 4.2 *For the spectral bound of $\tilde{\mathcal{A}}$, we have $s(\tilde{\mathcal{A}}) = 0$.*

Proof: Because Ω is bounded, $\tilde{\mathcal{A}}$ has compact resolvent and its spectrum consists only of eigenvalues.

Let $(u, v) \in H_{per}^4(\Omega) \times H_{per}^2(\Omega)$ and

$$\tilde{\mathcal{A}} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix},$$

then $v = \lambda u$ and it is enough to solve

$$\lambda^2 u + \partial_x^4 u + \partial_y^4 u + i e^{iy} \partial_x^2 u = 0.$$

¹cf. also [ABHN01, Definition 3.1.8.]

With $u = \sum_{m,n} u_{m,n} e^{imx} e^{iny} \neq 0$, this implies

$$(\lambda^2 + m^4 + n^4)u_{m,n} - im^2 u_{m,n-1} = 0 \quad (4.2)$$

for all $m, n \in \mathbb{Z}$. If for $m = 0, u_{m,n} \neq 0$, then by equation (4.2), $\lambda = \pm in^2$. Else, there exists $m \neq 0$ such that $u_{m,n} \neq 0$ and

$$u_{m,n-1} = \frac{\lambda^2 + m^4 + n^4}{im^2} u_{m,n}.$$

But since $\lim_{|n| \rightarrow \infty} u_{m,n} = 0$, there must exist m, n such that $\lambda^2 + m^4 + n^4 = 0$, so $\lambda \in i\mathbb{R}$.

Therefore $\sigma(\tilde{\mathcal{A}}) \subseteq i\mathbb{R}$ and $s(\tilde{\mathcal{A}}) = 0$.

□

Proposition 4.3 *The growth bound of $(T(t))_{t \geq 0}$ satisfies $\omega(T) \geq \frac{1}{2}$.*

Proof: We use the Gearhart-Prüss Theorem and show that $s_0(\tilde{\mathcal{A}}) \geq \frac{1}{2}$.

To this end, we construct a sequence λ_n such that $Re\lambda_n \rightarrow \frac{1}{2}$ and

$$\|R(\lambda_n, \tilde{\mathcal{A}})\|_H \rightarrow \infty \text{ as } n \rightarrow \infty.$$

We set

$$\lambda_n = \sqrt{-n^4 + in^2} = \sqrt[4]{n^8 + n^4} e^{\frac{i}{2}(\pi - \arctan \frac{1}{n^2})}.$$

Then $Re\lambda_n \rightarrow \frac{1}{2}$.

Let ϵ_n be a sequence such that

- $\epsilon_n \in (0, \pi)$,
- $\epsilon_n \rightarrow 0$ for $n \rightarrow \infty$,
- $n\epsilon_n^2 \rightarrow \infty$ for $n \rightarrow \infty$.

Let $\Phi \in C_c^\infty(-1, 1), \Phi \neq 0$ and define $u_n(x, y) := e^{inx} \Phi(\frac{y}{\epsilon_n})$ and $v_n(x, y) := \lambda_n u_n$.

We show $\frac{\|(\lambda_n - \tilde{\mathcal{A}})(u_n, v_n)\|_H}{\|(u_n, v_n)\|_H} \rightarrow 0$ for $n \rightarrow \infty$. This implies the desired result, since then we have that

$$\|R(\lambda_n, \tilde{\mathcal{A}})\|_H = \sup_{y \in \mathcal{D}(\tilde{\mathcal{A}})} \frac{\|y\|_H}{\|(\lambda_n - \tilde{\mathcal{A}})y\|_H} \rightarrow \infty \text{ as } n \rightarrow \infty.$$

We get the following estimates.

$$\|u_n\|_{H^2}^2 \geq C_1 n^4 \epsilon_n, \quad (4.3)$$

$$\lambda_n^2 u_n + (\partial_x^4 + \partial_y^4 + ie^{iy} \partial_x^2) u_n = in^2(1 - e^{iy})u_n + e^{inx} \partial_y^4 \Phi(\frac{y}{\epsilon_n}), \quad (4.4)$$

$$\|in^2(1 - e^{iy})u_n\|_{L^2}^2 \leq C_2 n^4 \epsilon_n^3, \quad (4.5)$$

$$\left\| e^{inx} \partial_y^4 \Phi\left(\frac{y}{\epsilon_n}\right) \right\|_{L^2}^2 = C_3 \epsilon_n^{-7}. \quad (4.6)$$

Combining (4.3), (4.4), (4.5) and (4.6), we obtain

$$\begin{aligned} \frac{\|(\lambda_n - \tilde{\mathcal{A}})(u_n, v_n)\|_H^2}{\|(u_n, v_n)\|_H^2} &= \frac{\|\lambda_n^2 u_n + (\partial_x^4 + \partial_y^4 + ie^{iy}\partial_x^2)u_n\|_{L^2}^2}{\|u_n\|_{H^2}^2 + \|v_n\|_{L^2}^2} \\ &\leq \frac{C_2 n^4 \epsilon_n^3 + C_3 \epsilon_n^{-7}}{C_1 n^4 \epsilon_n} \longrightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

□

We summarise our results in the following theorem.

Theorem 4.4 *Let $\Omega = (-\pi, \pi)^2 \subseteq \mathbb{R}^2$ and $H = H_{per}^2(\Omega) \times L_{per}^2(\Omega)$. Then the operator*

$$\tilde{\mathcal{A}} = \begin{pmatrix} 0 & I \\ -\partial_x^4 - \partial_y^4 - ie^{iy}\partial_x^2 & 0 \end{pmatrix}$$

with

$$\mathcal{D}(\tilde{\mathcal{A}}) = H_{per}^4(\Omega) \times H_{per}^2(\Omega)$$

generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ and

$$\omega(T) \geq \frac{1}{2} > s(\tilde{\mathcal{A}}) = 0.$$

This gives us another example arising from a partial differential equation where the growth and spectral bound differ.

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