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Published in:
International Journal of Group Theory

DOI:
10.22108/ijgt.2022.131838.1764

Publication date:
2023

Citation for published version (APA):
McDonough, T. P., & Pallikaros, C. A. (2023). Minimal determining sets for certain W-graph ideals. *International Journal of Group Theory*, *12*(3), 123-151. https://doi.org/10.22108/ijgt.2022.131838.1764

International Journal of Group Theory
ISSN (print): 2251-7650, ISSN (on-line): 2251-7669
Vol. 12 No. 3 (2023), pp. 123-151.

http://ijgt.ui.ac.ir
www.ui.ac.ir

MINIMAL DETERMINING SETS FOR CERTAIN W-GRAPH IDEALS

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ABSTRACT. We consider Kazhdan-Lusztig cells of the symmetric group S_n containing the longest element of a standard parabolic subgroup of S_n. Extending some of the ideas in [Beiträge zur Algebra und Geometrie, **59** (2018), no. 3, 523–547] and [Journal of Algebra and Its Applications, **20** (2021), no. 10, 2150181], we determine the rim of some additional families of cells and also of certain induced unions of cells. These rims provide minimal determining sets for certain W-graph ideals introduced in [Journal of Algebra, **361** (2012), 188–212].

1. Introduction

In [9] Kazhdan and Lusztig introduced the left cells, the right cells and the two-sided cells of a Coxeter group W as a means of investigating the representation theory of W and its associated Hecke algebra $\mathcal{H}$. It is also shown in [9] that in the case $W = S_n$ the Robinson-Schensted correspondence gives a combinatorial description of the Kazhdan-Lusztig cells. However, this does not lead to some straightforward way of obtaining reduced forms for the elements in these cells.

The present paper is a continuation of the work in [12, 13, 14, 15] and it is concerned with the problem of determining reduced expressions for all the elements in a given cell and also in certain induced unions of cells. (See [1], [18] and [3] for the induction of Kazhdan-Lusztig cells.) The focus is on (right) cells containing the longest element of a Young subgroup of S_n and also on the union of cells obtained by inducing such cells to S_{n+1}. By extending certain ideas in [14, 15] we are able to determine the rim of

Communicated by Patrizia Longobardi.
MSC (2020): Primary: 05E10; Secondary: 20C08; 20C30.
Keywords: W-graph ideal; Kazhdan-Lusztig cell; reduced form.
Article Type: Ischia Group Theory 2020/2021.
Received: 12 December 2021, Accepted: 09 April 2022.
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http://dx.doi.org/10.22108/IJGT.2022.131838.1764 .

some additional families of Kazhdan-Lusztig cells and also of the corresponding induced union of cells. As a result, reduced forms for all the elements in these subsets of S_n can be obtained directly.

Motivated by the W-graph structure with which the regular representation of $\mathcal{H}$ is endowed in [9], Howlett and Nguyen [6] introduced a notion of W-graph ideal in W. The work in this paper is closely connected with the work in [6, 16, 17, 7], as the elements of the rims obtained for the various subsets of $W = S_n$ we investigate in fact provide minimal determining sets for certain (right) W-graph ideals.

The paper is organized as follows: In Section 2 our aim is to investigate the connection between right ideals in W and root systems and, via this approach, in Proposition 2.10 we show how the minimal determining set of the right ideal $Z\mathfrak{X}_J$ in W can be obtained explicitly given the minimal determining set of a right ideal Z in W_J. (By W_J we denote a standard parabolic subgroup of W and by $\mathfrak{X}_J$ the set of distinguished right coset representatives of W_J in W.) In Section 3 we recall some background on ordered k-paths and admissible diagrams from [14, 15], while Section 4 is mainly concerned with the identification and investigation of certain key ordered k-paths having an admissible diagram as their support. Finally, in Section 5, using the ideas developed in the earlier parts of the paper, we obtain explicit descriptions for the minimal determining sets of certain W-graph ideals in S_n corresponding to Kazhdan-Lusztig cells and induced unions of such cells (see Theorem 5.1 and Remark 5.2).

2. Root systems and ideals in W

For a Coxeter system (W, S), Kazhdan and Lusztig [9] introduced the notion of a W-graph and used this notion to define three preorders $\leqslant_L$, $\leqslant_R$ and $\leqslant_{LR}$, with corresponding equivalence relations $\sim_L$, $\sim_R$ and $\sim_{LR}$, whose equivalence classes are called *left cells*, *right cells* and *two-sided cells*, respectively. Each cell of W provides a representation of W, with the C-basis of the Hecke algebra $\mathcal{H}$ of (W, S) playing an important role in the construction of this representation; see [9, § 1]. The C-basis equips the regular representation of $\mathcal{H}$ with a W-graph structure, one of the facts playing an important role in [9].

For the rest of this section we assume that (W, S) is a Coxeter system with W finite. Also let $J \subseteq S$. Then (W_J, J) is a Coxeter system, where $W_J = \langle J \rangle$ denotes the standard parabolic subgroup determined by a subset J of S. We denote by w_J the longest element of W_J and by $\mathfrak{X}_J$ the set of minimum length elements in the right cosets of W_J in W (the distinguished right coset representatives). Recall the *prefix relation* on the elements of W: if $x, y \in W$ we say that x is a *prefix* of y if y has a *reduced form* beginning with a reduced form for x. We then have that $\mathfrak{X}_J$ is the set of prefixes of d_J where d_J is the longest element of $\mathfrak{X}_J$ (see [4, Lemma 2.2.1]). Also recall that the right cell containing w_J is contained in $w_J\mathfrak{X}_J$ (see [10, 5.26.1]).

A *right ideal* in W is a subset in W which is closed under the taking of prefixes. Given a right ideal $\mathscr{I}$ in W, we call the set $Y(\mathscr{I}) = \{x \in \mathscr{I} : x \text{ is not the prefix of any other } y \in \mathscr{I}\}$ the *minimal determining set* for $\mathscr{I}$ since knowledge of $Y(\mathscr{I})$ leads directly to $\mathscr{I}$ by taking all prefixes.

The W-graphs introduced in [9] encode in a very concise way the structure of certain representations of $\mathcal{H}$. Motivated by the ideas in [9], Howlett and Nguyen in [6] introduced the notion of a W-graph ideal in W and produced, for any such ideal, a W-graph via an algorithm like the Kazhdan-Lusztig algorithm. A W-graph ideal is an ideal in W with the additional property that it admits a module structure in a very particular way (see [6, Definition 5.1]). In particular, the subsets $Z_J = \{d \in \mathfrak{X}_J : w_J d \sim_R w_J\}$ and $Z\mathfrak{X}_J$ (where Z is a W-graph ideal with respect to J in W_J) of W are W-graph ideals (with respect to J) in W (see [16, Theorem 5.4] and [6, Theorem 9.2]).

As a consequence, if $(\hat{W}, \hat{S})$ is a Coxeter system with $S \subseteq \hat{S}$ and $\hat{\mathfrak{X}}$ is the set of distinguished right coset representatives of W in $\hat{W}$, the set $Z_J\hat{\mathfrak{X}}$ is a W-graph ideal with respect to J in $\hat{W}$. Note that the set $w_J Z_J \hat{\mathfrak{X}}$ is the union of Kazhdan-Lusztig cells in $\hat{W}$ obtained from inducing to $\hat{W}$ the cell containing w_J in W.

Considering the connection between right ideals and root systems, our aim in this section is to relate explicitly, via this approach, the minimal determining sets of the right ideals Z and $Z\mathfrak{X}_J$ of W_J and W respectively (see Proposition 2.10).

Let $\mathbf{\Phi}$ be the root system corresponding to (W, S) and let $V = \langle \mathbf{\Phi} \rangle$; let $\mathbf{\Sigma}$ be a set of fundamental roots for $\mathbf{\Phi}$, let $\mathbf{\Phi}^+$ be the positive roots in $\mathbf{\Phi}$ and $\mathbf{\Phi}^- = -\mathbf{\Phi}^+$ the negative roots. Also let $\mathbf{\Phi}_J$ be the subsystem of $\mathbf{\Phi}$ corresponding to the subsystem (W_J, J) of (W, S).

For each $s \in S$, let α_s be the root corresponding to s and let ρ_s be the reflection corresponding to s. So $\alpha\rho_s = \alpha - \frac{2(\alpha,\alpha_s)}{(\alpha_s,\alpha_s)}\alpha_s$ for each $\alpha \in V$ (we suppose that $GL(V)$ acts on V on the right). There is an injective group homomorphism $\rho\colon W \to GL(V)$ defined by $x \mapsto \rho_x = \rho_{u_1} \cdots \rho_{u_r}$ where $u_1, \ldots, u_r \in S$ and $u_1 \cdots u_r$ is any reduced word for $x \in W$. It will be convenient to write vx for $v\rho_x$ where $v \in V$ and $x \in W$.

For any $\beta \in V$, $\beta \neq 0$, the reflection ρ_β is given by $\alpha\rho_\beta = \alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\beta$ for each $\alpha \in V$. So $\rho_s = \rho_{\alpha_s}$. For $x \in W$, let $N(x)$ be the set of positive roots of W which are mapped by x to negative roots. That is, $N(x) = \mathbf{\Phi}^+ \cap \mathbf{\Phi}^- x^{-1}$.

Below we collect some basic results on roots and the length function.

Result 1. [8, Theorem p.111, Lemma p.116] *Let $x \in W$, $s \in S$ and $\alpha, \beta \in \mathbf{\Phi}$.*

(i) If $l(sx) > l(x)$ then $\alpha_s x \in \mathbf{\Phi}^+$. If $l(sx) < l(x)$ then $\alpha_s x \in \mathbf{\Phi}^-$.
(ii) If $\alpha x = \beta$ then $x^{-1}\rho_\alpha x = \rho_\beta$.

Result 2. [4, Proposition 1.3.5] *Let $x \in W$ and write $x = u_1 \cdots u_r$ where $u_1, \ldots, u_r \in S$ and $r = l(x)$; that is, x is written as a reduced word. Let $\beta_i = \alpha_{u_i} u_{i-1} \cdots u_1$ for $1 \leqslant i \leqslant r$, interpreting β_1 to be α_{u_1}. Then $|N(x)| = l(x)$ and $N(x) = \{\beta_i : 1 \leqslant i \leqslant r\}$.*

Corollary 2.1. *Let $x \in W$ and let x' be a prefix of x. Then $N(x') \subseteq N(x)$.*

Proof. We may write $x' = u_1 \cdots u_p$ and $x = u_1 \cdots u_p \cdots u_r$ where $u_1, \ldots, u_r \in S$, $l(x') = p$ and $l(x) = r$. Using the notation in Result 2, we get $N(x) = \{\beta_i : 1 \leqslant i \leqslant r\}$ and $N(x') = \{\beta_i : 1 \leqslant i \leqslant p\}$. Hence, $N(x') \subseteq N(x)$. ☐

Corollary 2.2. *Let $x \in W$, $s \in S$ and suppose that $l(sx) < l(x)$. Then $N(x) = (N(sx))\, s \cup \{\alpha_s\}$.*

Proof. Write $x = u_1 \cdots u_r$ where $u_1, \ldots, u_r \in S$, $u_1 = s$ and $l(x) = r$. Let $\gamma_i = \alpha_{u_i} u_{i-1} \cdots u_2$, $2 \leqslant i \leqslant r$. By Result 2, $N(sx) = \{\gamma_i \colon 2 \leqslant i \leqslant r\}$. Since $\beta_i = \gamma_i s$, $2 \leqslant i \leqslant r$, and $\beta_1 = \alpha_{u_1} = \alpha_s$, we get the desired result. □

Corollary 2.3. *Let $x, x' \in W$ and suppose that $N(x') \subseteq N(x)$. Then x' is a prefix of x.*

Proof. Write $x = u_1 \cdots u_r$ and $x' = u'_1 \cdots u'_p$ where $u_1, \ldots, u_r, u'_1, \ldots, u'_p \in S$, $l(x) = r$ and $l(x') = p$. If $l(x') = 0$ then $x' = 1$ which is trivially a prefix of x. So we may suppose that $l(x') > 0$.

Let $\beta_i = \alpha_{u_i} u_{i-1} \cdots u_1$ for $1 \leqslant i \leqslant r$ and let $\gamma_i = \alpha_{u'_i} u'_{i-1} \cdots u'_1$, $1 \leqslant i \leqslant p$. By Result 2, $N(x) = \{\beta_i \colon 1 \leqslant i \leqslant r\}$ and $N(x') = \{\gamma_i \colon 1 \leqslant i \leqslant p\}$. Since $\alpha_{u'_1} = \gamma_1 \in N(x') \subseteq N(x)$, $\alpha_{u'_1} = \beta_i$ for some $1 \leqslant i \leqslant r$. That is, $\alpha_{u'_1} = \alpha_{u_i} u_{i-1} \cdots u_1$. By Result 1(ii), $u_1 \cdots u_{i-1} \cdot u_i \cdot u_{i-1} \cdots u_1 = u'_1$. So $u_1 \cdots u_i = u'_1 \cdot u_1 \cdots u_{i-1}$. Replacing $u_1 \cdots u_i$ in the original reduced word for x by $u'_1 \cdot u_1 \cdots u_{i-1}$, we get a reduced word for x starting with u'_1. That is, $l(u'_1 x) < l(x)$. By Corollary 2.2, $N(x) = (N(u'_1 x))\, u'_1 \cup \{\alpha_{u'_1}\}$.

Also by Corollary 2.2, $N(x') = (N(u'_1 x'))\, u'_1 \cup \{\alpha_{u'_1}\}$ since $l(u'_1 x') < l(x')$. Hence, $N(u'_1 x') \subseteq N(u'_1 x)$. By induction, $u'_1 x'$ is a prefix of $u'_1 x$. Hence, x' is a prefix of x. □

Combining Corollaries 2.1 and 2.3, we get the following proposition.

Proposition 2.4. *Let $x, x' \in W$. Then $N(x') \subseteq N(x)$ if, and only if, x' is a prefix of x. In particular, $N(x') = N(x)$ if, and only if, $x' = x$.*

Proposition 2.5. *Let $x = ud \in W$ with $u \in W_J$ and $d \in \mathfrak{X}_J$. Then $N(x) = N(u) \cup N(d)u^{-1}$ and $N(u) \cap N(d)u^{-1} = \varnothing$.*

Proof. Let $u = u_1 \cdots u_q$ and $d = u_{q+1} \cdots u_r$, where $u_i \in S$ for $1 \leqslant i \leqslant r$, be reduced words for $u \in W_J$ and $d \in \mathfrak{X}_J$. Then $x = u_1 \cdots u_r$ is also a reduced word. Let $\beta_i = \alpha_{u_i} u_{i-1} \cdots u_1$ for $1 \leqslant i \leqslant r$. By Result 2, $N(x) = \{\beta_i \colon 1 \leqslant i \leqslant r\}$ and $N(u) = \{\beta_i \colon 1 \leqslant i \leqslant q\}$. Let $\gamma_i = \alpha_{u_i} u_{i-1} \cdots u_{q+1}$ for $q+1 \leqslant i \leqslant r$. Then $\gamma_i u^{-1} = \beta_i$ for $q+1 \leqslant i \leqslant r$ and, again by Result 2, $N(d) = \{\gamma_i \colon q+1 \leqslant i \leqslant r\}$. Thus, $N(x)$ is the disjoint union of $N(u)$ and $N(d)u^{-1}$. □

Proposition 2.6. *Let $\mathbf{\Phi}_J^+ = \mathbf{\Phi}_J \cap \mathbf{\Phi}^+$. If $v \in W_J$ then $(\mathbf{\Phi}^+ - \mathbf{\Phi}_J^+)v \subseteq \mathbf{\Phi}^+ - \mathbf{\Phi}_J^+$.*

Proof. Recall that $N(v)$ is the set of positive roots of W which are mapped by v to negative roots; that is, $N(v) = \mathbf{\Phi}^+ \cap \mathbf{\Phi}^- v^{-1}$. Moreover, $l(v) = |N(v)| = |\mathbf{\Phi}^+ \cap \mathbf{\Phi}^- v^{-1}|$.

Let $\mathbf{\Psi} = \mathbf{\Phi}_J$, let $\mathbf{\Psi}^+$ be the positive roots of $\mathbf{\Phi}_J$ and $\mathbf{\Psi}^- = -\mathbf{\Psi}^+$ be the corresponding negative roots. Let $N_J(v)$ be the set of positive roots of W_J which are mapped by v to negative roots and let l_J be the length function of (W_J, J). So $N_J(v) = \mathbf{\Psi}^+ \cap \mathbf{\Psi}^- v^{-1}$ and $l_J(v) = |N_J(v)| = |\mathbf{\Psi}^+ \cap \mathbf{\Psi}^- v^{-1}|$. Since $\mathbf{\Psi} \subseteq \mathbf{\Phi}$, $\mathbf{\Psi}^+ \subseteq \mathbf{\Phi}^+$ and $\mathbf{\Psi}^- \subseteq \mathbf{\Phi}^-$, we have $N_J(v) = \mathbf{\Psi}^+ \cap \mathbf{\Psi}^- v^{-1} \subseteq \mathbf{\Phi}^+ \cap \mathbf{\Phi}^- v^{-1} = N(v)$. As $|N_J(v)| = l_J(v) = l(v) = |N(v)|$, it follows that $N_J(v) = N(v)$. So $\mathbf{\Psi}^+ \cap \mathbf{\Psi}^- v^{-1} = \mathbf{\Phi}^+ \cap \mathbf{\Phi}^- v^{-1}$, and $\mathbf{\Psi}^+ v \cap \mathbf{\Psi}^- = \mathbf{\Phi}^+ v \cap \mathbf{\Phi}^-$.

Now let $\alpha \in \mathbf{\Phi}^+ - \mathbf{\Psi}^+$. Then $\alpha v \in \mathbf{\Phi}^+ v$ and $\alpha v \notin \mathbf{\Psi}^+ v$. If $\alpha v \in \mathbf{\Phi}^-$, then $\alpha v \in \mathbf{\Phi}^+ v \cap \mathbf{\Phi}^- = \mathbf{\Psi}^+ v \cap \mathbf{\Psi}^-$, contrary to $\alpha v \notin \mathbf{\Psi}^+ v$. Hence $\alpha v \notin \mathbf{\Phi}^-$. As $\alpha \notin \mathbf{\Psi}$, $\alpha v \notin \mathbf{\Psi}$. So $\alpha v \in \mathbf{\Phi}^+ - \mathbf{\Psi} = \mathbf{\Phi}^+ - \mathbf{\Psi}^+$. □

http://dx.doi.org/10.22108/IJGT.2022.131838.1764

Proposition 2.7. *$N(d_J)v = N(d_J) = \mathbf{\Phi}^+ - \mathbf{\Phi}_J^+$ for all $v \in W_J$. In particular, $N(d)v \subseteq \mathbf{\Phi}^+ - \mathbf{\Phi}_J^+$ for all $d \in \mathfrak{X}_J$ and $v \in W_J$.*

Proof. We denote by w_S the element of maximum length in W. Then $w_S^2 = 1$ and $N(w_S) = \mathbf{\Phi}^+$ (see [4, p. 27]).

Continuing with the notation of Proposition 2.6, we also get $N(w_J) = \mathbf{\Psi}^+$. We can write $w_S = w_J d_J$. By Proposition 2.5, $\mathbf{\Phi}^+ = N(w_S) = N(w_J d_J) = N(w_J) \cup N(d_J)w_J^{-1}$ and $N(w_J) \cap N(d_J)w_J^{-1} = \varnothing$. So $N(d_J)w_J^{-1} = \mathbf{\Phi}^+ - \mathbf{\Psi}^+$. By Proposition 2.6, $(\mathbf{\Phi}^+ - \mathbf{\Psi}^+)w_J \subseteq \mathbf{\Phi}^+ - \mathbf{\Psi}^+$. Hence, $N(d_J) \subseteq \mathbf{\Phi}^+ - \mathbf{\Psi}^+$. Comparing the sizes of these sets, we get $N(d_J) = \mathbf{\Phi}^+ - \mathbf{\Psi}^+$.

Now let $v \in W_J$. From Proposition 2.6, $N(d_J)v = (\mathbf{\Phi}^+ - \mathbf{\Psi}^+)v \subseteq \mathbf{\Phi}^+ - \mathbf{\Psi}^+ = N(d_J)$. Again comparing sizes, we get $N(d_J)v = N(d_J)$. □

Corollary 2.8. *Suppose that $d \in \mathfrak{X}_J$ and $u_1, u_2 \in W_J$ with u_1 a prefix of u_2. Then $u_1 d$ is a prefix of $u_2 d_J$.*

Proof. From Proposition 2.5, $N(u_2 d_J) = N(u_2) \cup N(d_J)u_2^{-1}$. Hence $N(u_2 d_J) = N(u_2) \cup (\mathbf{\Phi}^+ - \mathbf{\Phi}_J^+)$ by Proposition 2.7. Again, by Proposition 2.5, $N(u_1 d) = N(u_1) \cup N(d)u_1^{-1}$. But $N(u_1) \subseteq N(u_2)$ and $N(d) \subseteq N(d_J)$ from Proposition 2.4. In view of Propositions 2.4 and 2.6, it follows that $N(d)u_1^{-1} \subseteq N(d_J)u_1^{-1} = \mathbf{\Phi}^+ - \mathbf{\Phi}_J^+$. This leads to $N(u_1 d) = N(u_1) \cup N(d)u_1^{-1} \subseteq N(u_2) \cup (\mathbf{\Phi}^+ - \mathbf{\Phi}_J^+) = N(u_2 d_J)$. The desired result is now immediate from Proposition 2.4. □

Result 3. *Compare with* [6, Lemma 9.1] *Let $x = ud \in W$ with $u \in W_J$ and $d \in \mathfrak{X}_J$, and let $x' = u'd'$ be a prefix of x with $u' \in W_J$ and $d' \in \mathfrak{X}_J$. Then u' is a prefix of u and d' is a prefix of d. In particular, if Z is a right ideal in W_J, then $Z\mathfrak{X}_J$ is a right ideal in W.*

Remark 2.9. *The converse of Result 3 is false. Let $W = S_3 = \langle S \rangle$ where $S = \{s_1, s_2\}$ with $s_1 = (1,2)$ and $s_2 = (2,3)$. Let $J = \{s_1\}$. Then $W_J = \{1, s_1\}$ and $\mathfrak{X}_J = \{1, s_2, s_2 s_1\}$. Consider $x = s_1 s_2$. Then $u = s_1$ and $d = s_2$. Let $x' = s_2 = u'd'$ where $u' = 1$ and $d' = s_2$. Then u' is a prefix of u and d' is a prefix of d. However, $(x')^{-1}x = s_2 s_1 s_2$ has length 3. So x' is not a prefix of x. In Corollary 2.8 we have seen that the converse of Result 3 is true in the special case $d = d_J$.*

Proposition 2.10. *Let Y be the minimal determining set of the right ideal Z of W_J. Then $Yd_J = \{xd_J \colon x \in Y\}$ is the minimal determining set of the right ideal $Z\mathfrak{X}_J$ of W.*

Proof. Let $t \in Z\mathfrak{X}_J$. Then $t = zd$ for some $z \in Z$ and $d \in \mathfrak{X}_J$, with z a prefix of $\hat{x}$ for some $\hat{x} \in Y$. By Corollary 2.8, $t = zd$ is a prefix of $\hat{x}d_J$. It follows that the set Yd_J contains the minimal determining set of $Z\mathfrak{X}_J$. In order to complete the proof it is enough to establish that $x_1 d_J$ is not a prefix of $x_2 d_J$ whenever $x_1, x_2 \in Y$ $(x_1 \neq x_2)$. Suppose, on the contrary, that x_1, x_2 are distinct elements of Y and $x_1 d_J$ is a prefix of $x_2 d_J$. By Proposition 2.4, $N(x_1 d_J) \subseteq N(x_2 d_J)$. But $N(x_1 d_J)$ (resp., $N(x_2 d_J)$) is the disjoint union of $N(x_1)$ (resp., $N(x_2)$) and $(\mathbf{\Phi}^+ - \mathbf{\Phi}_J^+)$ in view of Propositions 2.5 and 2.7. It follows that $N(x_1) \subseteq N(x_2)$, that is, x_1 is a prefix of x_2, which is the desired contradiction. □

3. Symmetric group background

For the rest of this paper we focus on the symmetric group. For the basic definitions and background concerning partitions, compositions, Young diagrams, Young tableaux and the Robinson-Schensted correspondence we refer to [19].

The symmetric group S_n (acting on the right) on $\{1, \ldots, n\}$ is a Coxeter group with Coxeter system (W, S) where $W = S_n$, $S = \{s_1, \ldots, s_{n-1}\}$, and s_i is the transposition $(i, i+1)$.

All our partitions and compositions will be assumed to be *proper* (that is, with no zero parts). We use the notation $\lambda \vDash n$ (respectively, $\lambda \vdash n$) to say that λ is a composition (respectively, partition) of n. If $\nu, \mu \vdash n$ with $\nu = (\nu_1, \ldots, \nu_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$, write $\nu \trianglelefteq \mu$ if $r \geqslant s$ and $\sum_{1\leqslant i\leqslant k} \nu_i \leqslant \sum_{1\leqslant i\leqslant k} \mu_i$, for all k with $1 \leqslant k \leqslant s$. This is the dominance order of partitions (see [19, p. 58]).

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a *composition* of n with r *parts*. Recall that the *conjugate* composition $\lambda' = (\lambda'_1, \ldots, \lambda'_{r'})$ of λ is defined by $\lambda'_i = |\{j : 1 \leqslant j \leqslant r \text{ and } i \leqslant \lambda_j\}|$ for $1 \leqslant i \leqslant r'$, where r' is the maximum part of the composition λ. It is immediate that λ' is a partition of n with r' parts. We also define the subset $J(\lambda)$ of S to be $S\backslash\{s_{\lambda_1}, s_{\lambda_1+\lambda_2}, \ldots, s_{\lambda_1+\cdots+\lambda_{r-1}}\}$. Thus, corresponding to the composition λ, there is a standard parabolic subgroup of W, also known as a Young subgroup, whose Coxeter generator set is $J(\lambda)$.

It was shown in [9] that in the case of the symmetric group S_n, the Robinson-Schensted correspondence gives a combinatorial method of identifying the Kazhdan-Lusztig cells. In describing the connection between the Kazhdan-Lusztig left and right cells of S_n and the tableaux arising from the Robinson-Schensted process one needs to be careful since this is affected by how the elements of the (abstract) Coxeter group act on the set $\{1, \ldots, n\}$, whether on the right or on the left.

At this point we recall briefly the generalizations of the notions of diagram and tableau, commonly used in the basic theory, see [13] for a more detailed description. A *diagram* D is a non-empty finite subset of $\mathbb{Z}^2$. We will assume that D has no empty rows or columns. These are the principal diagrams of [13]. We will also assume that both rows and columns of D are indexed consecutively from 1; a node in D will be given coordinates (a, b) where a and b are the indices respectively of the row and column which the node belongs to (rows are indexed from top to bottom and columns from left to right). The *row-composition* λ_D (respectively, *column-composition* μ_D) of D is defined by setting $\lambda_{D,k}$ (respectively, $\mu_{D,k}$) to be the number of nodes on the k-th row (respectively, column) of D. If λ and μ are compositions of n, we will write $\mathcal{D}^{(\lambda,\mu)}$ for the set of (principal) diagrams D with $\lambda_D = \lambda$ and $\mu_D = \mu$. We also define $\mathcal{D}^{(\lambda)} = \bigcup_{\mu\vDash n} \mathcal{D}^{(\lambda,\mu)}$. If $\nu \vdash n$, the *Young diagram* associated with ν is the unique element of $\mathcal{D}^{(\nu,\nu')}$. A *special diagram* is a diagram obtained from a Young diagram by permuting the rows and columns (see [13, Proposition 3.1] for a characterization of special diagrams).

We say that a diagram D has *size* n if it consists of precisely n nodes. We also define the *length of a column* of a diagram D to be the number of nodes D has on this column. If D has exactly m columns, we set $\alpha_D = (\alpha_1, \ldots, \alpha_m)$ where α_i equals the length of column i of D, for $1 \leqslant i \leqslant m$. We call the m-tuple α_D the *tuple of column-lengths* of D.

http://dx.doi.org/10.22108/IJGT.2022.131838.1764

If D is a diagram of size n, a *D-tableau* is a bijection $t : D \to \{1, \ldots, n\}$ and we refer to $(i, j)t$, where $(i, j) \in D$, as the *(i, j)-entry* of t. The group W acts on the set of D-tableaux in the obvious way—if $w \in W$, an entry i is replaced by iw and tw denotes the tableau resulting from the action of w on the tableau t. We denote by t^D and t_D the two D-tableaux obtained by filling the nodes of D with $1, \ldots, n$ by rows and by columns, respectively, and we write w_D for the element of W defined by $t^D w_D = t_D$.

Now let D be a diagram and let t be a *D-tableau.* We say t is *row-standard* if it is increasing on rows. Similarly, we say t is *column-standard* if it is increasing on columns. We say that t is *standard* if $(i', j')t \leqslant (i'', j'')t$ for any $(i', j'), (i'', j'') \in D$ with $i' \leqslant i''$ and $j' \leqslant j''$. Clearly a standard D-tableau is row-standard and column-standard, however the converse is not true, in general.

The following result will turn out to be useful in various arguments in Sections 4 and 5.

Result 4. [13, Proposition 3.5]. *See also* [15, Section 2]. *Compare* [2, Lemma 1.5]*] Let D be a diagram. Then the mapping $u \mapsto t^D u$ is a bijection of the set of prefixes of w_D to the set of standard D-tableaux.* □

Since $\mathfrak{X}_{J(\lambda_D)} = \{w \in S_n \colon t^D w \text{ is row-standard}\}$, see [2, Lemma 1.1], it follows that w_D and all its prefixes belong to $\mathfrak{X}_{J(\lambda_D)}$.

In general, an element of W has an expression of the form w_D for many different diagrams D of size n. If $\lambda \vDash n$ and $d \in \mathfrak{X}_{J(\lambda)}$, a way to locate suitable diagrams $D \in \mathcal{D}^{(\lambda)}$ with $d = w_D$ is given in [13, Proposition 3.7]. The proof involves the construction of a very particular diagram $D = D(d, \lambda) \in \mathcal{D}^{(\lambda)}$ with $w_D = d$. Moreover, in [13, Proposition 3.8] it is shown that among all diagrams $E \in \mathcal{D}^{(\lambda)}$ with $w_E = d$, diagram $D(d, \lambda)$ is the unique one with the minimum number of columns.

As in [14], for a composition λ of n, we define the following subsets of $\mathfrak{X}_{J(\lambda)}$ and $\mathcal{D}^{(\lambda)}$:

$$\begin{aligned}
Z(\lambda) &= \{e \in \mathfrak{X}_{J(\lambda)} \colon w_{J(\lambda)} e \sim_R w_{J(\lambda)}\}, \\
Z_s(\lambda) &= \{e \in Z(\lambda) \colon e = w_D \text{ for some special diagram } D \in \mathcal{D}^{(\lambda)}\}, \\
Y(\lambda) &= \{x \in Z(\lambda) \colon x \text{ is not a prefix of any other } y \in Z(\lambda)\}, \\
Y_s(\lambda) &= Y(\lambda) \cap Z_s(\lambda) = \{y \in Y(\lambda) \colon D(y, \lambda) \text{ is special}\}, \\
\mathcal{E}^{(\lambda)} &= \{D(y, \lambda) \colon y \in Y(\lambda)\} \text{ and } \mathcal{E}_s^{(\lambda)} = \{D \in \mathcal{E}^{(\lambda)} \colon D \text{ is special}\}.
\end{aligned}$$

As we have already seen in Section 2, $Z(\lambda)$ is a right ideal in W. Moreover, the set $w_{J(\lambda)} Z(\lambda)$ is the right cell of W containing $w_{J(\lambda)}$. We denote this right cell by $\mathfrak{C}(\lambda)$. The set $Y(\lambda)$ is the minimal determining set of the right ideal $Z(\lambda)$. We also call $Y(\lambda)$ the *rim* of the cell $\mathfrak{C}(\lambda)$. The map $y \mapsto D(y, \lambda)$ from $Y(\lambda)$ to $\mathcal{E}^{(\lambda)}$ is a bijection, so $Y(\lambda) = \{w_D \colon D \in \mathcal{E}^{(\lambda)}\}$. Hence, in order to give an explicit description of $Y(\lambda)$ or $\mathfrak{C}(\lambda)$ it is enough to locate the diagrams in $\mathcal{E}^{(\lambda)}$.

The work in [20] and [5] motivates the following definition.

Definition 3.1 (Compare with [14, Lemma 3.2], the definition before Remark 3.3 in [14], and [15, Definition 3.6])**.** *Let D be a diagram of size n.*

(i) A path *of* length *m in D is a non-empty sequence of nodes $((a_i, b_i))_{i=1}^m$ of D such that $a_i < a_{i+1}$ and $b_i \leqslant b_{i+1}$ for $i = 1, \ldots, m-1$.*

(ii) *For $k \in \mathbb{N}$, a k-*path *in D is a sequence of k mutually disjoint paths in D; the paths in this sequence are the* constituent paths *of the k-path. The* length *of a k-path is the sum of the lengths of its constituent paths; this is the total number of nodes in the k-path. The* type *of a k-path is the sequence of lengths of its paths in non-strictly decreasing order—in particular, the type of a k-path is a k-part partition. The* support *of a k-path Π, which we denote by $s(\Pi)$, is the set of nodes occurring in its paths.*

(iii) *Let Π be a k-path in D and let $k' \leqslant k$. A k'-*subpath *of Π is a k'-path in D whose constituent paths are also constituent paths of Π.*

(iv) *Let $\Pi = (\pi_1, \dots, \pi_k)$ be a k-path in D where $\pi_j = ((a_{i,j}, b_{i,j}))_{i=1}^{m_j}$, for $1 \leqslant j \leqslant k$. Π is said to be* ordered *if whenever $j, j' \in \{1, \dots, k\}$ with $j < j'$ and $(a_{i,j}, b_{i,j})$ and $(a_{i',j'}, b_{i',j'})$ are nodes of π_j and $\pi_{j'}$, respectively, with $a_{i,j} \leqslant a_{i',j'}$, then $b_{i,j} < b_{i',j'}$.*

(v) *A k-path and a k'-path in D are said to be* equivalent *to one another if they have the same support.*

(vi) *The diagram D is said to be of* subsequence type ν, *where $\nu = (\nu_1, \dots, \nu_r) \vdash n$, if the maximum length of a k-path in D is $\nu_1 + \cdots + \nu_k$ whenever $1 \leqslant k \leqslant r$. We call D* admissible *if it is of subsequence type λ'_D.*

Below we collect some results in [14] and [15] about paths and admissible diagrams which will play some part in Sections 4 and 5.

Result 5. *See* [14, Propositions 3.5 and 3.6 and Corollary 3.7] *Let D be a diagram of size n and let ν be a partition of n.*

(i) *If D is of subsequence type ν then $\mu''_D \trianglelefteq \nu \trianglelefteq \lambda'_D$.*

(ii) *We have $w_{J(\lambda_D)} w_D \sim_R w_{J(\lambda_D)}$ if, and only if, D is admissible.*

(iii) *If $D = s(\Pi)$ for some k-path Π in D of type λ'_D, then D is admissible. In particular, if D is a special diagram then D is admissible.*

Note, however, that for composition λ it is not true in general that every admissible diagram $D \in \mathcal{D}^{(\lambda)}$ is the support of some k-path in D of type λ' — consider for example the diagram $\begin{array}{cc} \times & \times \\ \times & \\ & \times \\ \times & \times \end{array}$ in $\mathcal{D}^{((2,1,1,2))}$.

Result 6. [15, Theorem 3.13] *Let $k \geqslant 1$ and suppose Π is a k-path in a diagram D. Then Π is equivalent to an ordered k-path in D.*

Result 7. [15, Corollary 3.16] *Let $\Pi = (\pi_1, \dots, \pi_k)$ be an ordered k-path in a diagram D, and let (a'_i, b'_i), $1 \leqslant i \leqslant l$, be l distinct nodes of D which are not in Π. If no path π_j, $1 \leqslant j \leqslant k$, contains a pair of nodes of the form $(a_{i,j,1}, b'_i)$, $(a_{i,j,2}, b'_i)$ with $a_{i,j,1} < a'_i < a_{i,j,2}$ for any i satisfying $1 \leqslant i \leqslant l$, then the paths $((a'_i, b'_i))$ may be inserted into the sequence Π to give an ordered $(k+l)$-path.*

Finally for this section we recall two results from [14] which relate the sets $\mathcal{E}^{(\lambda)}$ and $\mathcal{E}^{(\mu)}$ when composition μ is obtained from composition λ in some particular ways.

The *reverse* composition $\dot{\lambda}$ of a composition $\lambda = (\lambda_1,\ldots,\lambda_r)$ of n is the composition $(\lambda_r,\ldots,\lambda_1)$ of n obtained by reversing the order of the entries. For a diagram $D \in \mathcal{D}^{(\lambda)}$, the diagram $\dot{D} \in \mathcal{D}^{(\dot{\lambda})}$ is the diagram obtained by rotating D through 180°. If $D \in \mathcal{D}^{(\lambda,\mu)}$, then $\dot{D} \in \mathcal{D}^{(\dot{\lambda},\dot{\mu})}$.

Result 8. [14, Proposition 3.9] *and* [15, Remark 2.10] *Let $\lambda \vDash n$. The map $D \mapsto \dot{D}$ from $\mathcal{D}^{(\lambda)}$ to $\mathcal{D}^{(\dot{\lambda})}$ induces a bijection between the sets $\mathcal{E}^{(\lambda)}$ and $\mathcal{E}^{(\dot{\lambda})}$.*

Given a composition $\lambda = (\lambda_1, \ldots, \lambda_r) \vDash n$, let $\lambda_* = (\lambda_1, \ldots, \lambda_r, 1) \vDash n + 1$. In [14, Section 4], there is a well-defined mapping ψ from the set of admissible diagrams in $\mathcal{D}^{(\lambda)}$ to the set of admissible diagrams in $\mathcal{D}^{(\lambda_*)}$. For a given admissible diagram D in $\mathcal{D}^{(\lambda)}$, the diagram $D\psi$ is obtained by examining all diagrams constructed from D by appending an $(r+1)$-th row with a single node to D and selecting the diagram which is admissible and such that the column index of the new node is minimal.

Result 9. [15, Proposition 4.2] *Let $r \geqslant 2$, let $n \geqslant 2$ and let $\lambda = (\lambda_1, \ldots, \lambda_r) \vDash n$ be an r-part composition with $\lambda_r = 1$. Let ψ be the mapping described in* [14, Section 4]*. Then ψ induces a bijection from $\mathcal{E}^{(\lambda)}$ to $\mathcal{E}^{(\lambda_*)}$.*

4. Ordered k-path structure of admissible diagrams

Most of the work in this section is concerned with the identification and investigation of certain key ordered k-paths which have as their support an admissible diagram $D \in \mathcal{D}^{(\lambda)}$. Later on in the paper we show how these particular ordered k-paths lead to the determination of the set $\mathcal{E}^{(\lambda)}$ and thus to the determination of the rim of the Kazhdan-Lusztig cell $\mathfrak{C}(\lambda)$ (or, equivalently, to the determination of the minimal determining set of the W-graph ideal $Z(\lambda)$). Some motivation in taking this approach is given by the proof of [15, Theorem 4.6] as one of its main ingredients is that, in the case λ is a 3-part composition, any admissible diagram in $\mathcal{D}^{(\lambda)}$ is the support of an ordered k-path of type λ'.

Next, we focus on compositions λ of the form $(\lambda_1, \lambda_2, \lambda_3, 1^r)$. We begin by fixing some notation.

Hypothesis (*): Let $s \geqslant t \geqslant u \geqslant 1$. We say that the composition λ satisfies Hypothesis (*) if $\lambda = (\lambda_1, \lambda_2, \lambda_3, 1)$ is a composition of $s + t + u + 1$ where $\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is a permutation of (s, t, u).

We continue with a study of the ordered k-paths in an admissible diagram $D \in \mathcal{D}^{(\lambda)}$ where λ is a composition satisfying Hypothesis (*). Clearly, these diagrams have no paths of length greater than 4. If Π is a k-path in D, we let $z_i(\Pi)$ be the number of constituent paths in Π of length i for $1 \leqslant i \leqslant 4$. We make the following technical definition of two forms of ordered k-path in the diagram D. We justify this definition in Lemma 4.4.

Definition 4.1. *Suppose that composition λ satisfies Hypothesis $(*)$ and that $D \in \mathcal{D}^{(\lambda)}$ is an admissible diagram. An ordered s-path Π in D of length $s+t+u+1$ which contains a t-subpath of length $2t+u+1$ is said to be a* form-A *s-path if $z_1(\Pi) = s - t$, $z_2(\Pi) = t - u$, $z_3(\Pi) = u - 1$, and $z_4(\Pi) = 1$ and a* form-B *s-path if $z_1(\Pi) = s - t$, $z_2(\Pi) = t - u - 1$, $z_3(\Pi) = u + 1$, and $z_4(\Pi) = 0$.*

Remark 4.2. *Under the hypothesis and notation of Definition 4.1 we can make the following observations.*

(i) If the constituents of the t-subpath in Definition 4.1 are listed in the same order as they appear in the s-path then the t-path is also ordered.

(ii) Any form-A s-path in D has type λ'. In particular, if D has a form-A s-path then D is admissible (see Result 5(iii)).

(iii) If D has a form-B s-path then $t > u$.

It is possible for an admissible diagram $D \in \mathcal{D}^{(\lambda)}$, with λ satisfying Hypothesis (*), to have both form-A and form-B s-paths as the following example shows.

Example 4.3. *Let $D \in \mathcal{D}^{((4,6,3,1))}$ be the diagram* $\begin{array}{cccccccc} & \times & \times & & & \times & \times & \\ \times & & \times & \times & & \times & \times & \times \\ & & & & \times & \times & & \times \\ & & & & & \times & & \end{array}$. *The 6-paths $\Pi_1, \Pi_2 \in D$ where $\Pi_1 = \{\ \{\ \{(2,1)\}, \{(1,2), (2,3), (3,5), (4,6)\}, \{(1,3), (2,4), (3,6)\}, \{(2,6)\}, \{(1,6), (2,7), (3,8)\}, \{(1,7), (2,8)\}\ \}$ and $\Pi_2 = \{\ \{\ \{(2,1)\}, \{(1,2), (2,3), (4,6)\}, \{(1,3), (2,4), (3,5)\}, \{(1,6), (2,6), (3,6)\}, \{(1,7), (2,7), (3,8)\}, \{(2,8)\}\}$ are form-A and form-B, respectively.*

In the next two lemmas, which will play an important part in the discussion that follows, we investigate the existence of form-A or form-B s-paths in admissible diagrams $D \in \mathcal{D}^{(\lambda)}$ with λ satisfying Hypothesis (*).

Lemma 4.4. *Assume that composition λ satisfies Hypothesis $(*)$ and that $D \in \mathcal{D}^{(\lambda)}$ is an admissible diagram. Then D is the support of an ordered s-path Π of length $s+t+u+1$ which is either a form-A s-path or a form-B s-path. Moreover,*

(i) if there are paths in Π of length 1 then $s > t$ and all nodes occurring in such paths are in a row of D of length s,

(ii) if there are paths in Π of length 2 then $t > u$ and all nodes occurring in such paths are in the rows of D of lengths s and t,

(iii) all nodes occurring in paths of length 3 are in the first three rows of D, except in the case that Π is a form-B s-path, when one of these paths has its nodes on the rows of lengths s and t and on the fourth row of D.

Proof. We assume the hypothesis in the statement of the lemma. Choose distinct $i_1, i_2, i_3 \in \{1,2,3\}$ so that $\lambda_{i_1} = s$, $\lambda_{i_2} = t$ and $\lambda_{i_3} = u$. First note that an s-path in D of length $s+t+u+1$ contains all nodes of D. We will construct an s-path Π of this length in D with the stated properties.

Let $N = (4,l)$ be the fourth row node of D. Since D is an admissible diagram it has subsequence type $\lambda' = 4^1 3^{u-1} 2^{t-u} 1^{s-t}$. Thus D has a path of length 4 and every path in D of length 4 contains N.

Since D is admissible it has t-paths of length $2t+u+1$ and no t-paths of greater length (see Result 5(i)). Let $\Pi' = (\pi'_1, \ldots, \pi'_t)$ be one of these t-paths. Using Result 4 we may take Π' to be an ordered t-path of length $2t+u+1$. Using the notation introduced before Definition 4.1, let $z'_i = z_i(\Pi')$ for $1 \leqslant i \leqslant 4$.

Counting paths and nodes in Π',

$$z'_1 + z'_2 + z'_3 + z'_4 = t, \quad z'_1 + 2z'_2 + 3z'_3 + 4z'_4 = 2t + u + 1. \tag{4.1}$$

So,

$$z'_2 + 2z'_3 + 3z'_4 = t + u + 1, \quad 3z'_1 + 2z'_2 + z'_3 = 2t - u - 1. \tag{4.2}$$

Hence,

$$z'_2 + z'_3 + z'_4 \leqslant t, \quad z'_3 + 2z'_4 \geqslant u + 1. \tag{4.3}$$

Since every path of length 4 in D contains N, $z'_4 \leqslant 1$. Below we will consider the cases $z'_4 = 0$ and $z'_4 = 1$ separately.

As D contains no t-paths with more than $2t + u + 1$ nodes, none of the nodes of D which are not nodes of Π' can be inserted into a path of Π' to form a larger path. Hence, by Result 7, Π' may be extended to an ordered s-path Π by the appropriate insertion of the $s - t$ paths of length 1 determined by the remaining nodes.

Case $z'_4 = 0$: Then $t - u - 1 = 2z'_1 + z'_2 \geqslant 0$ from (4.2) and (4.3), so the choice of i_3 is unique. Moreover, from (4.3) we get $z'_3 \geqslant u + 1$. Since any path in Π of length 3, avoiding row i_3 must contain N, the u nodes of row i_3 must all lie in paths of length 3, thus giving us u of the paths of length 3. Hence, there is exactly one additional path of length 3, its nodes being on rows i_1, i_2 and 4 (all the remaining paths of length 3 necessarily have their nodes on rows i_1, i_2 and i_3). In particular, we have $z'_3 = u + 1$. From (4.3) we now get $z'_2 = t - u - 1$, hence $z'_2 = 2z'_1 + z'_2$. It follows that $z'_1 = 0$. Since all nodes on rows i_3 and 4 are on paths of length 3, the paths of length 2 only involve nodes on rows i_1 and i_2. The $u + 1$ paths of length 3 contain $u + 1$ nodes on row i_2. Hence the remaining $t - u - 1$ nodes on row i_2 are on the $t - u - 1$ paths of length 2. Moreover, the $s - t$ nodes of D which are not nodes of Π' are all on row i_1. If it is possible to choose i_1 in more than one way then $s = t$ and $z_1(\Pi) = 0$. So an apparent ambiguity arises concerning the rows of D containing the nodes of paths in Π of length 1 only if such paths do not exist.

Case $z'_4 = 1$: The path in Π' of length 4 contains one node on row i_3 and each path in Π' of length 3 contains a node of row i_3. Hence $z'_3 \leqslant u - 1$. From (4.3) we get $z'_3 = u - 1$, and from (4.2) we get $z'_2 = t - u$ and $z'_1 = 0$. Since all nodes on rows i_3 and 4 are on paths of lengths 3 and 4, the paths of length 2 involve only nodes on rows i_1 and i_2. If it is possible to choose i_3 in more than one way, then $t = u$ and $z'_2 = 0$. So an apparent ambiguity arises concerning the rows of D containing the nodes of paths in Π' of length 2 only if such paths do not exist. The path of length 4 and the $u - 1$ paths of length 3 contain u nodes on row i_2. Hence the remaining $t - u$ nodes on row i_2 are on the $t - u$ paths of length 2. Thus the $s - t$ nodes of D which are not nodes of Π' are all on row i_1. If it is possible to choose i_1 in more than one way then $s = t$ and $z_1(\Pi) = 0$. So an apparent ambiguity arises concerning the rows of D containing the nodes of paths in Π of length 1 only if such paths do not exist.

Since Π is ordered and either $z_1(\Pi) = s - t$, $z_2(\Pi) = t - u$, $z_3(\Pi) = u - 1$, and $z_4(\Pi) = 1$ or $z_1(\Pi) = s - t$, $z_2(\Pi) = t - u - 1$, $z_3(\Pi) = u + 1$, and $z_4(\Pi) = 0$, Π is either a form-A s-path or a form-B s-path. □

Remark 4.5. *Keeping the hypothesis of Lemma 4.4, it follows from the proof of Lemma 4.4 that if Γ is an ordered s-path in D of length $s + t + u + 1$ which is either a form-A s-path or a form-B s-path, then the distribution of the nodes of the paths in Γ in the rows of D are as set out in Lemma 4.4(i), (ii), (iii). This is because Γ necessarily contains an ordered t-subpath of length $2t+u+1$ (see also Remark 4.2(i)).*

Lemma 4.6. *Assume that composition λ satisfies Hypothesis $(*)$ and that $D \in \mathcal{D}^{(\lambda)}$ is an admissible diagram. If $\lambda_1 = s$ or t, then D is the support of a form-A s-path.*

Proof. We assume the hypothesis in the statement of the lemma and suppose further that D has no form-A s-paths. By Lemma 4.4, D has a form-B s-path Π. In particular, $t > u$ since $z_2(\Pi) = t - u - 1 \geqslant 0$. Moreover, the nodes of the paths in Π are distributed as set out in the statement again of Lemma 4.4. Write $\Pi = (\pi_1, \dots, \pi_s)$. Also since D is admissible it has a path π of length 4. Suppose $s(\pi) = \{(1, l_1), (2, l_2), (3, l_3), (4, l)\}$ where $N = (4, l)$ is the unique node of D on the fourth row; among all such paths of length 4 we will choose π to be the path which first minimizes l_1, then minimizes l_2 and finally minimizes l_3. Let π_j be the path in Π which contains N. Then π_j has length 3 and its remaining nodes are on rows of lengths s and t (recall $t > u$). Since $z_3(\Pi) = u+1 \geqslant 2$ and the remaining paths in Π of length 3 have their nodes on the first three rows there is a path $\pi_{j'}$ in Π of length 3 with $s(\pi_{j'}) = \{(1, l'_1), (2, l'_2), (3, l'_3)\}$. If $l'_3 \leqslant l$, we would get a form-A s-path in D by replacing the paths $\pi_{j'}$ and π_j in Π by the paths with support $s(\pi_{j'}) \cup \{N\}$ and $s(\pi_j) - \{N\}$, respectively. Since this is not so, every path in Π of length 3 ending on row 3 ends in a column strictly to the right of N.

If $u = \lambda_3$, the node $(3, l_3)$ is on a path in Π of length 3 by Lemma 4.4(iii). Since $l_3 \leqslant l$, this is excluded by the previous paragraph. Hence $u \neq \lambda_3$.

Suppose now that $u = \lambda_2$. Then the node $(2, l_2)$ is on a path $\pi_{j''}$ in Π of length 3 by Lemma 4.4 (iii) and $j'' \neq j$. Write $\pi_{j''} = \{(1, l''_1), (2, l_2), (3, l''_3)\}$ and $\pi_j = \{(1, l^*_1), (3, l^*_3), (4, l)\}$. As above $l < l''_3$ and so $j < j''$, $l^*_1 < l''_1 \, (\leqslant l_2)$ and $l^*_3 < l''_3$. Since $\{(1, l^*_1), (2, l_2), (3, l_3), (4, l)\}$ is the support of a path of length 4 in D, we have that $l_1 \leqslant l^*_1$ by the minimal choice of π. Hence $l_1 \leqslant l^*_1 \leqslant l^*_3 \leqslant l < l''_3$ and $l^*_1 < l''_1 \leqslant l_2 \leqslant l_3 \leqslant l < l''_3$.

Next we choose j' minimal subject to $j < j' \leqslant j''$ and $\pi_{j'}$ has length 3 and let $s(\pi_{j'}) = \{(1, l'_1), (2, l'_2), (3, l'_3)\}$. Combining with our observations in the last paragraph we get that $l_1 \leqslant l^*_1 < l'_1 \leqslant l'_2 \leqslant l_2 \leqslant l_3 \leqslant l < l''_3$. It follows that $\{(1, l_1),\ (2, l'_2),\ (3, l_3),\ (4, l)\}$ is the support of a path in D of length 4. The minimal choice of π forces $l'_2 \geqslant l_2$, and since $l'_2 \leqslant l_2$ from above, we get that $l'_2 = l_2$. We conclude that $j' = j''$, so $\pi_{j'} = \pi_{j''}$. In particular, the choice of $\pi_{j'}$ ensures that no path π_i with $j < i < j''$ has length 3. Hence, by Lemma 4.4 (i), (ii), no path π_i with $j < i < j''$ has a second row node.

Suppose for a moment that $l_2 \leqslant l^*_3$. Then there are paths $\widehat{\pi}_j$ and $\widehat{\pi}_{j''}$ in D with support $s(\pi_j) \cup \{(2, l_2)\}$ and $s(\pi_{j''}) - \{(2, l_2)\}$ respectively. The assumption that $l_2 \leqslant l^*_3$, together with the observation that no path π_i with $j < i < j''$ has a second row node, ensure that the s-path obtained by replacing the paths π_j and $\pi_{j''}$ in Π by the paths $\widehat{\pi}_j$ and $\widehat{\pi}_{j''}$, respectively, is a form-A s-path. Since this is excluded, we have $l_3^* < l_2$.

Let π_{j^0} be the path in Π containing the node $(3, l_3)$. Since $l_3^* < l_2$ and $l_2 \leqslant l_3$, we get $l_3^* < l_3 \leqslant l < l_3''$ (by combining with certain inequalities obtained above). It follows that $j < j^0 < j''$. Hence, from the discussion in the last-but-one paragraph, π_{j^0} has length at most 2 and does not contain any node in the second row.

Suppose first that π_{j^0} has length 2, so $s(\pi_{j^0}) = \{(1, l_1^0), (3, l_3)\}$ for some l_1^0 satisfying $l_1^* < l_1^0 < l_1''$. Recalling that $l_1'' \leqslant l_2 \leqslant l_3$, we see that the s-path obtained from Π by replacing

(i) π_j by $\tilde{\pi}_j$ where $s(\tilde{\pi}_j) = s(\pi_j) - \{(4, l)\}$,
(ii) π_{j^0} by $\tilde{\pi}_{j^0}$ where $s(\tilde{\pi}_{j^0}) = s(\pi_{j^0}) \cup \{(2, l_2), (4, l)\}$, and
(iii) $\pi_{j''}$ by $\tilde{\pi}_{j''}$ where $s(\tilde{\pi}_{j''}) = s(\pi_{j''}) - \{(2, l_2)\}$,

is an (ordered) form-A s-path in D, a contradiction.

It follows that π_{j^0} has length 1, so $s(\pi_{j^0}) = \{(3, l_3)\}$. Let Ψ be the ordered t-subpath of Π of length $2t + u + 1$ consisting precisely of the paths of length > 1 in Π (keeping the order these paths have in Π). Also let $\hat{\Psi}$ be the t-path of length $2t + u - 1$ in D obtained from Ψ by replacing

(i) π_j by $\breve{\pi}_j$ where $s(\breve{\pi}_j) = \{(1, l_1^*), (3, l_3)\}$, and
(ii) $\pi_{j''}$ by $\breve{\pi}_{j''}$ where $s(\breve{\pi}_{j''}) = \{(1, l_1''), (3, l_3'')\}$.

In particular, $s(\hat{\Psi}) \cup \{(3, l_3^*), (4, l), (2, l_2)\} = s(\Psi) \cup \{(3, l_3)\}$. Let Γ be the $(r+2)$-subpath of $\hat{\Psi}$ consisting of $\breve{\pi}_j$, $\breve{\pi}_{j''}$ and all the r paths (with $r \geqslant 0$) of Ψ of length 2 which lie strictly between π_j and $\pi_{j''}$ in the ordering of Ψ. Clearly Γ has length $2r + 4$. By Result 4 Γ is equivalent to an ordered $(r + 2)$-path Γ^* in D (of length $2r + 4$). Since $s(\Gamma^*)$ $(= s(\Gamma))$ does not contain any second row nodes, the maximum length of a path in Γ^* is 2. Hence Γ^* consists of precisely $r + 2$ paths each of length 2. Let $\pi_{j^0}^*$ be the path in Γ^* containing $(3, l_3)$, so $s(\pi_{j^0}^*) = \{(1, \bar{l}_1), (3, l_3)\}$ for some $\bar{l}_1$ with $\bar{l}_1 \leqslant l_1''$ $(\leqslant l_2 \leqslant l_3 \leqslant l)$. Also let π^* be the path in D of length 4 with $s(\pi^*) = s(\pi_{j^0}^*) \cup \{(2, l_2), (4, l)\}$.

Next, we construct the t-path $\hat{\Gamma}$ in D with $s(\hat{\Gamma}) = (s(\Psi) \cup \{(3, l_3)\}) - \{(3, l_3^*)\}$ as follows.

(i) Keeping the order the paths appear in Ψ, include all paths beginning from the first one up to and including the path immediately before π_j (but not including π_j).
(ii) Then include all the paths in Γ^*, in the order they appear in Γ^*, but with $\pi_{j^0}^*$ replaced by π^*.
(iii) Finally, include all the paths in Ψ appearing strictly after $\pi_{j''}$ keeping the order these paths have in Ψ.

By its construction, $\hat{\Gamma}$ is an ordered t-path in D of length $2t + u + 1$. Using Result 7, we can construct an ordered s-path $\hat{\Pi}$ of length $s + t + u + 1$ in D by inserting in the sequence $\hat{\Gamma}$ the $s - t$ paths of length 1, each having support a node of $D - s(\hat{\Gamma})$. To justify this, observe that if the node (a, b) belongs to $D - s(\hat{\Gamma})$, then the existence of nodes (a_1, b) and (a_2, b) in $\hat{\Gamma}$ with $a_1 < a < a_2$ would imply the existence of a t-path in D of length $2t+u+2$ which is not possible by Result 5(i). Clearly $\hat{\Pi}$ is a form-A s-path in D. Hence $u \neq \lambda_2$.

Summing up, we have shown that the assumption that D has no form-A s-paths implies that $t > u$ and $u = \lambda_1$. The required result now follows easily. □

In the following remark we recall a definition and some results in [15] which will turn out to be useful in the the discussion that follows.

Remark 4.7. *Let λ be a composition of n and let $D \in \mathcal{D}^{(\lambda)}$. Also assume that $D = s(\Pi)$ for some k-path $\Pi = (\pi_1, \dots, \pi_k)$ in D. As in [15, Definition 3.8] we denote by $D(\Pi)$ the diagram in $\mathcal{D}^{(\lambda)}$ constructed from D by replacing each node of π_j by a node on the same row but in column j, for $j = 1, \dots, k$. Then*

- *(i) If Π is ordered, we have from [15, Lemma 3.9] that $t^{D(\Pi)} w_D$ is a standard $D(\Pi)$-tableau.*
- *(ii) In the special case λ satisfies Hypothesis (*) and Π is a form-A s-path in D (then Π is ordered and has type λ'), we have that $D(\Pi)$ is a special diagram in $\mathcal{D}^{(\lambda)}$. Moreover, w_D is a prefix of $w_{D(\Pi)}$ since from item (i) of this remark, $t^{D(\Pi)} w_D$ is a standard $D(\Pi)$-tableau (see Result 4).*

Example 4.8. *Suppose the composition λ satisfies Hypothesis (*). Suppose further that $\lambda_1 = s$ or t. By Lemma 4.6 we know that any admissible diagram $D \in \mathcal{D}^{(\lambda)}$ has a form-A s-path, hence by Remark 4.7(ii) we know that w_D is a prefix of w_E for some special diagram $E \in \mathcal{D}^{(\lambda)}$. Given now an admissible diagram $D \in \mathcal{D}^{(\lambda)}$, below we consider some particular examples of special diagrams $E \in \mathcal{D}^{(\lambda)}$ which could serve this purpose.*

- *(i) If $\tilde{\lambda} = (s, u, t)$ and $C \subseteq \{1, \dots, t\}$ with $|C| = u$ and $v = \min(C)$, let $F_C = \{(1,i) \colon 1 \leqslant i \leqslant s\} \cup \{(2,i) \colon i \in C\} \cup \{(3,i) \colon 1 \leqslant i \leqslant t\} \cup \{(4,v)\}$. Clearly, the list of lengths of columns of F_C is a rearrangement of λ'. It follows that F_C is a special and hence admissible diagram in $\mathcal{D}^{(\lambda)}$.*

  If $\tilde{\lambda} = (8,3,5)$ and $C = \{2,3,4\}$, then F_C is the diagram
  $$\begin{array}{cccccccc} \times & \times & \times & \times & \times & \times & \times & \times \\ & \times & \times & \times & & & & \\ \times & \times & \times & \times & \times & & & \\ & \times & & & & & & \end{array}$$

  Let $C' \subseteq \{1, \dots, t\}$ with $|C'| = u$ and suppose that $w_{F_{C'}}$ is a prefix of w_{F_C}. Then the F_C-tableau $t^{F_C} w_{F_{C'}}$ is standard (and can be constructed by moving the entries of the $F_{C'}$-tableau $t_{F_{C'}}$ along the rows keeping their order, to the nodes of F_C). Observe that row 1 (resp., row 3) of F_C coincides with row 1 (resp., row 3) of $F_{C'}$ from the way these diagrams are constructed. Moreover, in order to preserve standardness, we see that the nodes in row 2 of F_C are in exactly the same positions as the nodes in row 2 of $F_{C'}$. This forces $C = C'$. We conclude that $F_C = F_{C'}$.
- *(ii) If $\tilde{\lambda} = (t, s, u)$, $C \subseteq \{1, \dots, s-t+u\}$ with $|C| = u$ and $v = \min(C)$, then $G_C = \{(1,i) \colon i \in (C \cup \{s-t+u+1, \dots, s\})\} \cup \{(2,i) \colon 1 \leqslant i \leqslant s\} \cup \{(3,i) \colon i \in C\} \cup \{(4,v)\}$ is a special and hence admissible diagram in $\mathcal{D}^{(\lambda)}$.*

If $\tilde{\lambda} = (5, 8, 3)$ *and* $C = \{2, 4, 5\}$*, then* G_C *is the diagram*

$$\begin{array}{cccccccc} & \times & & \times & \times & & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ & \times & & \times & \times & & & \\ & \times & & & & & & \end{array}$$

Suppose now that $C' \subseteq \{1, \ldots, s - t + u\}$ *with* $|C'| = u$ *and that* $t^{G_C} w_{G_{C'}}$ *is a standard* G_C*-tableau. From the way diagrams* G_C *and* $G_{C'}$ *are defined, we see that their second rows coincide. Moreover, the last* $t - u$ *nodes in row 1 of these diagrams are in exactly the same positions. In order to preserve standardness, the nodes in row 2 of these diagrams must also occupy the same positions. Hence* $C = C'$ *and this forces* $G_C = G_{C'}$*.*

(iii) If $\tilde{\lambda} = (t, u, s)$*, let* $C = \{v\} \cup \tilde{C}$ *with* $|\tilde{C}| = u - 1$*,* $\tilde{C} \subseteq \{s - t + 2, \ldots, s\}$ *and* $v \in \{1, \ldots, \min(\tilde{C}) - 1\}$*. (In particular,* $v = \min(C)$ *and* $|C| = u$*.) Let* $\tilde{v} = v$ *or* $s - t + 1$ *according as* $v < s - t + 1$ *or not. Then* $H_C = \{(1, i) \colon i = \tilde{v} \text{ or } s - t + 2 \leqslant i \leqslant s\} \cup \{(2, i) \colon i \in C\} \cup \{(3, i) \colon 1 \leqslant i \leqslant s\} \cup \{(4, v)\}$ *is a special and hence admissible diagram in* $\mathcal{D}^{(\lambda)}$*.*

If $\tilde{\lambda} = (5, 3, 8)$ *and* $C = \{3, 6, 8\}$*, then* H_C *is the diagram*

$$\begin{array}{cccccccc} & & \times & & \times & \times & \times & \times \\ & & \times & & & \times & & \times \\ \times & \times & \times & \times & \times & \times & \times & \times \\ & & \times & & & & & \end{array}$$

Suppose now that $t^{H_C} w_{H_{C'}}$ *is a standard* H_C*-tableau for permitted choices of* C *and* C' *above. From the construction of* H_C *and* $H_{C'}$*, the nodes in their third rows and moreover, the last* $t - 1$ *nodes in their first rows, occupy exactly the same positions. To preserve standardness, the last* $u - 1$ *nodes in row 2 of the two diagrams must also be in exactly the same positions. For the same reason, the node in row 4 (resp., first node in row 2) of* H_C *is in exactly the same position as the node in row 4 (resp., first node in row 2) of* $H_{C'}$*. Finally, from the way these diagrams are defined, their first nodes in row 1 are also forced to be in exactly the same position, proving that* $H_C = H_{C'}$*.*

Next, we introduce some more notation.

Definition 4.9. *Suppose composition* λ *satisfies Hypothesis (*) with the additional constraints* $t > u$ *and* $\lambda_1 = u$*. We then say that the diagram* $D \in \mathcal{D}^{(\lambda)}$ *satisfies Hypothesis (†) if in the associated tuple* $\alpha_D = (\alpha_1, \ldots, \alpha_m)$ *of column-lengths of* D *there is a single 4, exactly* $(u - 1)$ *3's and, in addition, the 4 occurs before all the 3's. We also define the* determining tuple $\hat{\alpha}_D$ *of* D *by* $\hat{\alpha}_D = (\hat{\alpha}_1, \ldots, \hat{\alpha}_m)$*, where* $\hat{\alpha}_j = \alpha_j$ *if* $\alpha_j \in \{2, 3, 4\}$*, and* $\hat{\alpha}_j = 1$ *or* $\bar{1}$ *according as the single node in a column* j *of length 1 in* D *is on row 2 or 3. [Clearly the tuple* $\hat{\alpha}_D$ *determines a diagram satisfying Hypothesis (†) uniquely, since all columns in* D *having length 2 (resp., length 3) necessarily have their nodes on rows 2 and 3 (resp., on rows 1, 2 and 3).]*

Remark 4.10. *Let* $E \in \mathcal{D}^{(\lambda)}$ *satisfy Hypothesis (†) (with composition* λ *as in Definition 4.9) and let* $\hat{\alpha}_E = (\hat{\alpha}_1, \ldots, \hat{\alpha}_m)$ *be the determining tuple for* E*. Suppose now that diagram* E' *has been obtained from* E *via any one of the operations (C1)–(C5) below.*

Operations (C1)–(C4): If $(\hat{\alpha}_j, \hat{\alpha}_{j+1}) = (1,2)$ *or* $(3,2)$ *or* $(2,\bar{1})$ *or* $(3,\bar{1})$ *for some* $j \geqslant 1$*, diagram* E' *is obtained from* E *by interchanging the* j*-th and* $(j+1)$*-th columns. Suppose for convenience that the first* $j-1$ *columns of* E *contain exactly* $\omega - 1$ *nodes (where* $\omega \geqslant 1$*). Then, from the way they are constructed, the tableaux* t_E *and* $t_{E'}$ *differ only on these two columns, which respectively take the form*

$\begin{array}{|cc|}\hline & \\ \omega & \omega+1 \\ & \omega+2 \\ \hline\end{array}$ *and* $\begin{array}{|cc|}\hline & \\ \omega & \omega+2 \\ \omega+1 & \\ \hline\end{array}$ *after operation (C1),* $\begin{array}{|cc|}\hline \omega & \\ \omega+1 & \omega+3 \\ \omega+2 & \omega+4 \\ \hline\end{array}$ *and* $\begin{array}{|cc|}\hline & \omega+2 \\ \omega & \omega+3 \\ \omega+1 & \omega+4 \\ \hline\end{array}$ *after operation (C2),*

$\begin{array}{|cc|}\hline & \\ \omega & \\ \omega+1 & \omega+2 \\ \hline\end{array}$ *and* $\begin{array}{|cc|}\hline & \\ & \omega+1 \\ \omega & \omega+2 \\ \hline\end{array}$ *after operation (C3),* $\begin{array}{|cc|}\hline \omega & \\ \omega+1 & \\ \omega+2 & \omega+3 \\ \hline\end{array}$ *and* $\begin{array}{|cc|}\hline & \omega+1 \\ & \omega+2 \\ \omega & \omega+3 \\ \hline\end{array}$ *after operation (C4). Since* $t^{E'}w_E$ *is standard,* w_E *is a prefix of* $w_{E'}$*.*

Operation (C5): If $\hat{\alpha}_j = 2$ *for some* j*, diagram* E' *is obtained from* E *after replacing the* j*-th column of* E *by two adjacent columns each having a single node; the single node of the first one (resp., second one) being on row 3 (resp., row 2). The difference in* t_E *and* $t_{E'}$ *can be described by* $\begin{array}{|c|}\hline \omega \\ \omega+1 \\ \hline\end{array}$ *and* $\begin{array}{|cc|}\hline & \omega+1 \\ \omega & \\ \hline\end{array}$*, respectively. In particular,* E' *has* $m+1$ *columns.*

Clearly, in all of the above cases $E' \in \mathcal{D}^{(\lambda)}$ *and* E' *satisfies Hypothesis (†). Moreover,* $t^{E'}w_E$ *is a standard* E'*-tableau, so* w_E *is a prefix of* $w_{E'}$*.*

Example 4.11. *Let composition* λ *satisfy Hypothesis (*). Suppose further that* $u < t$ *and* $\lambda_1 = u$*. In this example we introduce certain types of admissible diagrams* $D \in \mathcal{D}^{(\lambda)}$ *satisfying Hypothesis (†). In particular, such diagrams* D *cannot be transformed using operations (C1)–(C4) to an admissible diagram* K *such that* $t^K w_D$ *is a standard* K*-tableau. Moreover, in all cases, they are the support of a form-B s-path.*

(i) *If* $\tilde{\lambda} = (u,s,t)$*, let* ε*,* η*,* θ*,* ζ *and* ψ *be non-negative integers satisfying* $s = \varepsilon + \eta + 1 + \zeta + \psi$*,* $t = \varepsilon + \theta + \zeta + u$*,* $\psi \geqslant u-1$*, and* $\eta \geqslant \theta$*, let* $\mathcal{C}$ *be a* $(u-1)$*-subset of* $\{s+\theta-\psi+1, \ldots, s+\theta\}$*, and let* $\mathcal{S} = (\varepsilon, \eta, \theta, \zeta, \psi, \mathcal{C})$*. Define* $M^{(\mathcal{S})}$ *to be the diagram with nodes*

$(1,i)$: $i = \varepsilon + \eta + 1$ *or* $i \in \mathcal{C}$,

$(2,i)$: $1 \leqslant i \leqslant \varepsilon + \eta + 1$ *or* $\varepsilon + \eta + \theta + 2 \leqslant i \leqslant s + \theta$,

$(3,i)$: $1 \leqslant i \leqslant \varepsilon$ *or* $\varepsilon + \eta + 1 \leqslant i \leqslant \varepsilon + \eta + \theta + \zeta + 1$ *or* $i \in \mathcal{C}$,

$(4,i)$: $i = \varepsilon + \eta + 1$.

Then $M^{(\mathcal{S})} \in \mathcal{D}^{(\lambda)}$*, it has* $s+\theta$ *columns and it is easy to check that it satisfies Hypothesis (†). In the determining tuple of* $M^{(\mathcal{S})}$ *there is a single 4 and the tuple occurring before the 4 consists of an* ε*-tuple of 2's followed by an* η*-tuple of 1's. Following the 4 there is a* θ*-tuple of* $\bar{1}$*'s followed by a* ζ*-tuple of 2's followed by a* ψ*-tuple of 1's and 3's containing exactly* $(u-1)$ *3's.*

Schematically, $M^{(\mathcal{S})}$ takes the form

$$
\begin{array}{cccccc}
\overbrace{\qquad\qquad}^{\varepsilon} & \overbrace{\qquad\qquad}^{\eta} & \overbrace{\quad}^{1} & \overbrace{\qquad\qquad}^{\theta} & \overbrace{\qquad\qquad}^{\zeta} & \overbrace{\qquad\qquad\qquad\qquad\qquad}^{\psi} \\
 & & \times & & & \qquad\qquad \times \qquad\qquad\quad \times \\
\times \cdots \times & \times \cdots \times & \times & & \times \cdots \times & \times \cdots \times \cdots \times \cdots \times \\
\times \cdots \times & & \times & \times \cdots \times & \times \cdots \times & \qquad\qquad \times \qquad\qquad\quad \times \\
 & & \times & & &
\end{array} .
$$

From the construction of $M^{(\mathcal{S})}$ we see that $M^{(\mathcal{S})}$ is special if, and only if, $\theta = 0$. It is immediate that $M^{(\mathcal{S})}$ is special if $\theta = 0$. Conversely, if $M^{(\mathcal{S})}$ is special and $s > t$, it is clear that $\theta = 0$. Finally, if $s = t$, the relations in line 2 of this example imply that $\eta = \theta$ and $\psi = u - 1$, so the additional constraint that $M^{(\mathcal{S})}$ is special, now implies that $\eta = \theta = 0$. In particular we have that $M^{(\mathcal{S})}$ is admissible if $\theta = 0$.

If $\eta \geqslant \theta \geqslant 1$, let $\Pi = (\pi_1, \dots, \pi_s)$ and $\Pi' = (\pi'_1, \dots, \pi'_s)$ be the s-paths defined by

$$
\pi_j = \begin{cases}
\{(2,j),(3,j)\} & \textit{if} \quad 1 \leqslant j \leqslant \varepsilon, \\
\{(2,j),(3,j+\eta+1)\} & \textit{if} \quad \varepsilon+1 \leqslant j \leqslant \varepsilon+\theta, \\
\{(2,j)\} & \textit{if} \quad \varepsilon+\theta+1 \leqslant j \leqslant \varepsilon+\eta, \\
\{(1,j),(2,j),(3,j),(4,j)\} & \textit{if} \quad j = \varepsilon+\eta+1, \\
\{(2,j+\theta),(3,j+\theta)\} & \textit{if} \quad \varepsilon+\eta+2 \leqslant j \leqslant s-\psi, \\
\{(1,j+\theta),(2,j+\theta),(3,j+\theta)\} & \textit{if} \quad s-\psi+1 \leqslant j \leqslant s \textit{ and } j+\theta \in \mathcal{C}, \\
\{(2,j+\theta)\} & \textit{if} \quad s-\psi+1 \leqslant j \leqslant s \textit{ and } j+\theta \notin \mathcal{C}.
\end{cases}
$$

and

$$
\pi'_j = \begin{cases}
\{(2,j),(3,j)\} & \textit{if} \quad 1 \leqslant j \leqslant \varepsilon, \\
\{(2,j),(3,j+\eta),(4,j+\eta)\} & \textit{if} \quad j = \varepsilon+1, \\
\{(2,j),(3,j+\eta)\} & \textit{if} \quad \varepsilon+2 \leqslant j \leqslant \varepsilon+\theta, \\
\{(2,j)\} & \textit{if} \quad \varepsilon+\theta+1 \leqslant j \leqslant \varepsilon+\eta, \\
\{(1,j),(2,j),(3,j+\theta)\} & \textit{if} \quad j = \varepsilon+\eta+1, \\
\{(2,j+\theta),(3,j+\theta)\} & \textit{if} \quad \varepsilon+\eta+2 \leqslant j \leqslant s-\psi, \\
\{(1,j+\theta),(2,j+\theta),(3,j+\theta)\} & \textit{if} \quad s-\psi+1 \leqslant j \leqslant s \textit{ and } j+\theta \in \mathcal{C}, \\
\{(2,j+\theta)\} & \textit{if} \quad s-\psi+1 \leqslant j \leqslant s \textit{ and } j+\theta \notin \mathcal{C}.
\end{cases}
$$

Then Π has type λ', so that $M^{(\mathcal{S})}$ is an admissible diagram (see Remark 4.2). Also, Π' is a form-B s-path.

We look at a specific case. If $\tilde{\lambda} = (3,8,5)$ *and* $\mathcal{S} = (1,3,1,0,3,\{7,8\})$ *then*

$$M^{(\mathcal{S})} = \begin{array}{ccccccccc} & & & & \times & & \times & \times & \\ \times & \times & \times & \times & \times & & \times & \times & \times \\ \times & & & & \times & \times & \times & \times & \\ & & & & \times & & & & \end{array}.$$

Let $\Pi = \{\ \{(2,1),(3,1)\}, \{(2,2),(3,6)\}, \{(2,3)\}, \{(2,4)\}, \{(1,5),(2,5),(3,5),(4,5)\}, \{(1,7),(2,7),(3,7)\}, \{(1,8),(2,8),(3,8)\}, \{(2,9)\}\}$ *and* $\Pi' = \{\ \{(2,1),(3,1)\}, \{(2,2),(3,5),(4,5)\}, \{(2,3)\}, \{(2,4)\}, \{(1,5),(2,5),(3,6)\}, \{(1,7),(2,7),(3,7)\}, \{(1,8),(2,8),(3,8)\}, \{(2,9)\}\}$*. Then* Π *and* Π' *are* 8*-paths in the diagram* $M^{(\mathcal{S})}$ *of types* $\lambda' = 4^1 3^2 2^2 1^3$ *and* $3^4 2^1 1^3$*, respectively. Thus* $M^{(\mathcal{S})}$ *is admissible. Moreover* Π' *is a form-B* 8*-path.*

(ii) If $\tilde{\lambda} = (u,t,s)$*, let* η*,* ε*,* θ*,* φ*, and* ζ *be non-negative integers satisfying* $s = \eta + \varepsilon + \varphi + \zeta + u$ *and* $t = \varepsilon + \theta + \zeta + u$*, and* $\varphi \geqslant \theta$*. Let* $\mathcal{S} = (\eta, \varepsilon, \theta, \varphi, \zeta)$*. Define* $N^{(\mathcal{S})}$ *to be the diagram with nodes*

$$(1,i) \ : \ i = \eta + \varepsilon + \theta + 1 \text{ or } s + \theta - u + 2 \leqslant i \leqslant s + \theta,$$

$$(2,i) \ : \ \eta + 1 \leqslant i \leqslant \eta + \varepsilon + \theta + 1 \text{ or } s + \theta - u - \zeta + 2 \leqslant i \leqslant s + \theta,$$

$$(3,i) \ : \ 1 \leqslant i \leqslant \eta + \varepsilon \text{ or } \eta + \varepsilon + \theta + 1 \leqslant i \leqslant s + \theta,$$

$$(4,i) \ : \ i = \eta + \varepsilon + \theta + 1.$$

Then $N^{(\mathcal{S})} \in \mathcal{D}^{(\lambda)}$*, it has* $s + \theta$ *columns and it is easy to check that it satisfies Hypothesis (†). The determining tuple of* $N^{(\mathcal{S})}$ *consists, going from left to right, of an* η*-tuple of* $\bar{1}$*'s followed by an* ε*-tuple of* 2*'s, a* θ*-tuple of* 1*'s, a single* 4*, a* φ*-tuple of* $\bar{1}$*'s, a* ζ*-tuple of* 2*'s and, finally, a* $(u-1)$*-tuple of* 3*'s.*

Schematically, $N^{(\mathcal{S})}$ *takes the form*

$$\begin{array}{ccccccc} \overbrace{\qquad\qquad}^{\eta} & \overbrace{\qquad\qquad}^{\varepsilon} & \overbrace{\qquad\qquad}^{\theta} & \overbrace{\quad}^{1} & \overbrace{\qquad\qquad}^{\varphi} & \overbrace{\qquad\qquad}^{\zeta} & \overbrace{\qquad\qquad}^{u-1} \\ & & & \times & & & \times \qquad \times \\ & \times \cdots \times & \times \cdots \times & \times & & \times \cdots \times & \times \cdots \times \\ \times \cdots \times & \times \cdots \times & & \times & \times \cdots \times & \times \cdots \times & \times \qquad \times \\ & & & \times & & & \end{array}.$$

It is easy to see that $N^{(\mathcal{S})}$ *is special if, and only if,* $\theta = 0$*. It follows that* $N^{(\mathcal{S})}$ *is admissible if* $\theta = 0$*. Observe also that the relations* s *and* t *satisfy, force* $\varphi = 0$ *and* $\eta = 0$ *if* $s = t$*. So in the case* $s = t$ *the shape of* $M^{(\mathcal{S})}$ *in fact coincides with the shape of* $N^{(\mathcal{S})}$*.*

If $\varphi \geqslant \theta \geqslant 1$*, let* $\Pi = (\pi_1, \ldots, \pi_s)$ *and* $\Pi' = (\pi'_1, \ldots, \pi'_s)$ *be the* s*-paths defined by*

$$\pi_j = \begin{cases} \{(3,j)\} & \text{if} \quad 1 \leqslant j \leqslant \eta, \\ \{(2,j),(3,j)\} & \text{if} \quad \eta+1 \leqslant j \leqslant \eta+\varepsilon, \\ \{(2,j),(3,j+\theta+1)\} & \text{if} \quad \eta+\varepsilon+1 \leqslant j \leqslant \eta+\varepsilon+\theta, \\ \{(1,j),(2,j),(3,j),(4,j)\} & \text{if} \quad j = \eta+\varepsilon+\theta+1, \\ \{(3,j+\theta)\} & \text{if} \quad \eta+\varepsilon+\theta+2 \leqslant j \leqslant \eta+\varepsilon+\varphi+1, \\ \{(2,j+\theta),(3,j+\theta)\} & \text{if} \quad \eta+\varepsilon+\varphi+2 \leqslant j \leqslant \eta+\varepsilon+\varphi+\zeta+1, \\ \{(1,j+\theta),(2,j+\theta),(3,j+\theta)\} & \text{if} \quad \eta+\varepsilon+\varphi+\zeta+2 \leqslant j \leqslant \eta+\varepsilon+\varphi+\zeta+u, \end{cases}$$

and

$$\pi'_j = \begin{cases} \{(3,j)\} & \text{if} \quad 1 \leqslant j \leqslant \eta, \\ \{(2,j),(3,j)\} & \text{if} \quad \eta+1 \leqslant j \leqslant \eta+\varepsilon, \\ \{(2,j),(3,j+\theta),(4,j+\theta)\} & \text{if} \quad j = \eta+\varepsilon+1, \\ \{(2,j),(3,j+\theta)\} & \text{if} \quad \eta+\varepsilon+2 \leqslant j \leqslant \eta+\varepsilon+\theta, \\ \{(1,j),(2,j),(3,j+\theta)\} & \text{if} \quad j = \eta+\varepsilon+\theta+1, \\ \{(3,j+\theta)\} & \text{if} \quad \eta+\varepsilon+\theta+2 \leqslant j \leqslant \eta+\varepsilon+\varphi+1, \\ \{(2,j+\theta),(3,j+\theta)\} & \text{if} \quad \eta+\varepsilon+\varphi+2 \leqslant j \leqslant \eta+\varepsilon+\varphi+\zeta+1, \\ \{(1,j+\theta),(2,j+\theta),(3,j+\theta)\} & \text{if} \quad \eta+\varepsilon+\varphi+\zeta+2 \leqslant j \leqslant \eta+\varepsilon+\varphi+\zeta+u. \end{cases}$$

Then Π *has type* λ'*, so that* $N^{(\mathcal{S})}$ *is an admissible diagram. Also,* Π' *is a form-B s-path.*

We look at a specific case. If $\tilde{\lambda} = (3,5,8)$ *and* $\mathcal{S} = (3,0,1,1,1,3)$ *then*

$$N^{(\mathcal{S})} = \begin{array}{ccccccccc} & & & & \times & & & \times & \times \\ & & & \times & \times & & \times & \times & \times \\ \times & \times & \times & & \times & \times & \times & \times & \times \\ & & & & \times & & & & \end{array}.$$

Let $\Pi = \{\ \{(3,1)\},\ \{(3,2)\},\ \{(3,3)\},\ \{(2,4),\ (3,6)\},\ \{(1,5),\ (2,5),\ (3,5),\ (4,5)\},\ \{(2,7),\ (3,7)\},$ $\{(1,8),\ (2,8),\ (3,8)\},\ \{(1,9),\ (2,9),\ (3,9)\}\ \}$ *and* $\Pi' = \{\ \{(3,1)\},\ \{(3,2)\},\ \{(3,3)\},\ \{(2,4),\ (3,5),$ $(4,5)\},\ \{(1,5),\ (2,5),\ (3,6)\},\ \{(2,7),\ (3,7)\},\ \{(1,8),\ (2,8),\ (3,8)\},\ \{(1,9),\ (2,9),\ (3,9)\}\ \}$*.*
Then Π *and* Π' *are* 8*-paths in the diagram* $N^{(\mathcal{S})}$ *of types* $\lambda' = 4^1 3^2 2^2 1^3$ *and* $3^4 2^1 1^3$*, respectively. Thus* $N^{(\mathcal{S})}$ *is admissible. Moreover* Π' *is a form-B* 8*-path.*

In the course of the proof of Theorem 5.1 we will show that any diagram D satisfying Hypothesis (†), and which also satisfies some additional constraints, can be transformed using operations (C1)–(C5) to a diagram which either equals $M^{(\mathcal{S})}$ or $N^{(\mathcal{S})}$ for a suitable tuple S. The following two lemmas will also turn out to be useful.

Lemma 4.12. *Under the hypothesis and notation of Example 4.11(i), let* $K = M^{(\mathcal{S})}$ *and* $K' = M^{(\mathcal{S}')}$ *be the diagrams corresponding to* $\mathcal{S} = (\varepsilon, \eta, \theta, \zeta, \psi, \mathcal{C})$ *and* $\mathcal{S}' = (\varepsilon', \eta', \theta', \zeta', \psi', \mathcal{C}')$*, respectively, where*

$\eta \geqslant \theta$ and $\eta' \geqslant \theta'$. Suppose that the further constraints $\zeta = 0$ if $\eta > \theta$ and $\zeta' = 0$ if $\eta' > \theta'$ are in force. If $w_{K'}$ is a prefix of w_K, then $K = K'$.

Proof. We assume the hypothesis and we suppose that $w_{K'}$ is a prefix of w_K. Then $t^K w_{K'}$ is a standard K-tableau (see Result 4). Observe that $t^K w_{K'}$ is obtained from the K'-tableau $t_{K'}$ by moving the entries of each row of $t_{K'}$, keeping their order, to the nodes of K on the same row. This forces the entry on row 4 of $t_{K'}$ to move to the single node on row 4 of K and, moreover the first entry (starting from the left) on row 1 of $t_{K'}$ to move to the first node on row 1 of K.

Thus, in order to preserve standardness, the column of length 4 in $t_{K'}$ must move to the column of length 4 in K. Counting nodes on rows 3 and 2 of K and K' lying to the left of the column of length 4 in each of the two diagrams, we immediately get $\varepsilon = \varepsilon'$ and $\eta = \eta'$. On the other hand, counting nodes on rows 2 and 3 lying to the right of the column of length 4 in each of the two diagrams, we get $\zeta + \psi = \zeta' + \psi'$ and $\theta + \zeta = \theta' + \zeta'$ (since $\theta + \zeta + (u-1) = \theta' + \zeta' + (u-1)$).

In order to show that $\theta = \theta'$, $\zeta = \zeta'$ and $\psi = \psi'$, it is convenient to consider the four subcases (a) $\eta > \theta$, $\eta' > \theta'$, (b) $\eta > \theta$, $\eta' = \theta'$, (c) $\eta = \theta$, $\eta' > \theta'$, and (d) $\eta = \theta$, $\eta' = \theta'$. In (a) we have $\zeta = 0 = \zeta'$, so $\theta = \theta'$ and $\psi = \psi'$ as required. In (b), we have $\zeta = 0$ hence $\theta = \theta + \zeta = \theta' + \zeta' \geqslant \theta' = \eta' = \eta > \theta$, a contradiction, so this case cannot occur. Similarly in (c), we have $\theta' = \theta' + \zeta' = \theta + \zeta \geqslant \theta = \eta = \eta' > \theta'$, again a contradiction. Finally in (d) we have $\theta = \eta = \eta' = \theta'$, hence $\zeta = \zeta'$ and $\psi = \psi'$.

It remains to look at at the last $\psi\,(= \psi')$ columns of $t_{K'}$ and K, which contain precisely $u-1$ columns of length 3 in each case. A similar argument as for the case of the column of length 4 shows that the $u-1$ columns of length 3 in $t_{K'}$ move to the $u-1$ columns of length 3 in K. This completes the proof in the case $s = t$, since $\psi = u-1$ in this case as we have seen. If $s > t$, counting the number of columns of length 1 in K and K' occurring between the block of ζ columns each having length 2 and the first column of length 3, we see that the first column of length 3 occurs in exactly the same position in both diagrams. Similarly, by looking at the number of columns of length 1 between any pair of consecutive columns of length 3 in K and K', we conclude that $K = K'$. ☐

Lemma 4.13. *Under the hypothesis and notation of Example 4.11(ii), let $K = N^{(\mathcal{S})}$ and $K' = N^{(\mathcal{S}')}$ be the diagrams corresponding to $\mathcal{S} = (\eta, \varepsilon, \theta, \varphi, \zeta)$ and $\mathcal{S}' = (\eta', \varepsilon', \theta', \varphi', \zeta')$, respectively, where $\varphi \geqslant \theta$ and $\varphi' \geqslant \theta'$. Suppose that the further constraints $\varepsilon = 0$ if $\varphi > \theta$ and $\varepsilon' = 0$ if $\varphi' > \theta'$ are in force. If $w_{K'}$ is a prefix of w_K, then $K = K'$.*

Proof. We assume the hypothesis, and suppose that $w_{K'}$ is a prefix of w_K. Then $t^K w_{K'}$ is a standard K-tableau. Since this tableau is obtained by moving the entries of each row of $t_{K'}$ to the nodes of K on the same row, keeping the order these entries appear, the column of length 4 in $t_{K'}$ must move to the column of length 4 in K by a similar argument to that in the previous lemma. It is also clear that the last $u-1$ columns of $t_{K'}$, each having length 3, move to the last $u-1$ columns of K. Considering the second row nodes (resp., third row nodes) lying to the left of the column of length 4 in each diagram, we see that $\varepsilon + \theta = \varepsilon' + \theta'$ (resp., $\eta + \varepsilon = \eta' + \varepsilon'$). Similarly, considering the second row nodes lying to the right of the column of length 4 in each diagram, we see that $\zeta = \zeta'$ and $\varphi + \zeta = \varphi' + \zeta'$ from which it follows that $\varphi = \varphi'$. In order to obtain the desired result we will consider the four subcases (a) $\varphi > \theta$, $\varphi' > \theta'$, (b) $\varphi > \theta$, $\varphi' = \theta'$, (c) $\varphi = \theta$, $\varphi' > \theta'$, and (d) $\varphi = \theta$, $\varphi' = \theta'$. In (a), we get $\varepsilon = 0 = \varepsilon'$ from the hypothesis, so $\theta = \theta'$ and $\eta = \eta'$. In (b) we have $\varepsilon = 0$, hence $\theta = \theta' + \varepsilon' \geqslant \theta' = \varphi' = \varphi > \theta$, a contradiction, so this case cannot occur. Similarly, in (c), we have $\varepsilon' = 0$, hence $\theta' = \varepsilon + \theta \geqslant \theta = \varphi = \varphi' > \theta'$, again a contradiction. Finally in (d) we have $\theta = \varphi = \varphi' = \theta'$, so $\varepsilon = \varepsilon'$ and $\eta = \eta'$. We conclude that $K = K'$. □

5. Explicit results on minimal determining sets

In this section we obtain explicit descriptions of the minimal determining sets of certain W-graph ideals in the symmetric group. These are (right) W-graph ideals corresponding to the Kazhdan-Lusztig cell $\mathfrak{C}(\lambda)$ for $\lambda = (\lambda_1, \lambda_2, \lambda_3, 1^r)$ or $(1^r, \mu_1, \mu_2, \mu_3)$, or to the union of cells obtained by inducing such a Kazhdan-Lusztig cell $\mathfrak{C}(\lambda)$.

Theorem 5.1. *Let $r \geqslant 4$ and $s \geqslant t \geqslant u \geqslant 1$. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a composition where $\tilde{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ is a permutation of (s, t, u) and $\lambda_i = 1$ if $i > 3$. Then $\mathcal{E}(\lambda) = \mathcal{E}_s(\lambda)$ if, and only if, $\lambda_1 = s$ or t.*

Explicit descriptions of the elements of $\mathcal{E}(\lambda)$ and $\mathcal{E}_s(\lambda)$ are given in the proof and the values of $|\mathcal{E}_s(\lambda)|$ and $|\mathcal{E}(\lambda)| - |\mathcal{E}_s(\lambda)|$ are given in Tables 3 and 4.

Proof. Since the results for $r > 4$ can be obtained easily by induction on r using Result 9 assuming the results for $r = 4$, we consider the case $r = 4$. In particular, we have that λ satisfies Hypothesis (*). We suppose further that $d \in Z(\lambda)$ and we let $D = D(d, \lambda)\,(\in \mathcal{D}^{(\lambda)})$, so $d = w_D$. Then D is an admissible diagram by Result 5(ii). Below, the cases (I) $\lambda_1 = s$ or $\lambda_1 = t$, and (II) $\lambda_1 = u$ and $t > u$, will be considered separately.

Case (I): $\lambda_1 = s$ or t: By Lemma 4.6, we have $D = s(\Pi')$ for some form-A s-path Π'. Moreover, d is a prefix of $w_{D'}$ for some special diagram $D' \in \mathcal{D}^{(\lambda)}$ by Remark 4.7(ii). In particular, $Y(\lambda) = Y_s(\lambda)$ if $\lambda_1 = s$ or t. Diagram D' has s columns and the nodes on any row of D' of length s have column indices j, for $1 \leqslant j \leqslant s$. The node on row 4 of D' is $(4, l)$, where $1 \leqslant l \leqslant s$. Let A be the set of column indices of the nodes of D' of any row of D' of length t, and let B be the set of column indices of the nodes of D' on any one of its rows of length u. Then $l \in B \subseteq A \subseteq \{1, \ldots, s\}$. Clearly, $|A| = t$ and $|B| = u$. Finally, let $m = \min(B)$.

The case λ is a partition is already covered in [11, Lemma 3.3] and in this case $\mathcal{E}^{(\lambda)}$ has a single element , the Young diagram associated to λ. It will be convenient, in order to complete case (I), to consider the subcases (i) $\tilde{\lambda} = (s, u, t)$, (ii) $\tilde{\lambda} = (t, s, u)$, and (iii) $\tilde{\lambda} = (t, u, s)$ (even though these subcases are not disjoint and there are also intersections with the partition case already considered).

Subcase (I)(i): $\tilde{\lambda} = (s, u, t)$: Let $\delta\colon A \to \{1, \ldots, t\}$ be the order preserving bijection and let $C = B\delta$. Consider the diagram $E = F_C$ as in Example 4.8(i). Then E is a special, and hence admissible, diagram such that $t^E w_{D'}$ is a standard E-tableau. So $w_{D'}$ is a prefix of w_E. Also from the discussion in

Example 4.8(i) we can deduce that $\mathcal{E}^{(\lambda)}$ is precisely the set of diagrams F_C for all the different permitted choices of C. In particular, $|Y(\lambda)| = |Y_s(\lambda)| = \binom{t}{u}$ in this case. (For example in the case $t = u$ there is a unique choice for F_C and this is the Young diagram associated to the partition λ.)

Subcase (I)(ii): $\tilde{\lambda} = (t, s, u)$: Let C be the set of u smallest indices in A and consider the diagram $E = G_C$ as in Example 4.8(ii). Clearly, $w_{D'}$ is a prefix of w_E. From the discussion in Example 4.8(ii) we see that $\mathcal{E}^{(\lambda)}$ consists precisely of the diagrams G_C for all the different permitted choices of C. It follows that in this case $|Y(\lambda)| = |Y_s(\lambda)| = \binom{s-t+u}{u}$.

Subcase (I)(iii): $\tilde{\lambda} = (t, u, s)$: We will need to split this case into two subcases

(a) $m\,(= \min(B)) \geqslant s - t + 1$, and

(b) $m < s - t + 1$.

If $m \geqslant s - t + 1$, we set $C = B$ and $v = m$ (so $v = \min(B) = \min(C)$) and let $\tilde{C} = C - \{v\}$, (so $\tilde{C} \subseteq \{s-t+2, \ldots, s\}$). Comparing with Example 4.8(iii) we also let $\tilde{v} = s-t+1$ (since $v \geqslant s-t+1$). If $m < s-t+1$, let $\delta \colon A \to \{m\} \cup \{s-t+2, \ldots, s\}$ be the order preserving bijection. We also set $v = m$, $\tilde{C} = (B - \{m\})\delta$ and $C = \tilde{C} \cup \{v\}$. Hence $v\,(= m) = \min(C)$ and $\tilde{C} \subseteq \{s - t + 2, \ldots, s\}$. Comparing with Example 4.8(iii), we set $\tilde{v} = v$ in this subcase.

In either of the subcases (a) or (b) above we get, by setting $E = H_C$, that $t^E w_{D'}$ is a standard E-tableau, with $E \in \mathcal{D}^{(\lambda)}$ being a special (and hence admissible) diagram. Thus, combining with the last paragraph of Example 4.8(iii) we conclude that in the case $\tilde{\lambda} = (t, u, s)$, the set $\mathcal{E}^{(\lambda)}$ consists of the diagrams H_C for all the different permitted choices of C. For the subcase $m \geqslant s - t + 1$ there are $\binom{t}{u}$ such diagrams H_C, which is the number of u-sets in $\{s-t+1, \ldots, s\}$ as these determine v uniquely. For the subcase $m < s-t+1$ there are $(s-t)\binom{t-1}{u-1}$ such diagrams H_C since, here, we can combine each of the $s-t$ choices of v with each of the $\binom{t-1}{u-1}$ choices of $C - \{v\}$ in $\{s - t + 2, \ldots, s\}$. We conclude that for $\tilde{\lambda} = (t, u, s)$ we have $|Y(\lambda)| = |Y_s(\lambda)| = (s - t)\binom{t-1}{u-1} + \binom{t}{u}$. Observe that in the special case $t = s$ (here $s - t + 1 = 1$, so we cannot have $m < s - t + 1$), the $\binom{t}{u}$ different diagrams H_C which occur are precisely the $\binom{t}{u}$ different diagrams F_C occurring in subcase (I)(i) with $s = t$.

Case (II): $\lambda_1 = u$ and $t > u$ (which is equivalent to $\lambda_1 \neq s$ and $\lambda_1 \neq t$): Recall that d denotes an arbitrary element of $Z(\lambda)$ and $D = D(d, \lambda)$, but in this case diagram D may or may not have form-A s-paths. Our upshot is to show that d is a prefix of w_K for some admissible diagram K of shape $M^{(\mathcal{S})}$ or $N^{(\mathcal{S})}$ (as these are defined in Example 4.11) according as $\tilde{\lambda} = (u, s, t)$ or (u, t, s). As an intermediate goal, we aim to show that d is a prefix of w_E, for some diagram $E \in \mathcal{D}^{(\lambda)}$ satisfying Hypothesis (†), but also having some additional properties as we will see later.

First, we consider the case D does not have a form-A s-path. Since D is admissible, Lemma 4.4 ensures that D has a form-B s-path. Choose one such form-B s-path $\Pi = (\pi_1, \ldots, \pi_s)$ and define the s-tuple $\hat{a} = (a_1, \ldots, a_s)$ by setting a_j to be the length of π_j for $1 \leqslant j \leqslant s$. Let j_1 be the first j with $a_j = 3$ and let j_2 be the second such j.

Comparing with Lemma 4.4 we see that π_{j_1} is the path of length 3 in Π containing the node on row 4 (otherwise Π would be equivalent to a form-A s-path in D). Next, we form the diagram $\tilde{E} = D(\Pi)$, see Remark 4.7. From the same remark we see that $t^{\tilde{E}}d$ is a standard $\tilde{E}$-tableau, since Π is ordered. Clearly $\tilde{E}$ is not admissible since it has no path of length 4. The $\tilde{E}$-tableau $t^{\tilde{E}}d$ has exactly s columns and takes the form given in Table 1. In this table we denote by b_j or c_j the entries on rows 2 or 3, accordingly, of column j of $t^{\tilde{E}}d$ for $1 \leqslant j \leqslant s$. Symbol $\hat{C}$ denotes the columns with index $j > j_2$; these have entries either on rows 1, 2 and 3, or on rows 2 and 3 or on a single row (row 2 if $\tilde{\lambda} = (u, s, t)$ or row 3 if $\tilde{\lambda} = (u, t, s)$). See Lemma 4.4 for the distribution of the nodes of $\tilde{E}$ in its columns, taking into account the way $\tilde{E}$ has been constructed. In particular, $\tilde{E}$ has no columns of length 1 if $s = t$.

Entries b_j or c_j may be blank if $j \notin \{j_1, j_2\}$, however b_j (resp., c_j) cannot be blank if $\tilde{\lambda} = (u, s, t)$ (resp., $\tilde{\lambda} = (u, t, s)$) again by Lemma 4.4. None of $b_{j_1}, b_{j_2}, c_{j_1}, c_{j_2}$ is blank from our hypothesis that they belong to columns of length 3. Entry y is not blank and it is the sole entry on row 4 of $t^{\tilde{E}}d$. Moreover, x is not blank and it is the first entry on row 1 of $t^{\tilde{E}}d$. Hence, y is the entry occupying the node $(4, j_1)$ of $\tilde{E}$ and x is the entry occupying the node $(1, j_2)$ of $\tilde{E}$, with $j_1 < j_2$.

Table 1. $t^{\tilde{E}}d$

$$t^{\tilde{E}}d = \begin{array}{|cccccccc|} & & & & & & x & \\ b_1 & \cdots & b_{j_1} & b_{j_1+1} & \cdots & b_{j'} & \cdots & b_{j_2} \quad \hat{C} \\ c_1 & \cdots & c_{j_1} & c_{j_1+1} & \cdots & c_{j'} & \cdots & c_{j_2} \\ & & y & & & & & \end{array}$$

The fact that D has no form-A s-paths ensures that $x > b_{j_1}$ and $y < c_{j_2}$. (If $x < b_{j_1}$ this would mean that x lies in a column of tableau t_D of weakly smaller index than the index of the column containing b_{j_1}, which would mean in turn that D has a form-A s-path. We can exclude the possibility $y > c_{j_2}$ by using similar argument.)

Now let N_1 be the node in D with $N_1 t_D = x$. Clearly N_1 is the first node on row 1 of D from the way $t^{\tilde{E}}d$ is related to t_D. Since D is admissible, it contains a path π of length 4 with $N_1 \in s(\pi)$. Also let N_i, with N_i on row i of D for $2 \leqslant i \leqslant 4$, be the remaining nodes of π. So $s(\pi) = \{N_1, N_2, N_3, N_4\}$ with $N_1 t_D = x$ and $N_4 t_D = y$ (the last equality follows from the fact that N_4 is the only node on row 4 of D). We also have that $N_2 t_D = b_k$ and $N_3 t_D = c_l$ for some k, l with $1 \leqslant k, l \leqslant s$. Since π is a path in D, it follows from the way t_D is constructed, that $x < b_k < c_l < y$. Hence, the relations $b_{j_1} < x$ and $y < c_{j_2}$ obtained above, together with the standardness of $t^{\tilde{E}}d$, give that $j_1 < k$ and $l < j_2$.

Claim. There exist j', j'' with $j_1 \leqslant j'' \leqslant j' \leqslant j_2$ and $x < b_{j'} < c_{j''} < y$.

Proof of Claim. Recall that $x < b_k < c_l < y$ and also that $b_{j_1} < x$ where $j_1 < k$ and $y < c_{j_2}$ where $l < j_2$.

Suppose first that $k \leqslant l$. Then $j_1 < k \leqslant l < j_2$. Now at least one of the entries b_l or c_k is non-empty according as $\tilde{\lambda} = (u, s, t)$ or (u, t, s). The claim is now proved, using the standardness of $t^{\tilde{E}}d$, by setting $j' = j'' = l$ or $j' = j'' = k$ accordingly. [If b_l is non-empty, then $b_k \leqslant b_l < c_l$ since $k \leqslant l$

and $t^{\tilde{E}}d$ is standard. So $x < (b_k \leqslant)\, b_l < c_l < y$ and $j_1 < (k \leqslant)\, l < j_2$. If c_k is non-empty, then $x < b_k < c_k\, (\leqslant c_l) < y$ and $j_1 < k\, (\leqslant l) < j_2$.]

Suppose now that $k > l$. We will consider separately the four subcases

(a) $k \leqslant j_2$ and $l \geqslant j_1$,
(b) $k \leqslant j_2$ and $l < j_1$,
(c) $k > j_2$ and $l \geqslant j_1$, and
(d) $k > j_2$ and $l < j_1$.

In subcase (a), we have $j_1 \leqslant l < k \leqslant j_2$. Since $x < b_k < c_l < y$, the claim is proved by setting $k = j'$ and $l = j''$.

In subcase (b), we have $l < j_1 < k \leqslant j_2$, so $x < b_k < c_l < c_{j_1} < y$, again using the standardness of $t^{\tilde{E}}d$. By setting $j'' = j_1$ and $j' = k$, we see that $j_1 \leqslant j'' < j' \leqslant j_2$ and $x < b_{j'} < c_{j''} < y$ as required.

In subcase (c), we have $j_1 \leqslant l < j_2 < k$, so by setting $j' = j_2$ and $j'' = l$ we get $j_1 \leqslant j'' < j' = j_2$ and $x < b_{j_2}\,(= b_{j'}) < b_k < c_l\,(= c_{j''}) < y$ as required.

Finally, in subcase (d), we have $l < j_1 < j_2 < k$ so $b_{j_2} < b_k$ and $c_l < c_{j_1}$. Moreover, using the standardness of $t^{\tilde{E}}d$, we get $x < b_{j_2} < b_k < c_l < c_{j_1} < y$. By setting $j' = j_2$ and $j'' = j_1$, we have $j_1 \leqslant j'' < j' \leqslant j_2$ and $x < b_{j'} < c_{j''} < y$, thus completing the proof of the claim.

Now let $\hat{E} \in \mathcal{D}^{(\lambda)}$ be the underlying diagram of the tableau t obtained by moving the entries in tableau $t^{\tilde{E}}d$ according to the scheme in Table 2, where any blank column, that is one corresponding to a blank c_j (with $j'' < j \leqslant j'$) or b_j (with $j'' \leqslant j < j'$) according as $\tilde{\lambda} = (u, s, t)$ or $\tilde{\lambda} = (u, t, s)$, is removed.

Table 2. $t = t^{\hat{E}}d$

$$t = \boxed{\begin{array}{cccccccccccccc} & & & & & & x & & & & & & & \\ b_1 & \cdots & b_{j''-1} & b_{j''} & b_{j''+1} & \cdots & b_{j'} & & & & b_{j'+1} & \cdots & b_{j_2} & \hat{C} \\ c_1 & \cdots & c_{j''-1} & & & & c_{j''} & c_{j''+1} & \cdots & c_{j'} & c_{j'+1} & \cdots & c_{j_2} & \\ & & & & & & y & & & & & & & \end{array}}$$

Clearly, from the construction, $t = t^{\hat{E}}d$. Moreover, $\hat{E}$ satisfies Hypothesis (†) and t is a standard $\hat{E}$-tableau. So, by setting $E = \hat{E}$, the intermediate goal of showing that d is a prefix of w_E for some diagram $E \in \mathcal{D}^{(\lambda)}$ satisfying Hypothesis (†) has been achieved in the case D has no form-A s-path. So we assume now that D has a form-A s-path, say $\check{\Pi}$, and we aim to show that $D(\check{\Pi})$ can be transformed into a diagram $\check{E} \in \mathcal{D}^{(\lambda)}$ satisfying Hypothesis (†) and which also satisfies the additional requirement that d is a prefix of $w_{\check{E}}$. First observe that $D(\check{\Pi})$ is special and $t^{D(\check{\Pi})}d$ is a standard $D(\check{\Pi})$-tableau by Remark 4.7. The construction of $\check{E}$ from $D(\check{\Pi})$ is as follows: If the column of length 4 in $D(\check{\Pi})$ lies to the left of all columns of length 3 in $D(\check{\Pi})$, set $\check{E} = D(\check{\Pi})$. Otherwise, let $\check{E}$ be the diagram obtained from $D(\check{\Pi})$ by moving the single node on row 4 of $D(\check{\Pi})$ to the first column of $D(\check{\Pi})$ having length 3 (keeping this node on row 4). In either case diagram $\check{E}$ satisfies Hypothesis (†) and $t^{\check{E}}d$ is a standard $\check{E}$-tableau by its construction.

http://dx.doi.org/10.22108/IJGT.2022.131838.1764

For the rest of the proof we denote by E any diagram of shape $\hat{E}$ or $\check{E}$ which has been obtained from D via any of the above processes. In particular, irrespective of whether E is of 'type $\hat{E}$' or of 'type $\check{E}$', diagram E satisfies Hypothesis (†) and d is a prefix of w_E since $t^E d$ is standard. Moreover, the columns of length 1 in E (excluding the regions in $\hat{E}$ containing the nodes corresponding to the entries b_j in tableau t for $j'' \leqslant j < j'$ and entries c_j in tableau t for $j'' < j \leqslant j'$ — these regions can be considered to be 'empty' in $\check{E}$) have their single node on row 2 if $\tilde{\lambda} = (u, s, t)$, and on row 3 if $\tilde{\lambda} = (u, t, s)$. We can also observe that E has s' columns, where $s' = s$ in the case E is of 'type $\check{E}$', and $s' = s + n_2$ where $n_2 = |\{j : a_j = 2 \text{ and } j'' < j \leqslant j'\}|$ (resp., $n_2 = |\{j : a_j = 2 \text{ and } j'' \leqslant j < j'\}|$) if $\tilde{\lambda} = (u, s, t)$ (resp., $\tilde{\lambda} = (u, t, s)$) in the case E is of 'type $\hat{E}$'. (Note that in the special case $s = t$ we have $n_2 = j' - j''$.)

Let $\hat{\alpha}_D = (\hat{\alpha}_1, \dots, \hat{\alpha}_{s'})$ be the determining tuple for diagram E. For the rest of the proof it will be convenient to consider the subcases $\tilde{\lambda} = (u, s, t)$ and $\tilde{\lambda} = (u, t, s)$ separately. Operations (C1)–(C5) discussed in Remark 4.10 will play a key role.

Subcase (II)(i): $\tilde{\lambda} = (u, s, t)$: Applying operations of type (C1) to the columns in the region of E which lies to the left of the (unique) column of length 4 in E, and also operations of types (C2) and (C3) to the columns in the region of E which lies to the right of the column of length 4, we see that w_E is a prefix of $w_{K'}$ for some diagram $K' \in \mathcal{D}^{(\lambda)}$ satisfying Hypothesis (†) which can be described as follows: Diagram K' has s' columns (same number of columns as E) and its determining tuple $\hat{\alpha}_{K'}$ begins with an ε'-tuple of 2's, followed by an η'-tuple of 1's, then has a single 4 and, following the 4, it has a θ'-tuple of $\bar{1}$'s ($\theta' = 0$ if E is of 'type $\check{E}$'), followed by a ζ'-tuple of 2's, followed by a ψ'-tuple of 1's and 3's containing exactly $(u-1)$ 3's. From the construction, we see that $\eta' \geqslant \theta'$. [This is obvious if E is of 'type $\check{E}$' since $\theta' = 0$ in this case. If E is of 'type $\hat{E}$', observe that $\eta' \geqslant j' - j'' \geqslant \theta'$, since by Lemma 4.4 none of the b_j's for $j'' \leqslant j < j'$ is empty, whereas we could possibly have some empty c_j's for $j'' < j \leqslant j'$.] If $\eta' = \theta'$ or $\zeta' = 0$, we set $K = K'$. If $\eta' > \theta'$ and $\zeta' > 0$, we apply operation (C5) to the first column of length 2 occurring from the left in the block of ζ' columns of length 2, and then by repeated applications of operation (C1) we can 'carry' the column of length 1 (with a single node on row 2) which has resulted from the application of (C5), to the position immediately to the left of the block of ψ' columns of length 1 or 3. Next, we set $\zeta'(0) = \zeta'$, $\theta'(0) = \theta'$ and, for $k \geqslant 1$, $\zeta'(k) = \zeta'(k-1) - 1$ and $\theta'(k) = \theta'(k-1) + 1$, where k is the number of repetitions of the above routine. The process stops after r repetitions of the routine, where r is the smallest integer such that either $\zeta'(r) = 0$ or $\eta' = \theta'(r)$ and we let K be the diagram obtained from K' at this stage of the process. Clearly K has $s' + r$ columns and $\zeta'(r) = 0$ if $\eta' > \theta'(r)$. Moreover K satisfies Hypothesis (†) and d is a prefix of w_K (see Remark 4.10). Comparing with Example 4.11(i) we see that $K = M^{(\mathcal{S})}$ for some permitted tuple $\mathcal{S} = (\varepsilon, \eta, \theta, \zeta, \psi, \mathcal{C})$ with $\eta \geqslant \theta$ and the further constraint $\zeta = 0$ if $\eta > \theta$. (In fact we have $\varepsilon = \varepsilon'$, $\eta = \eta'$, $\theta = \theta'(r) = \theta' + r$, $\zeta = \zeta'(r) = \zeta' - r$, $\psi = \psi' + r$.) Also recall from Example 4.11(i) that for such tuples $\mathcal{S}$, diagram $M^{(\mathcal{S})}$ is an admissible diagram and, in addition, $M^{(\mathcal{S})}$ is special if, and only if, $\theta = 0$. It now follows from Lemma 4.12 that $\mathcal{E}^{(\lambda)}$ is precisely the set of diagrams $M^{(\mathcal{S})}$ where the tuple $\mathcal{S}$ of non-negative integers satisfies the above constraints $\eta \geqslant \theta$ and $\zeta = 0$ if $\eta > \theta$. (Clearly $\mathcal{E}_s^{(\lambda)}$ is the subset of $\mathcal{E}^{(\lambda)}$ obtained by imposing the further restriction $\theta = 0$.)

Counting nodes on the second and third rows, we get $s = \varepsilon + \eta + 1 + \zeta + \psi$ and $t = \varepsilon + \theta + \zeta + u$. So, $\theta + \zeta \leqslant t - u$ and $\psi = s - t - (u-1) - \bar{\eta}$, where $0 \leqslant \bar{\eta} = \eta - \theta \leqslant s - t$. Thus, by setting $v = s - t + u$, we see that for $M^{(\mathcal{S})}$ to belong to $\mathcal{E}^{(\lambda)} - \mathcal{E}_s^{(\lambda)}$ (resp., for $M^{(\mathcal{S})}$ to belong to $\mathcal{E}_s^{(\lambda)}$), the number of permitted determining tuples with $\zeta \geqslant 1$ is

$$\sum_{\theta=1}^{t-u-1} \sum_{\zeta=1}^{t-u-\theta} \binom{v-1}{u-1} = \binom{t-u}{2}\binom{v-1}{u-1} \quad \Big(\text{resp.}, \sum_{\zeta=1}^{t-u} \binom{v-1}{u-1} = (t-u)\binom{v-1}{u-1}\Big)$$

and the number of permitted determining tuples with $\zeta = 0$ is

$$\sum_{\bar{\eta}=0}^{s-t} \sum_{\theta=1}^{t-u} \binom{v-1-\bar{\eta}}{u-1} = (t-u)\binom{v}{u} \quad \Big(\text{resp.}, \sum_{\bar{\eta}=0}^{s-t} \binom{v-1-\eta}{u-1} = \binom{v}{u}\Big).$$

Thus, we have determined $|\mathcal{E}_s(\lambda)|$ and $|\mathcal{E}(\lambda) - \mathcal{E}_s(\lambda)|$ to be the values given in Tables 3 and 4 and the corresponding diagrams take the form $M^{(\mathcal{S})}$ where $\mathcal{S} = (\varepsilon, \eta, \theta, \zeta, \psi, \mathcal{C})$ and $\mathcal{C}$ denotes an arbitrary set of $u-1$ columns among the last ψ columns.

Subcase (II)(ii): $\tilde{\lambda} = (u, t, s)$: We use similar arguments as for subcase (II)(i) but this time we begin by first applying a sequence of operations from types (C2), (C3) and (C4) to the columns lying in the region of E which is to the right of the column of length 4 and a sequence of operations of type (C3) to the columns of E lying in the region to the left of the column of length 4, in order to obtain a diagram $K' \in \mathcal{D}^{(\lambda)}$ with the following properties: Diagram K' satisfies Hypothesis (†), the K'-tableau $t^{K'}d$ is standard, K' has exactly s' columns and the determining tuple $\hat{\alpha}_{K'}$ of K' begins with an η'-tuple of $\bar{1}$'s, followed by an ε'-tuple of 2's, followed by a θ'-tuple of 1's, followed by a single 4, and following the 4, a φ'-tuple of $\bar{1}$'s followed by a ζ'-tuple of 2's, followed by a $(u-1)$-tuple of 3's. From the construction of E and the types of operation used to obtain K' from E, we also see that $\varphi' \geqslant j' - j'' \geqslant \theta'$, if E is 'of type $\hat{E}$'. In the case E is 'of type $\check{E}$', the relation $\varphi' \geqslant \theta'$ holds trivially since $\theta' = 0$ in this case.

Finally, in a similar fashion as in case (II)(i), now applying operations of types (C5) and (C3) to the columns of K' corresponding to the ε'-tuple of 2's which lie in the region to the left of the column of length 4 (but working from right to the left on this block of columns) we obtain a diagram K from K' with $w_{K'}$ a prefix of w_K (hence with d a prefix of w_K) such that $K = N^{(\mathcal{S})}$ for some tuple $\mathcal{S} = (\eta, \varepsilon, \theta, \varphi, \zeta)$ of non-negative integers satisfying the further constraints $\varphi \geqslant \theta$ and $\varepsilon = 0$ if $\varphi > \theta$. (Compare with Example 4.11(ii).) Lemma 4.13 now ensures that the set $\mathcal{E}^{(\lambda)}$ is precisely the set of diagrams $N^{(\mathcal{S})}$ for which the conditions $\varphi \geqslant \theta$ and $\varepsilon > 0$ if $\varphi > \theta$ are satisfied by $\mathcal{S}$. For the subset $\mathcal{E}_s^{(\lambda)}$ we need the further restriction $\theta = 0$ since from Example 4.11(ii) we know that $N^{(\mathcal{S})}$ is special if, and only if, $\theta = 0$.

Counting nodes on the second and third rows, we get $t = \varepsilon + \theta + \zeta + u$ and $s = \eta + \varepsilon + \varphi + \zeta + u$. So, $\theta + \zeta \leqslant t - u$ and $\bar{\varphi} = s - t - \eta$ where $\bar{\varphi} = \varphi - \theta$. Given θ and $\bar{\varphi}$ with $0 \leqslant \theta \leqslant t - u$ and $0 \leqslant \bar{\varphi} \leqslant s - t$, the quantities η, φ and $\varepsilon + \zeta$ are determined. If additionally, $\bar{\varphi} > 0$ then $\varepsilon = 0$, so ζ is determined, whereas if $\bar{\varphi} = 0$ then $0 \leqslant \varepsilon \leqslant t - u - \theta$. Thus, for $N^{(\mathcal{S})}$ to belong to $\mathcal{E}^{(\lambda)} - \mathcal{E}_s^{(\lambda)}$ (resp., for $N^{(\mathcal{S})}$ to belong to $\mathcal{E}_s^{(\lambda)}$) the number of permitted determining tuples with $\bar{\varphi} \geqslant 1$ is

$$\sum_{\theta=1}^{t-u} \sum_{\bar{\varphi}=1}^{s-t} 1 = (t-u)(s-t) \qquad \Big(\text{resp., } \sum_{\varphi=1}^{s-t} 1 = s-t\Big)$$

and the number of permitted deternining tuples with $\bar{\varphi} = 0$ is

$$\sum_{\theta=1}^{t-u} \sum_{\varepsilon=0}^{t-u-\theta} 1 = \binom{t-u+1}{2} \qquad \Big(\text{resp., } \sum_{\varepsilon=0}^{t-u} 1 = t-u+1\Big).$$

Thus, we have determined $|\mathcal{E}_s(\lambda)|$ and $|\mathcal{E}(\lambda) - \mathcal{E}_s(\lambda)|$ to be the values given in Tables 3 and 4 and the corresponding diagrams take the form $N^{(\mathcal{S})}$ where $\mathcal{S} = (\eta, \varepsilon, \theta, \varphi, \zeta)$. □

TABLE 3. Values of $|\mathcal{E}_s^{(\lambda)}|$, $s \geqslant t \geqslant u$, $r > 3$. (Theorem 5.1)

$\tilde{\lambda}$	(s,t,u)	(s,u,t)	(t,s,u)	(t,u,s)	(u,s,t)	(u,t,s)
	1	$\binom{t}{u}$	$\binom{s-t+u}{u}$	$(s-t)\binom{t-1}{u-1} + \binom{t}{u}$	$(t-u)\binom{s-t+u-1}{u-1} + \binom{s-t+u}{u}$	$s-u+1$

TABLE 4. Values of $|\mathcal{E}^{(\lambda)} - \mathcal{E}_s^{(\lambda)}|$, $s \geqslant t \geqslant u$, $r > 3$. (Theorem 5.1)

$\tilde{\lambda}$	(u,s,t)	(u,t,s)
	$\binom{t-u}{2}\binom{s-t+u-1}{u-1} + (t-u)\binom{s-t+u}{u}$	$(t-u)(s-t) + \binom{t-u+1}{2}$.

Remark 5.2. *(i) In view of Result 8, we immediately get from Theorem 5.1 (and Tables 3 and 4) all the corresponding information about the set $\mathcal{E}^{(\mu)}$ where the composition μ has the form $(1^r, \mu_1, \mu_2, \mu_3)$.*

(ii) Let λ be a composition of n with r parts. Recall that $Z(\lambda)$ is a right ideal in $S_n = \langle s_1, \ldots, s_{n-1}\rangle$ and that $Z(\lambda)\hat{\mathfrak{X}}$ is a right ideal in $S_{n+1} = \langle s_1, \ldots, s_n\rangle$, where s_i is the basic transposition $(i,\ i{+}1)$, and $\hat{\mathfrak{X}}$ denotes the set of distinguished right coset representatives of S_n in S_{n+1}. The longest element of $\hat{\mathfrak{X}}$ is the element $s_n s_{n-1} \cdots s_1$ $(= (1,\ 2, \ldots, n{+}1)$ in cycle-notation). Given $D \in \mathcal{E}^{(\lambda)}$, we define $\hat{D} = \{(r{+}1, 1)\} \cup \{(i, j+1) \colon (i,j) \in D\}$, so $\hat{D}$ is a diagram of size $n+1$. By Proposition 2.10, the minimal determining set of the right ideal $Z(\lambda)\hat{\mathfrak{X}}$ in S_{n+1} is the set $\{w_{\hat{D}} \colon D \in \mathcal{E}^{(\lambda)}\}$. (Comparing with the discussion in Section 3, we can consider this set to be the rim of the induced union of cells $w_J Z(\lambda)\hat{\mathfrak{X}}$.) Thus, the explicit results in [14], [15] and also in Section 5 of the present paper on the minimal determining sets of various families of right ideals of the form $Z(\lambda)$ lead to an explicit description of the minimal determining sets of the corresponding induced right ideals $Z(\lambda)\hat{\mathfrak{X}}$.

(iii) The results of this paper together with the results of [14] *and* [15] *give complete information about the set* $\mathcal{E}^{(\lambda)}$ *for all compositions* λ *of* n *for* $n \leqslant 6$. *Using similar methods we have also completed the case* $n = 7$. *More detailed information about the sizes of the rims of the corresponding cells in the form of tables can be obtained from any of the authors on request.*

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http://dx.doi.org/10.22108/IJGT.2022.131838.1764

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http://dx.doi.org/10.22108/IJGT.2022.131838.1764