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An asymptotic method of factorization of a class of matrix functions

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A novel method of asymptotic factorization of $n \times n$ matrix functions is proposed. The considered class of matrices is motivated by certain problems originated from the elasticity theory. An example is constructed to illustrate efficiency of the proposed procedure. The quality of approximation and the role of the chosen small parameter are discussed.

1. Introduction

We consider here the problem of factorization of continuous matrix functions of the real variable. This means the representation of a given invertible square matrix $G \in (\mathcal{C}(\mathbb{R}))^{n \times n}$ in the following form:

$$G(x) = G^-(x)A(x)G^+(x), \quad (1.1)$$

where continuous invertible matrices $G^-(x)$ and $G^+(x)$ possess an analytic continuation in the lower $\Pi^- = \{z = x + iy : \text{Im} < 0\}$ and upper $\Pi^+ = \{z = x + iy : \text{Im} > 0\}$ half-planes, respectively, and

$$A(x) = \text{diag} \left(\left(\frac{x-i}{x+i} \right)^{\kappa_1}, \dots, \left(\frac{x-i}{x+i} \right)^{\kappa_n} \right), \quad \kappa_1, \dots, \kappa_n \in \mathbb{Z}. \quad (1.2)$$

The representation (1.1) is called *right (continuous or standard) factorization* and can be considered for any oriented curve Γ of a certain classes which divides the complex plane into two domain D^- and D^+ with a changing of diagonal entries in $A(x)$ for $((x-t^+)/(x-t^-))^{\kappa_j}$, $t^\mp \in D^\mp$, or for x^{κ_j} (if $0 \in D^+$). A similar representation

$$G(x) = G^+(x)A(x)G^-(x)$$

is called *left (continuous or standard) factorization*. If the right- (left-) factorization exists, then the integer numbers

$\kappa_1, \dots, \kappa_n$, called *partial indices*, are determined uniquely up to the order. In particular, there exist constant transformations of factors, such that $\kappa_1 \geq \dots \geq \kappa_n$. The factors G^- and G^+ are not unique. Relations of pairs of factors are described, for example, in [1]. The right- (left-) factorization is called *canonical factorization* if all partial indices are equal to 0, i.e. $\kappa_1 = \dots = \kappa_n = 0$.

Factorization of matrix functions was first studied in relation to the vector–matrix Riemann (or Riemann–Hilbert) boundary value problem [2]; this was formulated by Riemann in his work on construction of complex differential equations with algebraic coefficients having a prescribed monodromy group [3] (see also [4]). By using the method of the Cauchy type integral, the vector–matrix Riemann boundary value problem was reduced in [5,6] to a system of the Fredholm integral equations. A part of the theory of the factorization problem is based on the study of such systems (see also [7]), though this approach does not answer, in particular, the questions of when it is possible to get factorization, how to construct factors and how to determine partial indices.

Among other sources of interest to the factorization problems, one can point out the vector-valued Wiener–Hopf equations on a half-line [8–11] and their discrete analogies, namely the block Toeplitz equations [12,13]. The developed technique found several applications in diffraction theory, fracture mechanics, geophysics, financial mathematics, etc. (see a brief description given, for example, in [14] and references therein).

Theoretical background for the study of the matrix factorization and its numerous generalizations is presented in [1,3,6,15–17] (see also [18]).

The theory of the factorization is more or less complete [15], but the above-mentioned constructive questions about existence, factors and partial indices (which are very important for practical applications) have been answered only in a number of special cases. Among them, one can mention rational matrix functions [7], functional commutative matrix functions (those satisfying $G(t)G(s) = G(s)G(t), \forall t, s \in \Gamma$ [19]), upper- (lower-) triangular matrices with factorizable diagonal elements [20–23], certain classes of meromorphic matrix functions [24–26], special cases of 2×2 Daniele–Khrapkov matrix functions (with a small degree of deviator polynomial) [27–30], special cases of 2×2 matrix functions with three rationally independent entries (see [28,31,32] and references therein), special cases of $n \times n$ generalization of the Daniele–Khrapkov matrix functions [14,33–35] and special classes of matrices possessing certain symmetry property (see [36] and references therein). Several approximate and asymptotic methods for matrix factorization have been developed too [37–39]. In particular, in Abrahams [40], the method of Padé approximation has been developed and applied (see also [18] and references therein).

In this paper, we propose a new asymptotic method of construction of factors for a special class of $n \times n$ non-rational matrix functions. The essential property of the considered matrices is that they become close (after suitable transformation) to a unit matrix (a similar assumption is used in [31,41]). The idea to use such representation in factorization is similar to that in general operator theory and has been exploited since the seminal work by Gohberg & Krein [8] (see also [1,42,43]).

To the best of authors' knowledge, our class does not coincide with any of the above-mentioned classes. This class contains matrix functions which appears in the study of certain problems in fracture mechanics related to perturbation of the crack propagation [44–47]. Another motivation is the use of such matrix functions in the study, the inverse scattering problem [41]. In [41], it is considered the generalized factorization of 2×2 matrix functions which are similar to that in our special case in §3*b*. An analysis of these authors is based on the equivalence theorem, relating the studied matrices with a product of triangular matrices, followed by solution of the corresponding Riemann boundary value problems. The conditions required to realize this method, as well as the used techniques differ essentially from ours. The main idea of our study is to reduce determination of factors at each step of approximation to the solution of a so-called vector–matrix jump boundary value problem. The paper is organized as follows. In §2, we introduce necessary notation and formulate the problem. A constructive algorithm is presented in §3. We also find it interesting to present here the realization of the algorithm in a special case of matrices of practical importance. The method is illustrated by an example given in §4. We conclude our study by

showing the quality of the factorization approximation by restricting ourselves only to the first asymptotic term and discuss the role of the chosen small parameter.

2. A class of matrices: problem formulation

Let us introduce the following class of invertible continuous $n \times n$, $n \geq 2$, matrix functions $\mathcal{G}K_n$ depending on a real parameter $\varphi \in \mathbb{R}$, satisfying the following conditions:

- (1) $G_\varphi \in (C(\mathbb{R}))^{n \times n}$ belongs to $\mathcal{G}K_n$ if it can be represented in the form

$$G_\varphi = R_\varphi F R_\varphi^{-1}, \quad (2.1)$$

where bounded locally Hölder-continuous on \mathbb{R} (in general non-rational) invertible matrix R_φ is such that

- (2) $R_0 = R_\varphi|_{\varphi=0} = I$, (2.2)

- (3) matrix function F does not depend on parameter φ , has Hölder-continuous entries f_{kl} on the extended real line $\bar{\mathbb{R}}$, i.e. $\forall k, l = 1, \dots, n$,

$$|f_{kl}(x_1) - f_{kl}(x_2)| \leq C \left| \frac{1}{x_1 + i} - \frac{1}{x_2 + i} \right|^\mu, \quad \forall x_1, x_2 \in \bar{\mathbb{R}}, \quad 0 < \mu < 1, \quad (2.3)$$

satisfies the following asymptotic estimate at infinity

- (4) $F(x) \rightarrow I$ and $|x| \rightarrow \infty$, (2.4)

- (5) F admits a right canonical factorization, i.e.

$$F(x) = F^-(x)F^+(x), \quad (2.5)$$

where Hölder-continuous on $\bar{\mathbb{R}}$ matrix functions $F^-(x)$ and $F^+(x)$ possess an analytic continuation in the lower Π^- and the upper Π^+ half-plane, respectively.

The matrices of the following form constitute a simple subclass of class $\mathcal{G}K_2$:

$$G_\varphi(x) = \begin{pmatrix} p(x) & q(x) e^{i\varphi x} \\ q(x) e^{-i\varphi x} & p(x) \end{pmatrix}. \quad (2.6)$$

This function appears after Fourier transforms of the Wiener–Hopf equation, describing a problem in fracture mechanics.

We note that the matrix functions of this type do not belong to any known class of matrix functions which admit explicit factorization.

It can be seen below that in our algorithm, the factors retain one of the properties of the matrix function G_φ , namely,

$$G_\varphi^-(z), \quad G_\varphi^+(z) \rightarrow I, \quad \text{as } z \rightarrow \infty, \quad \mp \text{Im } z > 0. \quad (2.7)$$

3. An algorithm

(a) General construction

By assumption, any matrix $G_\varphi(x) \in \mathcal{G}K_n$ can be written in the form

$$G_\varphi(x) = F^-(x)(F^-(x))^{-1}G_\varphi(F^+(x))^{-1}F^+(x) = F^-(x)G_{1,\varphi}F^+(x), \quad (3.1)$$

where $F^-(x)$ and $F^+(x)$ are components of the canonical factorization of the corresponding matrix F , and (see (2.2) and (2.4)) the matrix $G_{1,\varphi}(x)$ is represented in the form

$$G_{1,\varphi}(x) = (F^-(x))^{-1}G_\varphi(F^+(x))^{-1} = (F^-(x))^{-1}R_\varphi F R_\varphi^{-1}(F^+(x))^{-1}.$$

In addition to (1)–(5), we assume that

- (6) There exist a small parameter $\varepsilon = \varepsilon(\varphi)$ (more exactly its value will be described later), such that for all $x \in \mathbb{R}$ and any finite φ

$$G_{1,\varphi}(x) = I + \varepsilon N_\varphi(x), \quad (3.2)$$

and matrix $N_\varphi(x)$ is bounded and locally Hölder continuous on \mathbb{R} .

Note that by assumption, each entry of the matrix $N_\varphi(x)$ has a limit as $|x| \rightarrow +\infty$, i.e. there exists the value $N_\varphi(\infty)$. Note also that the commutativity of the involved matrices is not assumed.

Let us look for the first-order factorization of the matrix $G_{1,\varphi}(x)$ in the form

$$G_{1,\varphi}(x) = I + \varepsilon N_\varphi(x) = (I + \varepsilon N_{1,\varphi}^-(x))(I + \varepsilon N_{1,\varphi}^+(x)). \quad (3.3)$$

Comparing terms for different powers of ε we get, in particular, the following relation for determination of factors $N_{1,\varphi}^-$ and $N_{1,\varphi}^+$:

$$N_{1,\varphi}^-(x) + N_{1,\varphi}^+(x) = N_\varphi(x), \quad x \in \mathbb{R}. \quad (3.4)$$

It is customary to denote

$$M_{0,\varphi}(x) \equiv N_\varphi(x).$$

The jump boundary value problem (3.4) has a solution represented in terms of a slight modification of the matrix-valued Cauchy type integral [1,5]

$$N_{1,\varphi}^\mp(z) = \frac{1}{2}M_{0,\varphi}(\infty) \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{M_{0,\varphi}(t) - M_{0,\varphi}(\infty)}{t - z} dt \equiv \frac{1}{2}M_{0,\varphi}(\infty) \mp (\mathbf{C}_0 M_{0,\varphi})(z). \quad (3.5)$$

The above modification is proposed in order to avoid extra discussion of the convergence of the above integrals. In this form, the integrals are convergent automatically. Moreover, its boundary values

$$(\mathbf{C}_0 M_{0,\varphi})^\mp(x) = \lim_{\text{Im } z \rightarrow \mp 0} (\mathbf{C}_0 M_{0,\varphi})(z)$$

satisfy Sokhotsky–Plemelj formulae, i.e.

$$\begin{aligned} (\mathbf{C}_0 M_{0,\varphi})^\mp(x) &= \frac{1}{2}M_{0,\varphi}(x) - \frac{1}{2}M_{0,\varphi}(\infty) \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{M_{0,\varphi}(t) - M_{0,\varphi}(\infty)}{t - x} dt \\ &= \frac{1}{2}M_{0,\varphi}(x) - \frac{1}{2}M_{0,\varphi}(\infty) \mp \frac{1}{2}(\mathbf{S}_0 M_{0,\varphi})(x) \end{aligned} \quad (3.6)$$

or

$$N_{1,\varphi}^\mp(x) = \frac{1}{2}M_{0,\varphi}(x) \mp \frac{1}{2}(\mathbf{S}_0 M_{0,\varphi})(x), \quad (3.7)$$

where \mathbf{S}_0 is the singular integral operator along the real line with density $M_{0,\varphi}(t) - M_{0,\varphi}(\infty)$. It follows from [2, n. 4.6] that both matrices $N_{1,\varphi}^-(x)$ and $N_{1,\varphi}^+(x)$ satisfy Hölder conditions on \mathbb{R} , are bounded there with $N_{1,\varphi}^\mp(\infty) = \frac{1}{2}M_{0,\varphi}(\infty)$ and possess an analytic continuation into lower Π^- and upper Π^+ half-planes, respectively. Surely, its product is also Hölder conditions on \mathbb{R} and bounded.

Let us refine the factorization of the matrix $G_{1,\varphi}(x)$, i.e. look for a presentation of $G_{1,\varphi}(x)$ in the form

$$G_{1,\varphi}(x) = I + \varepsilon N_\varphi(x) = (I + \varepsilon N_{1,\varphi}^-(x) + \varepsilon^2 N_{2,\varphi}^-(x))(I + \varepsilon N_{1,\varphi}^+(x) + \varepsilon^2 N_{2,\varphi}^-(x)), \quad (3.8)$$

where $N_{1,\varphi}^-(x)$ and $N_{1,\varphi}^+(x)$ are those found at the previous step.

Comparing terms at different powers of ε , we get, in particular, the following relation for determination of factors $N_{2,\varphi}^-$ and $N_{2,\varphi}^+$:

$$N_{1,\varphi}^-(x)N_{1,\varphi}^+(x) + N_{2,\varphi}^-(x) + N_{2,\varphi}^+(x) = 0. \quad (3.9)$$

Denoting

$$M_{1,\varphi}(x) \equiv -N_{1,\varphi}^-(x)N_{1,\varphi}^+(x),$$

we arrive at the following jump boundary value problem

$$N_{2,\varphi}^-(x) + N_{2,\varphi}^+(x) = M_{1,\varphi}(x), \quad x \in \mathbb{R}, \quad (3.10)$$

where the right-hand side is already known. The solution to this problem is given by the formula similar to (3.5)

$$N_{2,\varphi}^\mp(z) = \frac{1}{2}M_{1,\varphi}(\infty) \mp (\mathbf{C}_0 M_{1,\varphi})(z). \quad (3.11)$$

It has the same properties as the solution of (3.5), in particular, its boundary values satisfy the relation

$$N_{2,\varphi}^\mp(x) = \frac{1}{2}M_{1,\varphi}(x) \mp \frac{1}{2}(\mathbf{S}_0 M_{1,\varphi})(x) \quad (3.12)$$

and $N_{2,\varphi}^\mp(\infty) = -\frac{1}{8}N_{2,\varphi}^2(\infty)$.

One can proceed in the same manner. Thus on the k th step, we use the representation

$$\begin{aligned} G_{1,\varphi}(x) &= I + \varepsilon N_\varphi(x) = (I + \varepsilon N_{1,\varphi}^-(x) + \dots + \varepsilon^k N_{k,\varphi}^-(x)) \\ &\quad \times (I + \varepsilon N_{1,\varphi}^+(x) + \dots + \varepsilon^k N_{k,\varphi}^+(x)), \end{aligned} \quad (3.13)$$

where $N_{1,\varphi}^-(x), \dots, N_{k-1,\varphi}^-(x)$ and $N_{1,\varphi}^+(x), \dots, N_{k-1,\varphi}^+(x)$ are found in the previous steps. This leads to the jump boundary value problem

$$N_{k,\varphi}^-(x) + N_{k,\varphi}^+(x) = M_{k-1,\varphi}, \quad x \in \mathbb{R}, \quad (3.14)$$

where

$$M_{k-1,\varphi} = -[N_{1,\varphi}^-(x)N_{k-1,\varphi}^+(x) + N_{2,\varphi}^-(x)N_{k-2,\varphi}^+(x) + \dots + N_{k-1,\varphi}^-(x)N_{1,\varphi}^+(x)].$$

The solution to this problem is given by a formula similar to (3.5) (or to (3.11)).

Thus, the factorization of the matrix function $G_{1,\varphi}(x)$ is given in the form of an asymptotic series

$$G_{1,\varphi}(x) = \left(I + \sum_{k=1}^{\infty} \varepsilon^k N_{k,\varphi}^-(x) \right) \left(I + \sum_{k=1}^{\infty} \varepsilon^k N_{k,\varphi}^+(x) \right), \quad (3.15)$$

where the pair $N_{k,\varphi}^-(x)$ and $N_{k,\varphi}^+(x)$ is the unique solution to the jump problem (3.14) for any $k \in \mathbb{N}$.

The following theorem gives conditions when this asymptotic factorization becomes an explicit one, i.e. gives convergence conditions for the asymptotic series involved.

Theorem 3.1. Let G_φ be a matrix which meets conditions (1) to (6). Let the parameter ε (defined in (6)) satisfies the inequality

$$|\varepsilon| \leq \frac{1}{A} \quad (3.16)$$

with the constant $A = A(\varphi)$ being equal to

$$A = \|N_\varphi(\cdot)\|_\mu (1 + C_\mu)^2, \quad (3.17)$$

$\|N_\varphi(\cdot)\|_\mu$ being the norm of the matrix function $N_\varphi(x)$ in the Hölder space H_μ equal to the maximum of the norms of its entries, and C_μ being the norm of the singular integral operator $S_0: H_\mu \rightarrow H_\mu$.

Then both series in the right-hand side of (3.15) converge for all $x \in \mathbb{R}$.

Proof. It follows (e.g. [2, p. 48]) that singular integral operator S_0 is bounded in Hölder spaces since the 'standard' singular integral operator S is. The later is well-known, see [48] for the exact value of the norm of $\|S\|_{H_\mu \rightarrow H_\mu}$. Let us denote the norm of S_0 in Hölder space $H_\mu(\mathbb{R})$ by C_μ , i.e.

$$C_\mu = \|S_0\|_{H_\mu \rightarrow H_\mu}.$$

Then, we have the following series of estimates

$$\|N_{1,\varphi}^\mp(\cdot)\|_\mu \leq \alpha_1 \|N_\varphi(\cdot)\|_\mu (1 + C_\mu), \quad \text{where } \alpha_1 = \frac{1}{2},$$

$$\|N_{2,\varphi}^\mp(\cdot)\|_\mu \leq \frac{1}{2} \|M_{1,\varphi}(\cdot)\|_\mu (1 + C_\mu)$$

and

$$\|M_{1,\varphi}(\cdot)\|_\mu \leq (\alpha_1 \|N_\varphi(\cdot)\|_\mu (1 + C_\mu))^2,$$

i.e.

$$\|N_{2,\varphi}^\mp(\cdot)\|_\mu \leq \alpha_2 \|N_\varphi(\cdot)\|_\mu^2 (1 + C_\mu)^3, \quad \text{where } \alpha_2 = \frac{1}{2} \alpha_1^2.$$

Finally, for each $k \geq 2$

$$\|N_{k,\varphi}^\mp(\cdot)\|_\mu \leq \frac{1}{2} \|M_{k-1,\varphi}(\cdot)\|_\mu (1 + C_\mu)$$

and

$$\|M_{k-1,\varphi}(\cdot)\|_\mu \leq (\alpha_1 \alpha_{k-1} + \alpha_2 \alpha_{k-2} + \dots + \alpha_{k-1} \alpha_1) \|N_\varphi(\cdot)\|_\mu^k (1 + C_\mu)^{2k-2},$$

i.e.

$$\|N_{k,\varphi}^\mp(\cdot)\|_\mu \leq \alpha_k \|N_\varphi(\cdot)\|_\mu^k (1 + C_\mu)^{2k-1}, \quad \text{where } \alpha_k = \frac{1}{2} (\alpha_1 \alpha_{k-1} + \dots + \alpha_{k-1} \alpha_1).$$

We can calculate explicitly few first coefficients α_k , namely, $\alpha_1 = 1/2$, $\alpha_2 = 1/8$ and $\alpha_3 = 1/16$. As for coefficients with large enough indices, we can proof by induction that

$$\alpha_k < \frac{1}{16(k-3)}, \quad \forall k \geq 12.$$

Therefore,

$$\|\varepsilon^k N_{k,\varphi}^\mp(\cdot)\|_\mu \leq |\varepsilon|^k \frac{1}{32(k-3)(1+C_\mu)} (\|N_\varphi(\cdot)\|_\mu (1+C_\mu)^2)^k, \quad \forall k \geq 12.$$

As the sequence $\sqrt[k]{1/(32(k-3)(1+C_\mu))} \leq 1$ is increasing for sufficiently large k and

$$\lim_{k \rightarrow \infty} \sqrt[k]{\frac{1}{32(k-3)(1+C_\mu)}} = 1,$$

then the convergence of the series

$$\left(I + \sum_{k=1}^{\infty} \varepsilon^k N_{k,\varphi}^-(x) \right) \quad \text{and} \quad \left(I + \sum_{k=1}^{\infty} \varepsilon^k N_{k,\varphi}^+(x) \right)$$

for all $x \in \mathbb{R}$ follows from (3.16). ■

Remark 3.2. It follows from the standard properties of the Cauchy type integral and singular integral with Cauchy kernel that conditions of theorem 3.1 guarantee convergence of the series in the right-hand side of (3.15) in the half-planes Π^- , Π^+ , respectively.

Remark 3.3. In fact, the decay of the second term in the right-hand side of (3.2) at infinity follows from the asymptotic relations (2.7) which in turn follows from the properties of matrices of the considered class and the proposed construction.

Remark 3.4. If the number $A = A(\varphi)$ in theorem 3.1 is small enough, i.e.

$$A = \|N_\varphi(\cdot)\|_\mu (1 + C_\mu)^2 < 1, \tag{3.18}$$

then the results remain valid for $\varepsilon=1$ and the described procedure will work without any changes.

(b) Special case

Let us consider the problem of factorization of 2×2 invertible matrices from a subclass of $\mathcal{G}K_2$, namely

$$G_\varphi(x) = \begin{pmatrix} p(x) & q(x) e^{ix\varphi} \\ q(x) e^{-ix\varphi} & p(x) \end{pmatrix}, \quad (3.19)$$

given on the real line ($x \in \mathbb{R}$) and depending on the real parameter $\varphi \in \mathbb{R}$.

We assume that the following assumptions hold:

- (1) the entries $p(x)$ and $q(x)$ are real-valued Hölder continuous functions on $\bar{\mathbb{R}}$, i.e. $p, q \in H_\mu(\bar{\mathbb{R}})$;
- (2) the combinations of the functions p and q are positive

$$p(x) \pm q(x) > 0, \quad x \in \mathbb{R}; \quad \text{and} \quad (3.20)$$

- (3) the following limits exist

$$\lim_{|x| \rightarrow +\infty} p(x) = 1 \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} q(x) = 0. \quad (3.21)$$

As in the general case, the factorization of matrices of type (3.19) is motivated by certain problem of fracture mechanics. The considered matrices are similar to those which are studied and explicitly factorized in [31,41,49], but certain conditions of the above-cited papers are not satisfied in our case.

Note that even if one supposes that the functions $p(x)$ and $q(x)$ are meromorphically continued into semi-planes Π^- and Π^+ , then it does not mean that these extended functions have a finite number of zeroes and poles there.

Remark 3.5. Under conditions (1)–(3), matrix (3.19) admits the canonical factorization

$$G_\varphi(x) = G_\varphi^-(x) G_\varphi^+(x). \quad (3.22)$$

By (3.20), matrix (3.19) is positive definite and thus admits the canonical factorization [1].

Let us start with factorization of an auxiliary matrix $F(x) = G_0(x)$ having no exponential term in their entries

$$F(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & p(x) \end{pmatrix}. \quad (3.23)$$

Note that

$$F(x) = P \begin{pmatrix} p(x) + q(x) & 0 \\ 0 & p(x) - q(x) \end{pmatrix} P, \quad (3.24)$$

where the projector P is defined

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

It follows from conditions (1)–(3) that both diagonal elements of the middle diagonal matrix have index equal to zero. Hence, they admit the representation

$$\left. \begin{aligned} p(x) + q(x) &= (p(x) + q(x))^- \cdot (p(x) + q(x))^+ \\ p(x) - q(x) &= (p(x) - q(x))^- \cdot (p(x) - q(x))^+, \end{aligned} \right\} \quad (3.25)$$

and

with the factors of the form [2],

$$\left. \begin{aligned} (p(z) + q(z))^\mp &= \exp \left\{ \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log [p(\tau) + q(\tau)]}{\tau - z} d\tau \right\}, \quad z \in \Pi^\mp \\ (p(z) - q(z))^\mp &= \exp \left\{ \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log [p(\tau) - q(\tau)]}{\tau - z} d\tau \right\}, \quad z \in \Pi^\mp, \end{aligned} \right\} \quad (3.26)$$

and

and boundary values $(p(x) + q(x))^{\mp}$ of the functions $(p(z) + q(z))^{\mp}$ (and $(p(x) - q(x))^{\mp}$ of the functions $(p(z) - q(z))^{\mp}$ are determined by using Sokhotsky–Plemelj formulae [2].

Therefore, we immediately obtain the right canonical factorization of the matrix $G_0(x)$

$$G_0(x) = F(x) = F^-(x)F^+(x), \quad (3.27)$$

where

$$F^-(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} (p(x) + q(x))^- & (p(x) - q(x))^- \\ (p(x) + q(x))^- & -(p(x) - q(x))^- \end{pmatrix} \quad (3.28)$$

and

$$F^+(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} (p(x) + q(x))^+ & (p(x) + q(x))^+ \\ (p(x) - q(x))^+ & -(p(x) - q(x))^+ \end{pmatrix}. \quad (3.29)$$

Now we can represent the initial matrix $G_\varphi(x)$ in the form

$$G_\varphi(x) = F^-(x)(F^-(x))^{-1}G_\varphi(x)(F^+(x))^{-1}F^+(x) \equiv F^-(x)G_{1,\varphi}(x)F^-(x) \quad (3.30)$$

and proceed with factorization of the matrix $G_{1,\varphi}(x)$.

The inverse matrices $(F^-(x))^{-1}$ and $(F^+(x))^{-1}$ are equal, respectively,

$$\left. \begin{aligned} (F^-(x))^{-1} &= \frac{1}{\Delta_-} \begin{pmatrix} (p(x) - q(x))^- & (p(x) - q(x))^- \\ (p(x) + q(x))^- & -(p(x) + q(x))^- \end{pmatrix} \\ \Delta_- &= \frac{2}{\sqrt{2}}(p(x) + q(x))^- (p(x) - q(x))^- \end{aligned} \right\} \quad (3.31)$$

and

and

$$\left. \begin{aligned} (F^+(x))^{-1} &= \frac{1}{\Delta_+} \begin{pmatrix} (p(x) - q(x))^+ & (p(x) + q(x))^+ \\ (p(x) - q(x))^+ & -(p(x) + q(x))^+ \end{pmatrix} \\ \Delta_+ &= \frac{2}{\sqrt{2}}(p(x) + q(x))^+ (p(x) - q(x))^+ \end{aligned} \right\} \quad (3.32)$$

and

Hence (for shortness, we omit the argument x for functions p and q),

$$\left. \begin{aligned} G_{1,\varphi} &= \frac{1}{\Delta} \begin{pmatrix} (p - q)(2p + q(e^{i\varphi k} + e^{-i\varphi k})) & (p - q)^-(p + q)^+ q(e^{-i\varphi k} - e^{i\varphi k}) \\ (p - q)^+(p + q)^- q(e^{i\varphi k} - e^{-i\varphi k}) & (p + q)(2p - q(e^{i\varphi k} - e^{-i\varphi k})) \end{pmatrix} \\ \Delta &= 2(p^2 - q^2). \end{aligned} \right\} \quad (3.33)$$

It is not hard to see that $G_{1,\varphi}$ is a sum of the unit matrix and the matrix which is ‘small’ for appropriate choice of φ . Hence, following remark 3.4, we rewrite the right-hand side of (3.33) as the following sum:

$$G_{1,\varphi} = I + N_\varphi, \quad (3.34)$$

where

$$N_\varphi = \begin{pmatrix} -\frac{2q \sin^2(\varphi x/2)}{(p + q)} & -\frac{iq(p - q)^-(p + q)^+ \sin \varphi x}{(p^2 - q^2)} \\ \frac{iq(p - q)^+(p + q)^- \sin \varphi x}{(p^2 - q^2)} & \frac{2q \sin^2(\varphi x/2)}{(p - q)} \end{pmatrix}. \quad (3.35)$$

Note that the latter matrix can be written as the sum of two diagonal matrices, namely

$$N_\varphi = N_1 + N_2 \quad (3.36)$$

$$N_1 = 2 \sin^2 \frac{\varphi x}{2} \begin{pmatrix} -\frac{q}{(p + q)} & 0 \\ 0 & \frac{q}{(p - q)} \end{pmatrix} \quad (3.37)$$

and

$$N_2 = i \sin \varphi x \begin{pmatrix} 0 & -\frac{q(p-q)^-(p+q)^+}{(p^2-q^2)} \\ \frac{q(p-q)^+(p+q)^-}{(p^2-q^2)} & 0 \end{pmatrix}. \quad (3.38)$$

Let us denote

$$\varepsilon_1 = \varepsilon_1(\varphi) \equiv \max_{x \in \mathbb{R}} \left| q(x) \sin \frac{\varphi x}{2} \right|. \quad (3.39)$$

Lemma 3.6. *The parameter ε_1 can be taken smaller than any positive number δ by an appropriate choice of φ .*

Proof. Indeed, taking into account condition (3.21)₂, one concludes that there exists $x_\delta > 0$ such that for any $|x| \geq x_\delta$

$$\left| q(x) \sin \frac{\varphi x}{2} \right| \leq |q(x)| \leq \delta.$$

On the other hand, as $xq(x)$ belongs to the space $H_\mu[-x_\delta, x_\delta] \subset C[-x_\delta, x_\delta]$ there exists a constant $q_h > 0$ such that

$$|xq(x)| \leq q_h, \quad \text{for all } |x| \leq x_\delta.$$

Finally, this means

$$\left| q(x) \sin \frac{\varphi x}{2} \right| \leq |q(x)| \frac{\varphi x}{2} \leq \frac{\varphi}{2} q_h.$$

Choosing $\varphi = 2\delta/q_h$, we finish the proof. ■

It follows from the structure of the matrix N_φ from (3.35) that parameter $A = A(\varphi)$ discussed in theorem 3.1 is smaller than 1 for an appropriate choice of φ . It guarantees applicability of the general procedure in this special case.

4. An example

We present here an example of 2×2 matrix function $G_\varphi \in \mathcal{G}K_2$ for which the above-discussed factorization does not involve Cauchy type integration for the components and auxiliary matrices.

Let,

$$G_\varphi(x) = \begin{pmatrix} \frac{x^2+10}{x^2+1} & \frac{6}{x^2+1} e^{ix\varphi} \\ \frac{6}{x^2+1} e^{-ix\varphi} & \frac{x^2+10}{x^2+1} \end{pmatrix}, \quad x \in \mathbb{R}. \quad (4.1)$$

It is a special case of the matrix functions discussed in §3b. Here, $p(x) = (x^2+10)/(x^2+1)$ and $q(x) = 6/(x^2+1)$, and an auxiliary matrix $F(x) = G_0(x)$ has the form

$$G_0(x) = \begin{pmatrix} \frac{x^2+10}{x^2+1} & \frac{6}{x^2+1} \\ \frac{6}{x^2+1} & \frac{x^2+10}{x^2+1} \end{pmatrix}, \quad x \in \mathbb{R}. \quad (4.2)$$

The above matrix satisfies all conditions (1)–(3) of §3b, moreover, the functions p and q are not only Hölder continuous on \mathbb{R} but infinitely differentiable.

It follows from condition (2) that

$$\text{Ind det } G_\varphi(x) = \text{Ind det } G_0(x) = 0,$$

besides, owing to condition (2), the matrix $G_\varphi(x)$ is positive definite and thus admits the canonical factorization (if it exists).

Canonical factorization of the matrix $G_0(x)$

$$G_0(x) = G_0^-(x)G_0^+(x)$$

can be found in an explicit form

$$G_0^-(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{x-4i}{x-i} & \frac{x-2i}{x-i} \\ \frac{x-4i}{x-i} & -\frac{x-2i}{x-i} \end{pmatrix} \quad (4.3)$$

and

$$G_0^+(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{x+4i}{x+i} & \frac{x+4i}{x+i} \\ \frac{x+2i}{x+i} & -\frac{x+2i}{x+i} \end{pmatrix}. \quad (4.4)$$

Hence, one can calculate an auxiliary matrix

$$G_{1,\varphi}(x) = (G_0^-(x))^{-1}G_\varphi(x)(G_0^+(x))^{-1}$$

in the following form:

$$G_{1,\varphi}(x) = \frac{(x^2+1)^2}{2(x^2+16)(x^2+4)} \begin{pmatrix} g_{1,1}(x) & g_{1,2}(x) \\ g_{2,1}(x) & g_{2,2}(x) \end{pmatrix}, \quad (4.5)$$

where

$$g_{1,1}(x) = \frac{2(x^2+4)(x^2+10) + 6(x^2+4)(e^{i\varphi x} + e^{-i\varphi x})}{(x^2+1)^2}, \quad (4.6)$$

$$g_{1,2}(x) = \frac{12(x-2i)(x+4i)(-e^{i\varphi x} + e^{-i\varphi x})}{(x^2+1)^2}, \quad (4.7)$$

$$g_{2,1}(x) = \frac{12(x+2i)(x-4i)(e^{i\varphi x} - e^{-i\varphi x})}{(x^2+1)^2} \quad (4.8)$$

and

$$g_{2,2}(x) = \frac{2(x^2+16)(x^2+10) + 6(x^2+16)(-e^{i\varphi x} - e^{-i\varphi x})}{(x^2+1)^2}. \quad (4.9)$$

It leads to the following representation of the matrix:

$$G_{1,\varphi}(x) = I + N_1 + N_2, \quad (4.10)$$

where we can take $\varepsilon = 1$ due to theorem 3.1 (see also remark 3.4), and

$$N_1 = 6i \sin^2 \frac{\varphi x}{2} \begin{pmatrix} \frac{1}{x^2+16} & 0 \\ 0 & -\frac{1}{x^2+4} \end{pmatrix}$$

$$\text{and } N_2 = 12i \sin \varphi x \begin{pmatrix} 0 & -\frac{1}{(x+2i)(x-4i)} \\ \frac{1}{(x-2i)(x+4i)} & 0 \end{pmatrix}.$$

This follows from the consideration presented in the previous section.

Following general scheme of §3a, we factorize $G_{1,\varphi}(x)$ at first in the form (3.3) where the first terms of factorization should be computed by the formula (3.4). Following the asymptotic

procedure described above, it is sufficient at the first stage to factorize the matrix

$$M_{0,\varphi}(x) = 6i \begin{pmatrix} \frac{\sin^2(\varphi x/2)}{x^2 + 16} & \frac{2 \sin \varphi x}{(x + 2i)(x - 4i)} \\ \frac{2 \sin \varphi x}{(x - 2i)(x + 4i)} & -\frac{\sin^2(\varphi x/2)}{x^2 + 4} \end{pmatrix}, \quad (4.11)$$

For this particular case, instead of using the Cauchy integrals, one can factorize each entry $n_{ij}(x)$, $i, j = 1, 2$, of the matrix by using decomposition in simple fractions and the Taylor formula. Combining the obtained results, we get the following representation of matrix $N_{1,\varphi}^-(x)$ and $N_{1,\varphi}^+(x)$ (first components of asymptotic factorization)

$$N_{1,\varphi}^-(x) = 6i \begin{pmatrix} \frac{-e^{-i\varphi x} - e^{-4\varphi} + 2}{32i(x - 4i)} + \frac{e^{-i\varphi x} - e^{-4\varphi}}{32i(x + 4i)} & \frac{-ie^{-i\varphi x} + ie^{-4\varphi}}{6i(x - 4i)} + \frac{ie^{-i\varphi x} - ie^{-2\varphi}}{6i(x + 2i)} \\ \frac{ie^{-i\varphi x} - ie^{-2\varphi}}{6i(x - 2i)} - \frac{ie^{-i\varphi x} - ie^{-4\varphi}}{6i(x + 4i)} & \frac{e^{-i\varphi x} + e^{-2\varphi} - 2}{16i(x - 2i)} - \frac{e^{-i\varphi x} - e^{-2\varphi}}{16i(x + 2i)} \end{pmatrix} \quad (4.12)$$

and

$$N_{1,\varphi}^+(x) = 6i \begin{pmatrix} \frac{e^{i\varphi x} + e^{-4\varphi} - 2}{32i(x + 4i)} - \frac{e^{i\varphi x} - e^{-4\varphi}}{32i(x - 4i)} & \frac{-ie^{i\varphi x} + ie^{-2\varphi}}{6i(x + 2i)} + \frac{ie^{i\varphi x} - ie^{-4\varphi}}{6i(x - 4i)} \\ \frac{ie^{i\varphi x} - ie^{-4\varphi}}{6i(x + 4i)} - \frac{ie^{i\varphi x} - ie^{-2\varphi}}{6i(x - 2i)} & \frac{-e^{i\varphi x} - e^{-2\varphi} + 2}{16i(x + 2i)} + \frac{e^{i\varphi x} - e^{-2\varphi}}{16i(x - 2i)} \end{pmatrix}. \quad (4.13)$$

Note that both plus ($N_{1,\varphi}^+$) and minus ($N_{1,\varphi}^-$) matrix functions vanish at infinity in the corresponding half-plane. The procedure can be performed further as it was described in the general case.

5. Outlook and discussions

To illuminate the efficiency of the proposed procedure, we present here the numerical results related to the previous example (§4) showing the quality of the factorization if one decides to restrict the approximation to the first asymptotic term only.

In figures 1 and 2, we present the normalized absolute error of the factorization using only the first asymptotic term for different values of the parameter φ . We compute the errors for each component of remainder, ΔK , that is difference between the exact factorization and its asymptotic approximation along the real axis

$$\Delta K(x) = G_{1,\varphi}(x) - (I + N_{1,\varphi}^-)(I + N_{1,\varphi}^+) = N_{1,\varphi}^- N_{1,\varphi}^+.$$

As it follows from the properties of the Cauchy type integral [5], the obtained estimates are valid also into the upper and lower half-planes.

In figure 1, we depict the error related to the diagonal elements of the remainder while in figure 2, we show the result for the off-diagonal element. Note that as they are complex conjugate with respect to each other, it is enough to discuss only one of them. Unexpectedly, even for rather large value of the parameter $\varphi = 1$, the error is not too high, while for smaller magnitudes of φ , it decays quickly with the argument and its larger value is concentrated only near the centre of the coordinate. Moreover, one can observe that for $j = 1, 2$,

$$\Delta k_{jj}(0) = 4\varphi^2 + O(\varphi^3), \quad \varphi \rightarrow 0.$$

This result can also be verified analytically.

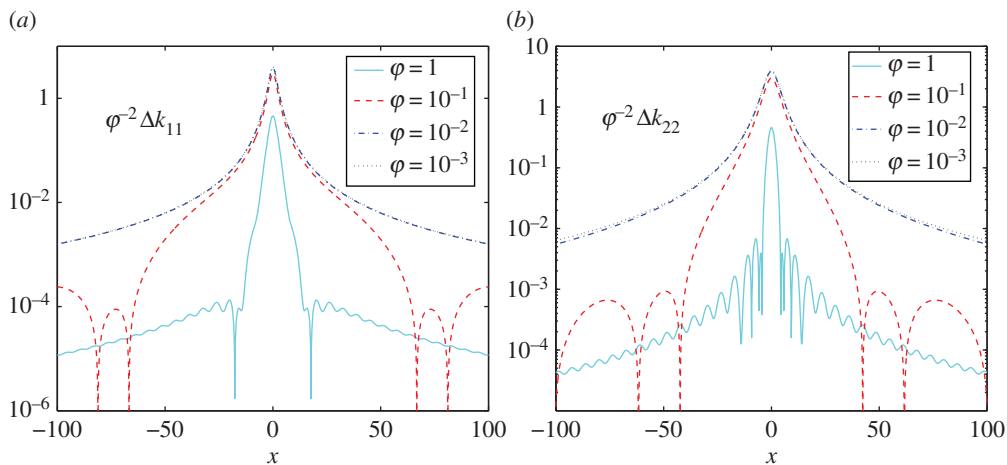


Figure 1. (a,b) Absolute error in the diagonal elements appeared by replacing the exact factorization with the only first asymptotic terms (4.12) and (4.13) for various values of the parameter φ . (Online version in colour.)

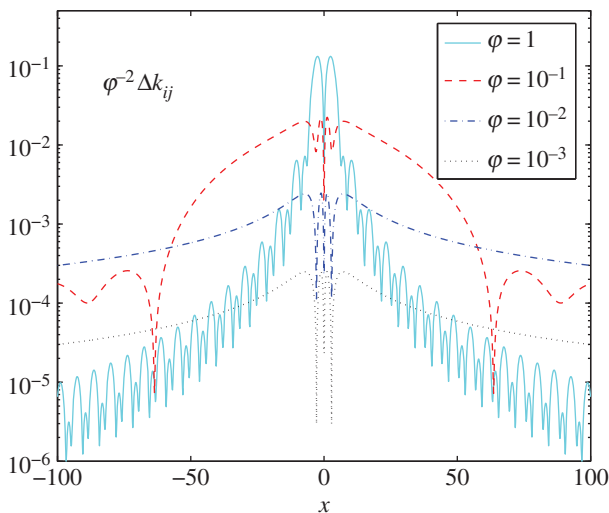


Figure 2. Absolute error in the off-diagonal elements appeared by replacing the exact factorization with the only first terms (4.12) and (4.13) for various values of the parameter φ . (Online version in colour.)

With a decrease in the parameter φ , the amplitude of the error oscillations increases but their support moves out from the coordinate centre to infinity. Thus, such oscillations play a minor role and the numerical computation of the Cauchy integrals in the described procedure would not affect the accuracy of the computations.

It is not correct to say, however, that φ is an optimal small parameter in the problem under consideration. As it follows from theorem 3.1, not only the parameter φ but the decay of the corresponding functions (e.g. the function $q(x)$ in the special case) play an important role in the analysis and it is rather δ which should be taken as the appropriate small parameter.

To clarify this, let us note that two terms N_1 and N_2 are of different orders with respect to the small parameter φ . Indeed: $N_1(\varphi) = O(\varphi^2)$ and $N_2(\varphi) = O(\varphi)$, as $\varphi \rightarrow 0$. As a result, one could construct another first-order approximation of the factorization based only on the term N_2 instead of $N_1 + N_2$. This gives the same first-order estimate in terms of the small parameter φ . Simple

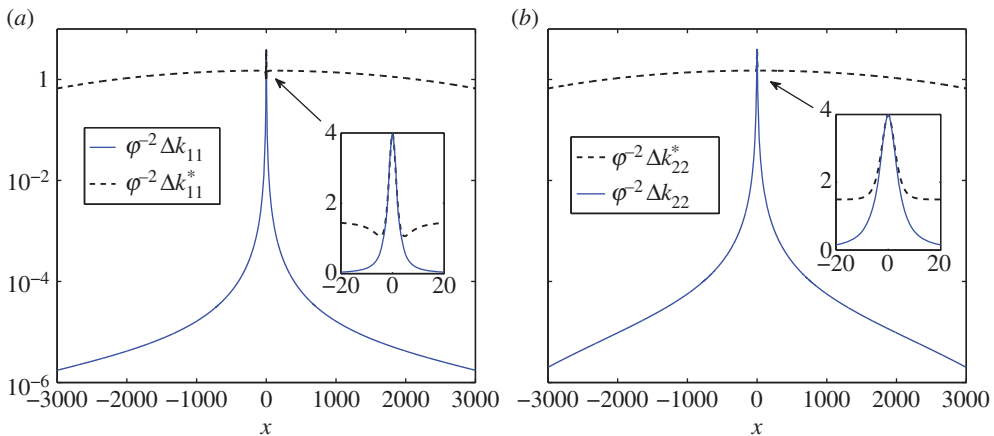


Figure 3. (a,b) Comparison of the errors in the diagonal elements and different approximations for the value of the parameter $\varphi = 10^{-3}$. The errors corresponding to the factorizations of matrix functions (5.1) and (5.2) are given by dotted lines, whereas those related to (4.12) and (4.13) are depicted by the solid line. (Online version in colour.)

calculations give the following (first-order) factorization terms in this case:

$$N_{1,\varphi}^{*-}(x) = \begin{pmatrix} 0 & -\frac{i(e^{-i\varphi x} - e^{-4\varphi})}{x - 4i} + \frac{i(e^{-i\varphi x} - e^{-2\varphi})}{x + 2i} \\ \frac{i(e^{-i\varphi x} - e^{-2\varphi})}{x - 2i} - \frac{i(e^{-i\varphi x} - e^{-4\varphi})}{x + 4i} & 0 \end{pmatrix} \quad (5.1)$$

and

$$N_{1,\varphi}^{*+}(x) = \begin{pmatrix} 0 & -\frac{i(e^{i\varphi x} - e^{-2\varphi})}{x + 2i} + \frac{i(e^{i\varphi x} - e^{-4\varphi})}{x - 4i} \\ \frac{i(e^{i\varphi x} - e^{-4\varphi})}{x + 4i} - \frac{i(e^{i\varphi x} - e^{-2\varphi})}{x - 2i} & 0 \end{pmatrix}. \quad (5.2)$$

We estimate then the quality of the approximation for the factorization of the matrix function $G_{1,\varphi}(x)$ based now solely on the first-order term with respect to the parameter φ . The new reminder is

$$\Delta K^*(x) = G_{1,\varphi}(x) - (I + N_{1,\varphi}^{*-})(I + N_{1,\varphi}^{*+}) = N_1(x) - N_{1,\varphi}^{*-}N_{1,\varphi}^{*+}.$$

Note that the off-diagonal terms give exact (identical) results for the terms (5.1) and (5.2) as opposite to the factorization provided by the terms (4.12) and (4.13). Thus at the first glance, the latter approach looks less beneficial than the former.

In figure 3, we compare the errors related to the diagonal elements of the remainders for the value of the parameter $\varphi = 10^{-3}$. The errors corresponding to matrix functions (5.1) and (5.2) are given by dotted lines while those related to (4.12) and (4.13) are depicted by the solid line. One can observe a striking difference in the accuracy of the two approximations. While for a small values of the variable $|x| < 1$ they are identical, for larger values of the argument, the approximation given by the general procedure provides much better accuracy than that based on the only first-order term, N_2 , with respect to the small parameter φ . Moreover, if one decides to continue asymptotic expansion further, it may become a real issue in numerical computations of the next asymptotic terms as the decay is very slow. Thus, the procedure suggested here is in a sense optimal.

Note that proposed procedure is working not only in the case described in §3b, but also in the case when entries of matrices are quasi-polynomials, i.e. the sum of different exponentials with meromorphic coefficients, provided that all conditions of the class $\mathcal{G}K_n$ are satisfied.

In the case of non-canonical factorization, our algorithm becomes more cumbersome. This situation is the subject of further publication.

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