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On the Expectation of a Ratio of Quadratic Forms in Normal Variables

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Using relatively recent results from multivariate distribution theory, the expectation of a ratio of quadratic forms in normal variables is obtained. Infinite series expressions involving the invariant polynomials of matrix argument are derived. Convergence of the solution depends upon the choice made for two positive, but upper bounded, constants. The same methodology is used to obtain the expectation of multiple ratios of quadratic forms in normal variables.

1. INTRODUCTION

Let y be an $(n \times 1)$ normally distributed vector with mean ω and positive definite variance–covariance matrix Ω, i.e., $y \sim N(\omega, \Omega)$. For the present, let r denote a ratio of quadratic forms in normal variables, that is,

$$r = (y'Ay)^p \, (y'By)^{-q}. \tag{1.1}$$

In (1.1), p and q are non-negative real numbers and $p \neq q$ unless stated otherwise. Both A and B are $(n \times n)$ symmetric matrices. Furthermore, allow $y'Ay$ and $y'By$ to be dependent, that is, $A\Omega B \neq 0$.

Many estimators and test statistics take the form of (1.1). The exact moments of such statistics are of interest, and the purpose of this paper is to examine the expectation of r, $E(r)$. There are numerous examples of studies which have considered particular aspects of this problem. Sawa [26] and Mehta and Swamy [21] examine moments of the k-Class estimator in simultaneous equations models. Srivastava and Tiwari [29] and Dwivedi and Srivastava [12] examine moments of the Double k-Class

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estimator. Sawa [27], Nankervis and Savin [23], and Grubb and Symons [13] examine the moments of the least squares estimator in the autoregressive model. DeGooijer [10] examines the moments of sample autocorrelations in time series ARIMA models. And moments of the test statistic of various tests of restrictions on the coefficients and of the covariance matrix in the general linear regression model have been examined by many authors, Durbin and Watson [11], for instance.

In this paper, a direct approach to evaluating $E(r)$ is taken. After a simple transformation to independent components, the expectation is obtained by averaging with respect to the transformed variables.

A distinguishing feature of the solution for $E(r)$ which contrasts with earlier work on similar expectations (see, in particular, Sawa [26, 27], Magnus [20], and Jones [18]) is that unresolved integrals do not appear. The solution for $E(r)$ takes the form of an infinite series, the convergence of which depends upon the choice made for two positive, but upper bounded, constants. The constants appear as a result of expressing the quadratic forms as convergent generalized hypergeometric functions.

Technically, the exposition of theoretical results depends heavily on the great economy of notation afforded by the use of the zonal polynomials introduced by James, and the invariant polynomials with multiple matrix arguments introduced by Davis. Zonal polynomial theory is surveyed in James [17], and Muirhead [22] also contains many useful references. For invariant polynomial theory see Davis [7–9], Chikuse [2], and Chikuse and Davis [4, 5].

Section 2 of the paper deals with the simplest case, that in which A and B in (1.1) are positive definite ($A, B > 0$). This enables a straightforward derivation of $E(r)$. Applications to problems concerning time series models and linear regression models are examined in Section 3.

The fourth section of the paper relaxes the assumption on B to positive semidefiniteness ($B \geqslant 0$). After further transformation to ensure that the denominator quadratic form involves a strictly positive definite matrix, the solution for $E(r)$ is shown to involve a mixture of expectations identical in form to those considered in Section 2.

The generalization to an expectation of a product of distinct ratios is examined briefly in Section 5. An application involving the expectation of products of time series autocorrelation coefficients is undertaken.

2. Positive Definite A and B

It will be convenient to transform r such that the normal variables have independent components. Define $x = L^{-1}y \sim N(\mu, I_n)$, where L is some

$(n \times n)$ non-singular matrix such that $\Omega = LL'$ and $\mu = L^{-1}\omega$. Let $D = L'AL$ and $F = L'BL$, and assuming A, $B > 0$ then D, $F > 0$. Therefore,

$$r = r^{p;q;n}_{D;F;\mu} = (x'Dx)^p \, (x'Fx)^{-q}, \tag{2.1}$$

where the extension in notation is obvious.

The expectation of the ratio, $E(r)$, is given by

$$\begin{aligned} E(r^{p;q;n}_{D;F;\mu}) &= \int_{\mathbb{R}^n} (x'Dx)^p \, (x'Fx)^{-q} \operatorname{pdf}(x) \, dx \\ &= (2\pi)^{-n/2} \, e^{-\lambda/2} \int_{\mathbb{R}^n} (x'Dx)^p \\ &\quad \times (x'Fx)^{-q} \, e^{(-x'x/2 + x'\mu)} \, dx, \end{aligned} \tag{2.2}$$

where $\lambda = \mu'\mu$ and $\mathbb{R}^n$ denotes the set of $(n \times 1)$ real vectors. The integration over x is conveniently accomplished by transforming $x \to (v, s)$ according to the decomposition $x = vs^{1/2}$, where $v'v = 1$ and scalar $s > 0$. The set of $(n \times m)$ matrices V such that $V'V = I_m$ is known as the Stiefel manifold which is denoted by $V(m, n)$; thus $v \in V(1, n)$. The Jacobian of transformation is $\frac{1}{2}(s)^{n/2-1}$; cf. Herz [14, Lemma 1.4]. Thus

$$\begin{aligned} E(r^{p;q;n}_{D;F;\mu}) = \frac{1}{2} (2\pi)^{-n/2} \, e^{-\lambda/2} \int_{V(1,n)} \int_0^\infty (v'Dv)^p \, (v'Fv)^{-q} \\ \times e^{(-s/2 + v'\mu s^{1/2})} s^{n/2+p-q-1} \, ds \, dv. \end{aligned} \tag{2.3}$$

Let $O(m) = V(m, m)$ denote the group of $(m \times m)$ orthogonal matrices. The integral over $V(1, n)$ is invariant under the transformation $\mu \to h\mu$, where $h \in O(1)$, because the further transformation $v \to hv$ would leave the integral unaltered. Hence the resulting expression may be averaged with respect to h. If James [17, Eq. (27)] is applied, (2.3) becomes

$$\begin{aligned} E(r^{p;q;n}_{D;F;\mu}) = \frac{1}{2} (2\pi)^{-n/2} \, e^{-\lambda/2} \int_{V(1,n)} \int_0^\infty (v'Dv)^p \, (v'Fv)^{-q} \\ \times e^{-s/2} \, {}_0F_1(\tfrac{1}{2}; sv'\mu\mu'v/4) s^{n/2+p-q-1} \, ds \, dv. \end{aligned} \tag{2.4}$$

In (2.4),

$${}_aF_b(\alpha_1, ..., \alpha_a; \beta_1, ..., \beta_b; \delta) = \sum_{i=0}^{\infty} \frac{(\alpha_1)_i \cdots (\alpha_a)_i}{(\beta_1)_i \cdots (\beta_b)_i} \frac{\delta^i}{i!}$$

denotes the generalized hypergeometric function of scalar argument (see Rainville [24]) and $(c)_d = (c)(c+1) \cdots (c+d-1)$. The series in (2.4) is uniformly convergent for all values of its argument so that it may be integrated term by term to obtain

$$E(r^{p;q;n}_{D;F;\mu})=\frac{1}{2}(2\pi)^{-n/2}\,e^{-\lambda/2}\int_{V(1,n)}(v'Dv)^p\,(v'Fv)^{-q}$$

$$\times\sum_{i=0}^{\infty}\left(2^{2i}\left(\frac{1}{2}\right)_i i!\right)^{-1}(v'\mu\mu'v)^i$$

$$\times\left[\int_0^{\infty} e^{-s/2}s^{n/2+p-q+i-1}\,ds\right]dv$$

$$=2^{p-q}\left(\frac{1}{2}n\right)_{p-q}e^{-\lambda/2}\int_{V(1,n)}(v'Dv)^p\,(v'Fv)^{-q}$$

$$\times\,{}_1F_1(\tfrac{1}{2}n+p-q;\,\tfrac{1}{2};\,\tfrac{1}{2}v'\mu\mu'v)(dv), \qquad (2.5)$$

where (dv) denotes the unit-normalized measure on the surface of the n-dimensional unit sphere, that is, $dv=2\pi^{n/2}(dv)/\Gamma(\frac{1}{2}n)$; cf. James [16]. The integration with respect to s exists for all i provided

$$n>2(q-p). \qquad (2.6)$$

Thus $E(r)$ will exist provided (2.6) is satisfied. If $A\Omega B=0$ (or, equivalently, $DF=0$) the two quadratic forms are independent, in which case $E(r)=E(r^{p;0;n}_{D;\cdot;\mu})\,E(r^{0;q;n}_{\cdot;F;\mu})$ which will exist provided $n>2q$.

Now select two positive constants, α and β say, such that $-1<v'(I-\alpha D)\,v<1$ and $-1<v'(I-\beta F)\,v<1$ for all $v\in V(1,n)$. It is straightforward to show that the allowable ranges of choice are $0<\alpha<2/d$ and $0<\beta<2/f$, where d and f denote the largest eigenvalues of D and F, respectively. For α and β within these bounds, it is possible to represent each quadratic form as an infinite binomial series. That is, for any real p, write

$$(v'Dv)^p=\alpha^{-p}(1-v'(I-\alpha D)v)^p$$

$$=\alpha^{-p}\,{}_1F_0(-p;\,v'(I-\alpha D)v); \qquad (2.7)$$

similarly, $(v'Fv)^{-q}=\beta^q\,{}_1F_0(q;\,v'(I-\beta F)v)$ for any real q. Notice that both generalized hypergeometric functions are uniformly convergent by construction. This device has been implicitly used by Ruben [25] in the context of the distribution function of a quadratic form in normal variables.

If the series representations are substituted into (2.5), $E(r)$ becomes

$$E(r^{p;q;n}_{D;F;\mu}) = 2^{p-q}\left(\frac{1}{2}n\right)_{p-q} \alpha^{-p}\beta^{q}e^{-\lambda/2}$$
$$\times \int_{V(1,n)} {}_1F_0(-p; v'(I-\alpha D)v)$$
$$\times {}_1F_0(q; v'(I-\beta F)v)$$
$$\times {}_1F_1(\tfrac{1}{2}n+p-q; \tfrac{1}{2}; \tfrac{1}{2}v'\mu\mu'v)(dv). \tag{2.8}$$

Integrating term by term yields

$$E(r^{p;q;n}_{D;F;\mu}) = 2^{p-q}(\tfrac{1}{2}n)_{p-q}\alpha^{-p}\beta^{q}e^{-\lambda/2}$$
$$\times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)_i\,(q)_j\,(\tfrac{1}{2}n+p-q)_k}{(\tfrac{1}{2})_k\,2^k\,i!\,j!\,k!}$$
$$\times \int_{V(1,n)} (v'(I-\alpha D)v)^i$$
$$\times (v'(I-\beta F)v)^j\,(v'\mu\mu'v)^k\,(dv). \tag{2.9}$$

Since v may be thought of as the first column of a matrix $H \in O(n)$, integration over $V(1,n)$ may be interpreted as an integration over $O(n)$. Write $v = He_1$, where $e_1 = (1, 0, ..., 0)'$ and $H \in O(n)$. Using James [17, Eq. (17)] and Davis [9, Eq. (1.1)] the integral in (2.9) is solved as

$$\int_{V(1,n)} (v'(I-\alpha D)v)^i\,(v'(I-\beta F)v)^j\,(v'\mu\mu'v)^k\,(dv)$$
$$= \int_{O(n)} (\operatorname{tr}(I-\alpha D)\,He_1e_1'H')^i\,(\operatorname{tr}(I-\beta F)\,He_1e_1'H')^j$$
$$\times (\operatorname{tr}\,\mu\mu'He_1e_1'H')^k\,(dH)$$
$$= \sum_{\gamma}\sum_{\kappa}\sum_{\nu}\int_{O(n)} C_{\gamma}((I-\alpha D)\,He_1e_1'H')$$
$$\times C_{\kappa}((I-\beta F)\,He_1e_1'H')\,C_{\nu}(\mu\mu'He_1e_1'H')(dH)$$
$$= \sum_{\gamma}\sum_{\kappa}\sum_{\nu}\sum_{\varphi\in\gamma\cdot\kappa\cdot\nu} C^{\gamma,\kappa,\nu}_{\varphi}((I-\alpha D), (I-\beta F), \mu\mu')$$
$$\times C^{\gamma,\kappa,\nu}_{\varphi}(e_1e_1', e_1e_1', e_1e_1')/C_{\varphi}(I_n). \tag{2.10}$$

In this expression (dH) denotes the unit-normalized Haar measure on the $O(n)$ group, and γ, κ, ν, and φ are ordered partitions of i, j, k, and $i+j+k$, respectively. Considerable simplification of (2.10) is possible. Using Chikuse [2, Eq. (3.4)],

$$C_{\varphi}^{\gamma,\kappa,\nu}(e_1e_1', e_1e_1', e_1e_1') = \theta_{\varphi}^{\gamma,\kappa,\nu} C_{\varphi}(e_1e_1'),$$

where $\theta_{\varphi}^{\gamma,\kappa,\nu} = C_{\varphi}^{\gamma,\kappa,\nu}(I, I, I)/C_{\varphi}(I)$ is a constant which is independent of n and may be zero. Now the zonal polynomial $C_{\varphi}(e_1e_1')$ is zero for all partitions of $i+j+k$ other than the top partition, $\varphi = (i+j+k, 0, ..., 0) = [i+j+k]$, when it is unity. In turn this implies that the only partitions of i, j, and k for which (2.10) is non-zero are the top order partitions $\gamma = (i, 0, ..., 0) = [i]$, $\kappa = (j, 0, ..., 0) = [j]$, and $\nu = (k, 0, ..., 0) = [k]$, respectively. Also note that $\theta_{[i+j+k]}^{[i],[j],[k]} = 1$ and $C_{[i+j+k]}(I_n) = (\frac{1}{2}n)_{i+j+k}/(\frac{1}{2})_{i+j+k}$. Thus, only the leading term in the series (2.10) is non-zero, and so the integral is finally solved as

$$\frac{(\frac{1}{2})_{i+j+k}}{(\frac{1}{2}n)_{i+j+k}} C_{[i+j+k]}^{[i],[j],[k]}(I-\alpha D, I-\beta F, \mu\mu'). \tag{2.11}$$

If (2.11) is substituted into (2.9), $E(r)$ is given by

$$E(r_{D;F;\mu}^{p;q;n}) = 2^{p-q}(\tfrac{1}{2}n)_{p-q}\, \alpha^{-p}\beta^{q}e^{-\lambda/2} \times \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-p)_i\,(q)_j\,(\frac{1}{2}n+p-q)_k}{(\frac{1}{2})_k\, 2^k\, i!\, j!\, k!} \times \frac{(\frac{1}{2})_{i+j+k}}{(\frac{1}{2}n)_{i+j+k}} C_{[i+j+k]}^{[i],[j],[k]}(I-\alpha D, I-\beta F, \mu\mu'). \tag{2.12}$$

Provided α and β are chosen within the prescribed bounds, (2.12) is a convergent series involving an invariant polynomial with three matrix arguments. The invariant polynomial in (2.12) appears in the special form of a top order polynomial; thus the methods of Chikuse [3] enable the computation of the invariant polynomial for all terms in the series (note that "$\sum_{i=1}^{r}$" in [3, Eq. (2.8)] should be replaced by $\prod_{j=1}^{r}$).

Should p be an integer the series representation (2.7) terminates after $p+1$ terms and is valid for any symmetric A (and hence D). In this case the index i in (2.12) need only increment to p; also, α is no longer bounded from above. Furthermore, $(v'Dv)^p$ can also be replaced by $C_{[p]}(Dvv')$, leading to $E(r)$ as

$$E(r_{D;F;\mu}^{p;q;n}) = 2^{p-q}(\tfrac{1}{2}n)_{p-q}\, \beta^{q}e^{-\lambda/2} \times \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(q)_j\,(\frac{1}{2}n+p-q)_k}{(\frac{1}{2})_k\, 2^k\, j!\, k!} \times \frac{(\frac{1}{2})_{p+j+k}}{(\frac{1}{2}n)_{p+j+k}} C_{[p+j+k]}^{[p],[j],[k]}(D, I-\beta F, \mu\mu'). \tag{2.13}$$

3. Applications

3.1. *Time Series Models*

Consider the stationary first-order autoregressive process

$$y_t + \phi y_{t-1} = e_t, \qquad t = 1, ..., n, \tag{3.1}$$

with $\text{abs}(\phi) < 1$. Assuming $e = (e_1 \cdots e_n)' \sim N(0, \sigma^2 I)$ implies that the observations $y = (y_1 \cdots y_n)' \sim N(0, \Omega)$. As in Anderson [1, Chap. 6], Ω^{-1} may be well approximated by $W^{-1} = (\gamma_0 I + \gamma_1 A_1)$, where $\gamma_0 = (1 + \phi^2)/\sigma^2$, $\gamma_1 = 2\phi/\sigma^2$, and A_1 is the standard first differencing matrix. Define the dth-order sample autocorrelation coefficient of the observations as

$$r_d = y' A_d y / y'y, \qquad d = 1, 2, ..., \tag{3.2}$$

where A_d is the standard dth-order differencing matrix. Let $M \in O(n)$ be such that $MA_1M' = D_1$ is a diagonal matrix with (t, t)th element $\cos \pi t/(n+1)$ (see [1, Sect. 6.5.4]); then $x = D^{1/2}My \sim N(0, I)$, where $D = MW^{-1}M' = (\gamma_0 I + \gamma_1 D_1) > 0$. Consequently, $r_d = x'\bar{A}_d x / x' D^{-1} x$, where $\bar{A}_d = D^{-1/2} M A_d M' D^{-1/2} > 0$. Applying (2.13) yields the hth moment of r_d ($h = 1, 2, ...$) as

$$\begin{aligned} E(r_d^h) &= E(r^{h;h;n}_{\bar{A}_d; D^{-1}; 0}) \\ &= \beta^h \sum_{j=0}^{\infty} \frac{(h)_j}{j!} \frac{(\frac{1}{2})_{j+h}}{(\frac{1}{2}n)_{j+h}} C^{[h],[j]}_{[h+j]}(\bar{A}_d, I - \beta D^{-1}) \end{aligned} \tag{3.3}$$

with $0 < \beta < 2\,(\gamma_0 - \text{abs}(\gamma_1) \cos \pi/(n+1))$.

Setting $\phi = 0$ in (3.1) yields the noise model; thus r_d is equivalent to the residual autocorrelation coefficient. Using Davis [7, Eq. (2.2)], (3.3) simplifies to

$$E(r_d^h) = \frac{(\frac{1}{2})_h}{(\frac{1}{2}n)_h} C_{[h]}(A_d), \tag{3.4}$$

which conveniently expresses many of the expectations given in Ljung and Box [19].

3.2. *Regression Model*

Consider the m-parameter classical linear regression model

$$y = X\beta + e, \tag{3.5}$$

where the $(T \times m)$ matrix X is of rank $m < T$. Let $M = (I - X(X'X)^{-1} X')$, which can be decomposed into $M = PP'$ for some $P \in V(n, T)$, where

$n = T - m$. For $\hat{e}$ the vector of ordinary least squares residuals, finds $\hat{e} = My = Me$. The Durbin and Watson [11] statistic used to test for autocorrelated disturbances is

$$r = \hat{e}' A_1 \hat{e}/\hat{e}'\hat{e} = e' M A_1 M e/e' M e = x' A_1^* x/x'x, \tag{3.6}$$

where $x = P'e/\sigma$ and $A_1^* = P' A_1 P$ $(n \times n)$. Under the null hypothesis of independence and assuming $e \sim N(0, \sigma^2 I_T)$, one finds $x \sim N(0, I_n)$, and the hth moment $(h = 1, 2, \ldots)$ of r is

$$E(r^h) = E(r^{h;h;n}_{A_1^*;I;0}) = (\tfrac{1}{2})_h \, C_{[h]}(A_1^*)/(\tfrac{1}{2}n)_h. \tag{3.7}$$

A further application is examination of the robustness of $E(r^h)$ to non-normal disturbances using Davis' [6] two step procedure. Under the null hypothesis of independence, the first step requires $E(r^h)$ to be computed under $e \sim N(\theta, \sigma^2 I)$ for fixed θ. The vector θ is a $(T \times 1)$ set of independent pseudo-random variables which have the same third and higher order cumulants and product cumulants as the desired non-normal distribution. Applying a linear operator, say E_θ as defined in [6], is step two of the procedure and will generate the true $E(r^h)$. Setting $\theta^* = P'\theta/\sigma$ and applying (2.13),

$$\begin{aligned} E(r^h) &= E_\theta E(r^{h;h;n}_{A_1^*;I;\theta^*}) \\ &= E_\theta e^{-\theta' M \theta/2\sigma^2} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}+k)_h}{(n/2+k)_h \, 2^k k!} \, C^{[h],[k]}_{[h+k]}(A_1^*, \theta^*\theta^{*\prime}), \end{aligned} \tag{3.8}$$

where β is selected as unity. Hillier and Smith [15] and Smith [28] have examined the robustness of $E(r^h)$ $(h = 1, 2, 3, 4)$ when the non-normal disturbances have moderate kurtosis and combined kurtosis and skewness, respectively.

4. Positive Semidefinite B

It is of interest to examine the situation occurring when the denominator matrix in (1.1) is positive semidefinite, that is, $B \geqslant 0$, and assume $\operatorname{rank}(B) = \operatorname{rank}(F) = m < n$. As $(n-m)$ of the eigenvalues of $(I - \beta F)$ must be unity regardless of β, then the representation of $(v'Fv)^{-q}$ in (2.5) as a convergent binomial series is invalidated. Nevertheless, after further transformations, $E(r)$ will be seen to involve a sum of expectations each evaluated exactly as in Section 2.

Let $F = H_1 \Lambda H_1'$, where the $(m \times m)$ diagonal matrix Λ contains all the positive eigenvalues of F and the columns of $H_1 \in V(m, n)$ are the corresponding eigenvectors. Let the columns of $H_2 \in V(n-m, n)$ be the $n-m$ eigenvectors corresponding to the $n-m$ zero eigenvalues of F. Obviously $H_1'H_2 = 0$ and $H_1H_1' + H_2H_2' = I_n$. Now transform to independent vectors with independent components: $z_1 = H_1'x = H_1'L^{-1}y \sim N(\gamma_1, I_m)$, where $\gamma_1 = H_1'L^{-1}\omega$; and $z_2 = H_2'x = H_2'L^{-1}y \sim N(\gamma_2, I_{n-m})$, where $\gamma_2 = H_2'L^{-1}\omega$. The ratio

$$\frac{y'Ay}{y'By} = \frac{x'(H_1H_1' + H_2H_2')\,D(H_1H_1' + H_2H_2')x}{x'Fx}$$

$$= \frac{z_1'\Omega_1 z_1 + 2z_1'\Omega_2 z_2 + z_2'\Omega_3 z_2}{z_1'Az_1}. \tag{4.1}$$

In (4.1), $\Omega_1 = H_1'DH_1$ $(m \times m)$, $\Omega_2 = H_1'DH_2$ $(m \times (n-m))$, and $\Omega_3 = H_2'DH_2$ $((n-m) \times (n-m))$. Observe that the matrix in the denominator of (4.1) is positive definite. By the multinomial expansion (throughout this section assume p is an integer; however, those instances when this can be relaxed to real p will be indicated) the expectation of (1.1) is

$$E(r) = \sum_{i_1=0}^{p} \sum_{i_2=0}^{i_1} \binom{p}{i_1}\binom{i_1}{i_2} \times E_{z_1}E_{z_2} \frac{(z_1'\Omega_1 z_1)^{p-i_1}(2z_1'\Omega_2 z_2)^{i_1-i_2}(z_2'\Omega_3 z_2)^{i_2}}{(z_1'Az_1)^q}. \tag{4.2}$$

As z_1 and z_2 are independent, the expectation over z in (4.2) is equivalent to the product of the expectations over z_1 and z_2, E_{z_1} and E_{z_2}, respectively. As (4.2) involves sums of expectations in a mixture of powers of quadratic and bilinear forms, the solution and existence requirements for $E(r)$ must be revised. Four situations will be examined in the following sub-sections, the first three of which have $\Omega_2 = 0$ in common. When $\Omega_2 = 0$ the bilinear term in (4.2) is eliminated and $E(r)$ can be solved using the results of Section 2.

4.1. $\Omega_2 = 0, \Omega_3 = 0$

This case arises if $DH_2 = 0$ and can occur when the quadratic forms $y'Ay$ and $y'By$ are dependent, that is, $A\Omega B \neq 0$. For example, the Durbin–Watson statistic (3.6) falls within this section's ambit. When $\Omega_2 = 0$ and $\Omega_3 = 0$, (4.2) simplifies to

$$E(r) = E_{z_1}(z_1'\Omega_1 z_1)^p (z_1'Az_1)^{-q} = E(r^{p;q;m}_{\Omega_1;A;\gamma_1}), \tag{4.3}$$

the solution for which can be obtained from (2.12) after suitable replacements. From (2.6), the requirement for the existence of $E(r)$ is $m > 2(q-p)$. In this case $p > 0$ may be real.

4.2. $\Omega_2 = 0$

This case can arise if the quadratic forms are dependent, for example, when $D = I$. When $\Omega_2 = 0$, (4.2) simplifies to

$$\begin{aligned} E(r) &= \sum_{i_1=0}^{p} \binom{p}{i_1} E_{z_1}\left(\frac{(z_1'\Omega_1 z_1)^{p-i_1}}{(z_1' \Lambda z_1)^q}\right) E_{z_2}(z_2'\Omega_3 z_2)^{i_1} \\ &= \sum_{i_1=0}^{p} \binom{p}{i_1} E(r^{p-i_1;q;m}_{\Omega_1;\Lambda;\gamma_1})\, E(r^{i_1;0;n-m}_{\Omega_3;\cdot;\gamma_2}). \end{aligned} \tag{4.4}$$

The solution to each of the expectations on the right-hand side of (4.4) is obtained from (2.12) or (2.13) after suitable replacements. The conditions for existence of $E(r)$ are identical to the conditions for existence of the expectation over z_1. The most stringent requirement occurs when the index $i_1 = p$, requiring $m > 2q$.

4.3. $\Omega_1 = 0$, $\Omega_2 = 0$

If the two quadratic forms are independent then $A\Omega B = 0$, which implies that $\Omega_1 = 0$ and $\Omega_2 = 0$. In this case (4.2) simplies to

$$\begin{aligned} E(r) &= E_{z_1}(z_1'\Lambda z_1)^{-q}\, E_{z_2}(z_2'\Omega_3 z_2)^p \\ &= E(r^{0;q;m}_{\cdot;\Lambda;\gamma_1})\, E(r^{p;0;n-m}_{\Omega_3;\cdot;\gamma_2}). \end{aligned} \tag{4.5}$$

Again $p > 0$ may be real and, provided $m > 2q$, $E(r)$ will exist.

4.4. *Non-zero Ω_1, Ω_2, and Ω_3*

This is the most general case and it applies to some dependent quadratic forms. The expression for $E(r)$ in (4.2) cannot be simplified. To determine the existence requirement for $E(r)$ it is necessary to establish the term of the sum (4.2) that imposes the greatest restriction on m. The particular term occurs when $i_1 = i_2 = p$ and requires $m > 2q$ for its existence. Therefore, $E(r)$ will exist provided $m > 2q$. The complete solution for $E(r)$ in this general case remains a matter of future research.

5. Multiple Ratios

So far only the expectation of a single ratio of quadratic forms has been considered; however, extending consideration to an expectation involving multiple ratios requires no theory beyond that already utilized. Adopt the nomenclature: $a(A)$ to denote $a_1, ..., a_A$ for scalars a_i ($i = 1, ..., A$); $a[A]$ to denote $[a_1], ..., [a_A]$ for top order partitions of scalars a_i ($i = 1, ..., A$); $A_{[a]}$

to denote $A_1, ..., A_a$ for symmetric matrices A_i $(i=1, ..., a)$. With the n-vector $y \sim N(\omega, \Omega)$, generalize (1.1) to

$$r = \prod_{l=1}^{P} (y'A_l y)^{p_l} \prod_{m=1}^{Q} (y'B_m y)^{-q_m}. \tag{5.1}$$

As before, with $LL' = \Omega$, let $D_l = L'A_l L > 0$ $(l=1, ..., P)$, $F_m = L'B_m L > 0$ $(m=1, ..., Q)$, $\mu = L^{-1}\omega$ and $x = L^{-1}y \sim N(\mu, I)$. Note that for purposes of illustration $A_l > 0$ $(l=1, ..., P)$ and $B_m > 0$ $(m=1, ..., Q)$ are assumed throughout this section. Given these transformations, denote r by $r^{p(P);q(Q);n}_{D[P];F[Q];\mu}$, a multiple ratio of quadratic forms in the standardized normal vector x.

Given the method of Section 2, and supposing that every quadratic form in r depends upon all other quadratic forms, $E(r)$ will exist provided $n > 2(\sum_1^Q q_m - \sum_1^P p_l)$. If r involved individual (or even blocks of) independent quadratic forms then the condition upon n for the existence of $E(r)$ may change. For example, suppose that the first denominator quadratic form in r is independent of all other $(P+Q-1)$ quadratic forms, provided the more stringent of the requirements $n > 2(\sum_2^Q q_m - \sum_1^P p_l)$ and $n > 2q_1$ is satisfied; then $E(r)$ will exist.

The infinite series representation of each quadratic form, as in (2.7), followed by a similar integration as performed to obtain (2.11), yields the solution for $E(r)$ as

$$\begin{aligned} E(r^{p(P);q(Q);n}_{D[P];F[Q];\mu}) &= 2^{\sum p_l - \sum q_m} (\tfrac{1}{2}n)_{\sum p_l - \sum q_m} \\ &\quad \times e^{-\lambda/2} \Big(\prod \beta_m^{q_m}\Big)\Big(\prod \alpha_l^{-p_l}\Big) \\ &\quad \times \sum{}^{*} \frac{(\frac{1}{2}n + \sum p_l - \sum q_m)_k}{2^k (\frac{1}{2})_k\, k!} \frac{(\prod (-p_l)_{i_l})}{(\prod i_l!)} \\ &\quad \times \frac{(\prod (q_m)_{j_m})}{(\prod j_m!)} \frac{(\frac{1}{2})_{\sum i_l + \sum j_m + k}}{(\frac{1}{2}n)_{\sum i_l + \sum j_m + k}} \\ &\quad \times C^{i[P], j[Q], [k]}_{[\sum i_l + \sum j_m + k]}((I - \alpha D)_{[P]}, (I - \beta F)_{[Q]}, \mu\mu'). \end{aligned} \tag{5.2}$$

The symbol $\sum^*$ denotes $\sum_{i_1=0}^{\infty} \cdots \sum_{i_P=0}^{\infty} \sum_{j_1=0}^{\infty} \cdots \sum_{j_Q=0}^{\infty} \sum_{k=0}^{\infty}$. If in (5.1) any p_l is integer then the index i_l would need only increment to p_l in (5.2). Alternatively $(v'D_l v)^{p_l}$ may be replaced by $C_{[p_l]}(D_l vv')$, which would eliminate the index i_l and leave D_l as an argument matrix in the invariant polynomial (see the discussion preceding (2.13)). This method is adopted in the examples below.

5.1. *Time Series Models* (*Continued*)

Consider the noise model, that is, when $\phi=0$ in (3.1). Ljung and Box [19] examine the mean and variance of the statistic $s=\sum_1^P r_l$, where r_l is the lth order residual autocorrelation coefficient as defined in (3.2). Let r denote the product of P residual autocorrelation coefficients each raised to some non-negative integer power, that is,

$$\begin{aligned} r = r_1^{h_1} \cdots r_P^{h_P} &= (y'y)^{-q} \prod_{l=1}^{P} (y'A_l y)^{h_l} \\ &= (x'x)^{-q} \prod_{l=1}^{P} (x'A_l x)^{h_l}, \end{aligned} \tag{5.3}$$

where $q=\sum h_l$ and $x=y/\sigma \sim N(0, I)$. In the multinomial expansion of s^q, a typical expectation required in $E(s^q)$ is equivalent to $E(r)=E(r^{h(P);q;n}_{A[P];I;0})$. In this case (5.2) simplifies considerably to yield $E(r)$ as

$$E(r^{h(P);q;n}_{A[P];I;0}) = \frac{(\frac{1}{2})_q}{(\frac{1}{2}n)_q} C^{h[P]}_{[q]}(A_{[P]}). \tag{5.4}$$

Thus $E(s^q)$ is given by a weighted sum of terms of the form (5.4).

Now consider the moments of products of sample autocorrelations in the autoregressive model (3.1) (ϕ is non-zero). Let r denote the product of P sample autocorrelation coefficients each raised to some non-negative integer power, that is,

$$\begin{aligned} r = r_1^{h_1} \cdots r_P^{h_P} &= (y'y)^{-q} \sum_{l=1}^{P} (y'A_l y)^{h_l} \\ &= (x'D^{-1}x)^{-q} \prod_{l=1}^{P} (x'\bar{A}_l x)^{h_l} \end{aligned} \tag{5.3}$$

where $x=D^{1/2}My \sim N(0, I)$. $E(r)$ is equivalent to $E(r^{h(P);q;n}_{\bar{A}[P];D^{-1};0})$, and equals

$$\begin{aligned} E(r^{h(P);q;n}_{\bar{A}[P];D^{-1};0}) = \beta^q \sum_{j=0}^{\infty} \frac{(q)_j}{j!} \frac{(\frac{1}{2})_{q+j}}{(\frac{1}{2}n)_{q+j}} \\ \times C^{h[P],[j]}_{[q+j]}(\bar{A}_{[P]}, I-\beta D^{-1}) \end{aligned} \tag{5.6}$$

with $0<\beta<2$ $(\gamma_0 - \text{abs}(\gamma_1)\cos \pi/(n+1))$. It is straightforward to show that (5.6) is equivalent to (5.4) when $\phi=0$.

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