

Aberystwyth University

An asymptotic model for a thin bonded elastic layer coated with an elastic membrane

Argatov, Ivan; Mishuris, Gennady

Published in:

Applied Mathematical Modelling

DOI:

[10.1016/j.apm.2015.09.109](https://doi.org/10.1016/j.apm.2015.09.109)

Publication date:

2016

Citation for published version (APA):

Argatov, I., & Mishuris, G. (2016). An asymptotic model for a thin bonded elastic layer coated with an elastic membrane. *Applied Mathematical Modelling*, 40(4), 2541-2548. <https://doi.org/10.1016/j.apm.2015.09.109>

General rights

Copyright and moral rights for the publications made accessible in the Aberystwyth Research Portal (the Institutional Repository) are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Aberystwyth Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Aberystwyth Research Portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

tel: +44 1970 62 2400

email: is@aber.ac.uk

Accepted Manuscript

An asymptotic model for a thin bonded elastic layer coated with an elastic membrane

I. Argatov, G. Mishuris

PII: S0307-904X(15)00652-6
DOI: [10.1016/j.apm.2015.09.109](https://doi.org/10.1016/j.apm.2015.09.109)
Reference: APM 10817

To appear in: *Applied Mathematical Modelling*

Received date: 20 November 2014
Revised date: 26 June 2015
Accepted date: 30 September 2015

Please cite this article as: I. Argatov, G. Mishuris, An asymptotic model for a thin bonded elastic layer coated with an elastic membrane, *Applied Mathematical Modelling* (2015), doi: [10.1016/j.apm.2015.09.109](https://doi.org/10.1016/j.apm.2015.09.109)

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



Highlights

- A transversely isotropic elastic layer coated with a thin layer is considered
- Flexural stiffness of the coating layer is neglected
- Leading-order asymptotic models for the layer deformation are constructed

ACCEPTED MANUSCRIPT

An asymptotic model for a thin bonded elastic layer coated with an elastic membrane

I. Argatov, G. Mishuris *

*Institute of Mathematics and Physics, Aberystwyth University, Ceredigion SY23
3BZ, Wales, UK*

Abstract

The deformation problem for a transversely isotropic elastic layer bonded to a rigid substrate and coated with a very thin elastic layer made of another transversely isotropic material is considered. The leading-order asymptotic models (for compressible and incompressible layers) are constructed based on the simplifying assumptions that the generalized plane stress conditions apply to the coating layer, and the flexural stiffness of the coating layer is negligible compared to its tensile stiffness.

Key words: Deformation problem, thin layer, elastic coating, transversely isotropic, asymptotic solution

1 Introduction

Some natural biological tissues such as articular cartilage possess an inhomogeneous, layered structure with anisotropic material properties. In particular, morphological studies of adult articular cartilage [13,16] show three different zones of preferred collagen fiber bundle orientation. The superficial zone formed by tangentially oriented collagen fibrils provides a thin layer with a high tensile stiffness in the direction parallel to articular surface. It was shown [20] that the high transverse stiffness of the superficial tissue layer (characterized by tangentially oriented collagen fibrils) is important in controlling the deformation response of articular cartilage. Generally speaking, the surface layer in biomaterials usually has different mechanical properties than the underlying bulk material. Due to this circumstance, the mechanical deformation behavior is strongly influenced by the complex interaction of these layers [30].

In the present paper, we consider the deformation problem for a transversely isotropic elastic layer reinforced with a thin elastic membrane ideally attached to one surface,

* Corresponding author.

Email address: ggm@aber.ac.uk (G. Mishuris).

while the other surface is bonded to a rigid substrate. Following [2,29], it is assumed that the reinforcing layer is very thin (with respect to a characteristic size of the applied load) so that its deformation can be treated in the framework of the generalized plane stress state. Moreover, it is assumed that the flexural stiffness of the coating layer is negligible compared to its tensile stiffness. Thus, the reinforcing thin layer is regarded as an elastic membrane.

The rest of the paper is organized as follows. In Section 2, we formulate the three-dimensional boundary conditions for a coated elastic layer. The deformation problem formulation itself is given in Section 3. Asymptotic analysis of the deformation problem is presented in Section 4. The main result of the present paper is presented by the leading-order asymptotic models for the local indentation of the coated elastic layer developed in Section 5 for the cases of compressible and incompressible layer. Finally, Section 7 contains some discussion of the obtained asymptotic models and outlines our conclusion.

2 Boundary conditions for a coated elastic layer

We consider a very thin transversely isotropic elastic coating layer (of uniform thickness \hat{h}) ideally attached to an elastic layer (of thickness h) made of another transversely isotropic material (see Fig. 1). Let the five independent elastic constants of the elastic layer and its coating be denoted by A_{11} , A_{12} , A_{13} , A_{33} , A_{44} and \hat{A}_{11} , \hat{A}_{12} , \hat{A}_{13} , \hat{A}_{33} , \hat{A}_{44} , respectively.



Figure 1. An elastic coated layer as a model for articular cartilage (the histological image of articular cartilage is taken from the paper [26]). The regions $-\hat{h} \leq z \leq 0$ and $z \geq h$ represent the superficial zone and the subchondral bone, respectively.

Under the assumption that the two layers are in perfect contact with one another along their common interface, $z = 0$, the following boundary conditions of continuity (interface conditions of perfect bonding) should be satisfied:

$$\hat{\mathbf{v}}(\mathbf{y}, 0) = \mathbf{v}(\mathbf{y}, 0), \quad \hat{w}(\mathbf{y}, 0) = w(\mathbf{y}, 0), \quad (1)$$

$$\hat{\sigma}_{3j}(\mathbf{y}, 0) = \sigma_{3j}(\mathbf{y}, 0), \quad j = 1, 2, 3. \quad (2)$$

Here, (\mathbf{v}, \mathbf{w}) and $(\hat{\mathbf{v}}, \hat{w})$ are the displacement vectors of the elastic layer $z \in (0, h)$ and the elastic coating layer $z \in (-\hat{h}, 0)$, respectively, σ_{ij} and $\hat{\sigma}_{ij}$ are the corresponding components of stress. In what follows, we make use of the Cartesian coordinate system (\mathbf{y}, z) , where $\mathbf{y} = (y_1, y_2)$ are the in-plane coordinates.

On the upper surface of the two-layer system, $z = -\hat{h}$, we impose the boundary conditions of normal loading with no tangential tractions

$$\hat{\sigma}_{31}(\mathbf{y}, -\hat{h}) = \hat{\sigma}_{32}(\mathbf{y}, -\hat{h}) = 0, \quad \hat{\sigma}_{33}(\mathbf{y}, -\hat{h}) = -p(\mathbf{y}), \quad (3)$$

where $p(\mathbf{y})$ is a specified function of external loading.

Following Rahman and Newaz [30], we simplify the deformation analysis of the elastic coating layer based on the following two assumptions: (1) the coating layer is assumed to be very thin, so that the generalized plane stress conditions apply; (2) the flexural stiffness of the coating layer in the z -direction is negligible compared to its tensile stiffness.

In the absence of body forces, the equilibrium equations for an infinitesimal element of the coating layer are

$$\frac{\partial \hat{\sigma}_{j1}}{\partial y_1} + \frac{\partial \hat{\sigma}_{j2}}{\partial y_2} + \frac{\partial \hat{\sigma}_{j3}}{\partial z} = 0, \quad j = 1, 2, 3. \quad (4)$$

The stress-strain relationship for the transversely isotropic elastic coating layer is given by

$$\begin{pmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{33} \\ \hat{\sigma}_{23} \\ \hat{\sigma}_{13} \\ \hat{\sigma}_{12} \end{pmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & \hat{A}_{13} & 0 & 0 & 0 \\ \hat{A}_{12} & \hat{A}_{11} & \hat{A}_{13} & 0 & 0 & 0 \\ \hat{A}_{13} & \hat{A}_{13} & \hat{A}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\hat{A}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\hat{A}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\hat{A}_{66} \end{bmatrix} \begin{pmatrix} \hat{\epsilon}_{11} \\ \hat{\epsilon}_{22} \\ \hat{\epsilon}_{33} \\ \hat{\epsilon}_{23} \\ \hat{\epsilon}_{13} \\ \hat{\epsilon}_{12} \end{pmatrix}, \quad (5)$$

where $(\hat{\epsilon}_{11}, \hat{\epsilon}_{22}, \dots, \hat{\epsilon}_{12})^T$ is the vector of strains in the coating layer, the superscript T denotes the transposition operation, and $2\hat{A}_{66} = \hat{A}_{11} - \hat{A}_{12}$.

Integrating Eqs. (4) through the thickness of coating layer and taking into account the interface and boundary conditions (2) and (3), we get

$$\hat{h} \left(\frac{\partial \hat{\sigma}_{j1}}{\partial y_1} + \frac{\partial \hat{\sigma}_{j2}}{\partial y_2} \right) = -\sigma_{j3} \Big|_{z=0}, \quad j = 1, 2, \quad (6)$$

$$\hat{h} \left(\frac{\partial \hat{\sigma}_{13}}{\partial y_1} + \frac{\partial \hat{\sigma}_{23}}{\partial y_2} \right) = -\sigma_{33} \Big|_{z=0} - p. \quad (7)$$

Here, $\hat{\sigma}_{ij}$ are the averaged stresses, i.e.,

$$\hat{\sigma}_{ij}(\mathbf{y}) = \frac{1}{\hat{h}} \int_{-\hat{h}}^0 \hat{\sigma}_{ij}(\mathbf{y}, z) dz.$$

Under the simplifying assumptions made above, we have

$$\hat{\sigma}_{13} = \hat{\sigma}_{23} = \hat{\sigma}_{33} = 0. \quad (8)$$

Hence, Eq. (7) immediately implies that

$$\sigma_{33} \Big|_{z=0} = -p. \quad (9)$$

Moreover, in view of (8), the averaged strain $\hat{\varepsilon}_{33}$ must satisfy the equation

$$\hat{A}_{13}\hat{\varepsilon}_{11} + \hat{A}_{13}\hat{\varepsilon}_{22} + \hat{A}_{33}\hat{\varepsilon}_{33} = 0.$$

Therefore, the in-plane averaged stress-strain relationship takes the form

$$\begin{pmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\sigma}_{12} \end{pmatrix} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} & 0 \\ \hat{A}_{12} & \hat{A}_{11} & 0 \\ 0 & 0 & 2\hat{A}_{66} \end{bmatrix} \begin{pmatrix} \hat{\varepsilon}_{11} \\ \hat{\varepsilon}_{22} \\ \hat{\varepsilon}_{12} \end{pmatrix}, \quad (10)$$

where we introduced the notation

$$\hat{\hat{A}}_{11} = \hat{A}_{11} - \frac{\hat{A}_{13}^2}{\hat{A}_{33}}, \quad \hat{\hat{A}}_{12} = \hat{A}_{12} - \frac{\hat{A}_{13}^2}{\hat{A}_{33}}, \quad 2\hat{\hat{A}}_{66} = \hat{A}_{11} - \hat{A}_{12}. \quad (11)$$

On the other hand, in view of the interface conditions (1), we will have

$$\hat{\varepsilon}_{11} = \varepsilon_{11}|_{z=0}, \quad \hat{\varepsilon}_{22} = \varepsilon_{22}|_{z=0}, \quad \hat{\varepsilon}_{12} = \varepsilon_{12}|_{z=0}, \quad (12)$$

where ε_{11} , ε_{22} , and ε_{12} are the in-plane strains in the coated elastic layer $z \in (0, h)$.

Therefore, taking Eqs. (10) and (12) into account, we transform the boundary conditions (6) as follows:

$$\begin{aligned} -\frac{1}{h}\sigma_{31}|_{z=0} &= \frac{\partial}{\partial y_1} \left(\hat{\hat{A}}_{11} \frac{\partial v_1}{\partial y_1} + \hat{\hat{A}}_{12} \frac{\partial v_2}{\partial y_2} \right) + \hat{\hat{A}}_{66} \frac{\partial}{\partial y_2} \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right), \\ -\frac{1}{h}\sigma_{32}|_{z=0} &= \hat{\hat{A}}_{66} \frac{\partial}{\partial y_1} \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) + \frac{\partial}{\partial y_2} \left(\hat{\hat{A}}_{12} \frac{\partial v_1}{\partial y_1} + \hat{\hat{A}}_{11} \frac{\partial v_2}{\partial y_2} \right). \end{aligned}$$

Finally, the above boundary conditions can be rewritten in the matrix form as

$$\sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2|_{z=0} = -\hat{\mathcal{L}}(\nabla_y)\mathbf{v}|_{z=0}, \quad (13)$$

where $\hat{\mathcal{L}}(\nabla_y)$ is a 2×2 matrix differential operator such that

$$\begin{aligned} \hat{\mathcal{L}}_{\alpha\alpha}(\nabla_y) &= \hat{h}\hat{\hat{A}}_{11} \frac{\partial^2}{\partial y_\alpha^2} + \hat{h}\hat{\hat{A}}_{66} \frac{\partial^2}{\partial y_{3-\alpha}^2}, \\ \hat{\mathcal{L}}_{\alpha\beta}(\nabla_y) &= \hat{h}(\hat{\hat{A}}_{12} + \hat{\hat{A}}_{66}) \frac{\partial^2}{\partial y_\alpha \partial y_\beta}, \quad \alpha, \beta = 1, 2, \quad \alpha \neq \beta. \end{aligned} \quad (14)$$

Thus, the deformation problem for an elastic layer coated with a very thin flexible elastic layer is reduced to that for the elastic layer without coating, but subjected to a different set of boundary conditions (9) and (13) on the surface $z = 0$.

3 Deformation problem formulation

Now, let us consider a relatively thin transversely isotropic elastic layer of uniform thickness, h , coated with an infinitesimally thin elastic membrane and bonded to a rigid substrate, so that

$$\mathbf{v}|_{z=h} = \mathbf{0}, \quad w|_{z=h} = 0. \quad (15)$$

In the absence of body forces, the vector (\mathbf{v}, w) of displacements in the elastic layer satisfies the Lamé system

$$\begin{aligned} A_{66}\Delta_y \mathbf{v} + (A_{11} - A_{66})\nabla_y \nabla_y \cdot \mathbf{v} + A_{44}\frac{\partial^2 \mathbf{v}}{\partial z^2} + (A_{13} + A_{44})\frac{\partial}{\partial z} \nabla_y w &= \mathbf{0}, \\ A_{44}\Delta_y w + A_{33}\frac{\partial^2 w}{\partial z^2} + (A_{13} + A_{44})\frac{\partial}{\partial z} \nabla_y \cdot \mathbf{v} &= 0. \end{aligned} \quad (16)$$

Assuming that the coated layer is loaded by a normal load and denoting the load density by p , we require that

$$\sigma_{33}|_{z=0} = -p. \quad (17)$$

Based on the analysis performed in Section 2, the influence of the elastic membrane (coating layer) is introduced by the boundary condition

$$\sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2|_{z=0} = -\hat{\mathcal{L}}(\nabla_y)\mathbf{v}|_{z=0}, \quad (18)$$

where $\hat{\mathcal{L}}(\nabla_y)$ is the matrix differential operator defined by formulas (14).

Taking into account the stress-strain relationship

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{12} & A_{11} & A_{13} & 0 & 0 & 0 \\ A_{13} & A_{13} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2A_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2A_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix},$$

we rewrite Eqs. (17), (18) as follows:

$$A_{13}\frac{\partial v_1}{\partial y_1} + A_{13}\frac{\partial v_2}{\partial y_2} + A_{33}\frac{\partial w}{\partial z}\Big|_{z=0} = -p, \quad (19)$$

$$A_{44}\left(\nabla_y w + \frac{\partial \mathbf{v}}{\partial z}\right)\Big|_{z=0} = -\hat{\mathcal{L}}(\nabla_y)\mathbf{v}\Big|_{z=0}. \quad (20)$$

Equations (15), (16), (19), and (20) comprise the deformation problem for the coated transversely isotropic elastic layer bonded to a rigid substrate.

Observe that the effect of the superficial tangential zone (modeled as an elastic membrane) has been introduced as an effective boundary condition (18). Such a surface membrane can also be considered as a material surface (with elasticity) in the framework of the surface elasticity theory of Gurtin and Murdoch [15]. The latter approach was recently used in a number of papers [12,21,33].

4 Asymptotic analysis of the deformation problem

Let h_* be a characteristic length of the external load distribution. Denoting by ε a small positive parameter, we require that

$$h = \varepsilon h_* \quad (21)$$

and introduce the so-called “stretched” dimensionless normal coordinate

$$\zeta = \frac{z}{\varepsilon h_*}.$$

In addition, we non-dimensionalize the in-plane coordinates by the formulas

$$\eta_i = \frac{y_i}{h_*}, \quad i = 1, 2, \quad \boldsymbol{\eta} = (\eta_1, \eta_2),$$

so that

$$\frac{\partial}{\partial z} = \frac{1}{\varepsilon h_*} \frac{\partial}{\partial \zeta}, \quad \nabla_y = \frac{1}{h_*} \nabla_\eta.$$

Moreover, we assume that the tensile stiffness of the coating layer is relatively high, i.e., $\hat{A}_{11} \gg A_{11}$, and so on. **For the sake of definiteness**, we consider the situation when

$$\hat{\mathcal{L}}(\nabla_y) = \varepsilon^{-1} \hat{\mathcal{L}}^*(\nabla_y), \quad (22)$$

that is, in particular, the ratio A_{11}/\hat{A}_{11} is of the order of $\varepsilon \hat{h}/h_*$.

Employing the perturbation algorithm [14], we represent the solution to the deformation problem (15), (16), (19), (20) as follows:

$$\mathbf{v} = \varepsilon^2 \mathbf{v}^1(\boldsymbol{\eta}, \zeta) + \dots, \quad (23)$$

$$w = \varepsilon w^0(\boldsymbol{\eta}, \zeta) + \varepsilon^3 w^2(\boldsymbol{\eta}, \zeta) + \dots. \quad (24)$$

Here, for the sake of brevity, we include only non-vanishing terms.

It can be shown (see, in particular, [3]) that the leading-order term in (24) is given by

$$w^0(\boldsymbol{\eta}, \zeta) = \frac{h_* p}{A_{33}} (1 - \zeta), \quad (25)$$

whereas the first non-trivial term of the expansion (23) satisfies the problem

$$A_{44} \frac{\partial^2 \mathbf{v}^1}{\partial \zeta^2} = -(A_{13} + A_{44}) \nabla_\eta \frac{\partial w^0}{\partial \zeta}, \quad \zeta \in (0, 1),$$

$$A_{44} \frac{\partial \mathbf{v}^1}{\partial \zeta} + \frac{1}{h_*} \hat{\mathbf{L}}^*(\nabla_\eta) \mathbf{v}^1 \Big|_{\zeta=0} = -A_{44} \nabla_\eta w^0 \Big|_{\zeta=0}, \quad \mathbf{v}^1 \Big|_{\zeta=1} = \mathbf{0}.$$

Substituting the expansion (25) for w^0 into the above equations, we obtain

$$A_{44} \frac{\partial^2 \mathbf{v}^1}{\partial \zeta^2} = \frac{A_{13} + A_{44}}{A_{33}} h_* \nabla_\eta p, \quad \zeta \in (0, 1),$$

$$A_{44} \frac{\partial \mathbf{v}^1}{\partial \zeta} + \frac{1}{h_*} \hat{\mathbf{L}}^*(\nabla_\eta) \mathbf{v}^1 \Big|_{\zeta=0} = -\frac{A_{44}}{A_{33}} h_* \nabla_\eta p, \quad \mathbf{v}^1 \Big|_{\zeta=1} = \mathbf{0}.$$
(26)

The solution to the boundary-value problem (26) is represented in the form

$$\mathbf{v}^1 = -\frac{A_{13} + A_{44}}{2A_{33}A_{44}} \zeta(1 - \zeta) h_* \nabla_\eta p + (1 - \zeta) \mathbf{V}^1(\boldsymbol{\eta}),$$
(27)

where $\mathbf{V}^1(\boldsymbol{\eta})$ satisfies the equation

$$\frac{1}{h_*} \hat{\mathbf{L}}^*(\nabla_\eta) \mathbf{V}^1 - A_{44} \mathbf{V}^1 = \frac{A_{13} - A_{44}}{2A_{33}} h_* \nabla_\eta p$$
(28)

on the entire plane $\zeta = 0$.

For the second non-trivial term of the expansion (24), we derive the problem

$$A_{33} \frac{\partial^2 w^2}{\partial \zeta^2} = -(A_{13} + A_{44}) \nabla_\eta \cdot \frac{\partial \mathbf{v}^1}{\partial \zeta} - A_{44} \Delta_\eta w^0, \quad \zeta \in (0, 1),$$

$$A_{33} \frac{\partial w^2}{\partial \zeta} \Big|_{\zeta=0} = -A_{13} \nabla_\eta \cdot \mathbf{v}^1 \Big|_{\zeta=0}, \quad w^2 \Big|_{\zeta=1} = 0.$$

Now, substituting the expressions (25) and (27) for w^0 and \mathbf{v}^1 , respectively, into the above equations, we arrive at the problem

$$\frac{\partial^2 w^2}{\partial \zeta^2} = -\left[(A_{13} + A_{44})^2 (2\zeta - 1) + 2A_{44}^2 (1 - \zeta) \right] \frac{h_* \Delta_\eta p}{2A_{33}^2 A_{44}}$$

$$+ \frac{A_{13} + A_{44}}{A_{33}} \nabla_\eta \cdot \mathbf{V}^1, \quad \zeta \in (0, 1),$$
(29)

$$\frac{\partial w^2}{\partial \zeta} \Big|_{\zeta=0} = -\frac{A_{13}}{A_{33}} \nabla_\eta \cdot \mathbf{V}^1, \quad w^2 \Big|_{\zeta=1} = 0.$$
(30)

Integrating Eq. (29) twice with respect to ζ and taking into account the boundary condition (30)₂, we obtain

$$w^2 = -\left[(A_{13} + A_{44})^2 (2\zeta^3 - 3\zeta^2 + 1) + 2A_{44}^2 (1 - \zeta)^3 \right] \frac{h_* \Delta_\eta p}{12A_{33}^2 A_{44}}$$

$$+ \frac{A_{13} + A_{44}}{2A_{33}} (1 - \zeta)^2 \nabla_\eta \cdot \mathbf{V}^1(\boldsymbol{\eta}) + C_2(\boldsymbol{\eta})(1 - \zeta),$$
(31)

where $C_2(\boldsymbol{\eta})$ is an arbitrary function.

The substitution of (31) into the boundary condition (30)₁ yields

$$C_2 = \frac{A_{44}}{2A_{33}^2} h_* \Delta_\eta p - \frac{A_{44}}{A_{33}} \nabla_\eta \cdot \mathbf{V}^1.$$

Hence, in view of this relation, formula (31) implies

$$w^2|_{\zeta=0} = - \left[(A_{13} + A_{44})^2 - 4A_{44}^2 \right] \frac{h_* \Delta_\eta p}{12A_{33}^2 A_{44}} + \frac{A_{13} - A_{44}}{2A_{33}} \nabla_\eta \cdot \mathbf{V}^1, \quad (32)$$

where \mathbf{V}^1 is the solution of Eq. (28).

5 Local indentation of the coated elastic layer: Leading-order asymptotics for the compressible and incompressible cases

Recall that the local indentation of an elastic layer is defined as

$$w_0(\mathbf{y}) \equiv w(\mathbf{y}, 0),$$

where $w(\mathbf{y}, 0)$ is the normal displacement of the layer surface.

In the case of compressible layer, Eqs. (24) and (25) yield

$$w_0(\mathbf{y}) \simeq \frac{h}{A_{33}} p(\mathbf{y}), \quad (33)$$

so that the deformation response of the coated elastic layer is analogous to that of a Winkler elastic foundation [19,28] with the foundation modulus $k = A_{33}/h$. In other words, the deformation of the elastic coating does not contribute substantially to the deformation of a thin compressible layer.

When the layer material approaches the incompressible limit, the right-hand side of (33) decreases to zero and the first term in the asymptotic expansion (24) disappears. At that, based on the known results [17], it can be shown that the ratios A_{13}/A_{33} and A_{44}/A_{33} tend to 1 and 0, respectively.

Therefore, in the incompressible limit situation formula (32) reduces to

$$w^2|_{\zeta=0} = - \frac{h_*}{12a_{44}} \Delta_\eta p(\boldsymbol{\eta}) + \frac{1}{2} \nabla_\eta \cdot \mathbf{V}^1(\boldsymbol{\eta}), \quad (34)$$

where $a_{44} = A_{44}$ is the out-of-plane shear modulus of the elastic layer, while $\mathbf{V}^1(\boldsymbol{\eta})$ satisfies the equation

$$\frac{1}{h_*} \hat{\mathcal{L}}^*(\nabla_\eta) \mathbf{V}^1(\boldsymbol{\eta}) - a_{44} \mathbf{V}^1(\boldsymbol{\eta}) = \frac{h_*}{2} \nabla_\eta p(\boldsymbol{\eta}), \quad \boldsymbol{\eta} \in \mathbb{R}^2. \quad (35)$$

Thus, in the case of incompressible bonded elastic layer, formulas (22)–(24), (34), and (35) give

$$w_0(\mathbf{y}) \simeq -\frac{h^3}{12a_{44}}\Delta_y p(\mathbf{y}) + \frac{h}{2}\nabla_y \cdot \mathbf{v}_0(\mathbf{y}), \quad (36)$$

where the vector $\mathbf{v}_0(\mathbf{y})$ solves the equation

$$h\hat{\mathcal{L}}(\nabla_y)\mathbf{v}_0(\mathbf{y}) - a_{44}\mathbf{v}_0(\mathbf{y}) = \frac{h^2}{2}\nabla_y p(\mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^2. \quad (37)$$

Observe that in view of (27), the vector-function $\mathbf{v}_0(\mathbf{y})$ has the meaning of the tangential displacement of the surface point $(\mathbf{y}, 0)$ of the elastic layer. Recall also that the matrix differential operator $\hat{\mathcal{L}}(\nabla_y)$ is defined by formulas (14). **It is also appropriate to comment here that the differential operator $\hat{\mathcal{L}}^*(\nabla_y)$ was introduced by formula (22) in such a way that both terms on the left-hand side of Eq. (37) turn out to be of the same order of magnitude as $\varepsilon \rightarrow 0$.**

6 Axisymmetric, periodic, and limit cases

Observe that in the axisymmetric case (in view of (12)), we have

$$\hat{\sigma}_{rr} = \hat{A}_{11}\varepsilon_{rr} + \hat{A}_{12}\varepsilon_{\theta\theta}, \quad \hat{\sigma}_{\theta\theta} = \hat{A}_{12}\varepsilon_{rr} + \hat{A}_{11}\varepsilon_{\theta\theta}, \quad \hat{\sigma}_{r\theta} = 0,$$

where

$$\varepsilon_{rr} = \frac{\partial v_r}{\partial r}, \quad \varepsilon_{\theta\theta} = \frac{v_r}{r},$$

while Eqs. (6) should be replaced with the following one:

$$\frac{\hat{h}}{r} \left(\frac{\partial(r\hat{\sigma}_{rr})}{\partial r} - \hat{\sigma}_{\theta\theta} \right) = -\sigma_{zr}|_{z=0}.$$

Correspondingly, the boundary condition (13) takes the following form:

$$\sigma_{zr}|_{z=0} = -\hat{h}\hat{A}_{11} \left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right). \quad (38)$$

It is to note here that the axisymmetric boundary condition (38) was previously derived in a number of papers [2,8,29,30].

Further, as a simple example (which is relevant to the contact of wavy and rough surfaces [19,27,32]), we consider an example of periodic loading

$$p(y_1) = p_0 \left(1 + \cos \frac{\pi y_1}{l} \right), \quad (39)$$

where p_0 is the maximum of distributed normal stresses with the spatial period $2l$.

In this case, Eqs. (36) and (37) reduce to the form

$$w_0 = -\frac{h^3}{12a_{44}} \frac{d^2 p}{dy_1^2} + \frac{h}{2} \frac{dv_0}{dy_1}, \quad (40)$$

$$h\hat{h}\hat{A}_{11} \frac{d^2 v_0}{dy_1^2} - a_{44}v_0 = \frac{h^2}{2} \frac{dp}{dy_1}. \quad (41)$$

Substituting (39) into Eq. (41), we readily get

$$v_0(y_1) = \frac{h p_0}{2\pi\hat{h}\hat{A}_{11}(1+\mu)} \sin \frac{\pi y_1}{l}, \quad (42)$$

where we have introduced the notation

$$\mu = \frac{l^2 a_{44}}{\pi^2 h \hat{h} \hat{A}_{11}}. \quad (43)$$

Now, the substitution of (39) and (42) into Eq. (40) yields

$$w_0(y_1) = \frac{\pi^2 h^3 p_0 (1+4\mu)}{12l^2 a_{44} (1+\mu)} \cos \frac{\pi y_1}{l}. \quad (44)$$

The variation of the maximum indentation $w_0(0)$ with the change of the dimensionless parameter (43) is represented in Fig. 2.

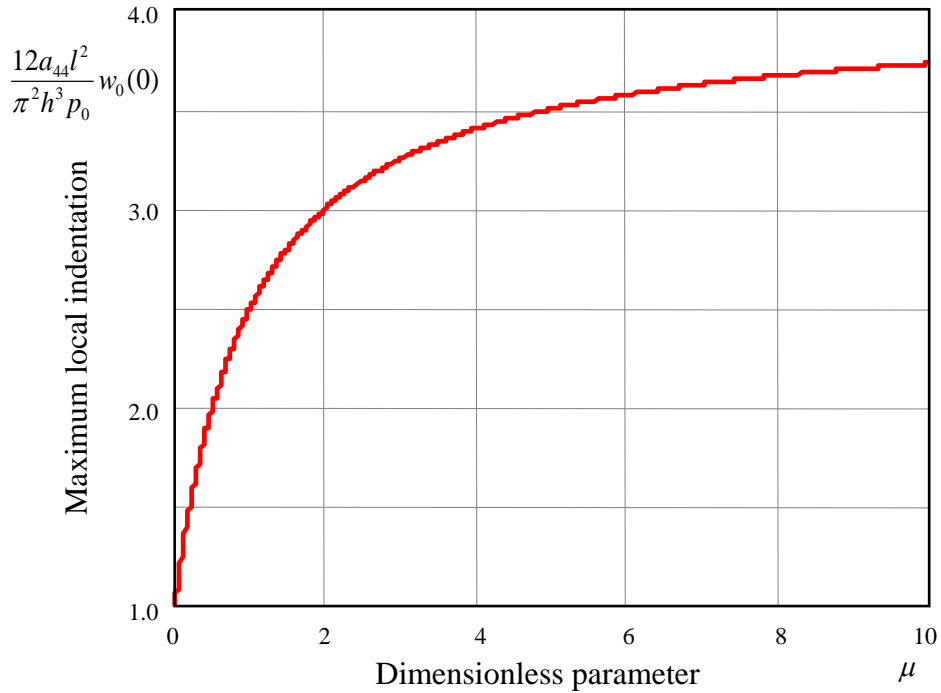


Figure 2. The normalized maximum local indentation.

Finally, let us consider two opposite limit situations in the incompressible case. First, when the coating is absent and $\hat{\mathbf{L}}(\nabla_y) \equiv 0$, Eq. (37) implies

$$\mathbf{v}_0(\mathbf{y}) = -\frac{h^2}{2a_{44}}\nabla_y p(\mathbf{y}).$$

The substitution of this expression into formula (36) gives

$$w_0(\mathbf{y}) \simeq -\frac{h^3}{3a_{44}}\Delta_y p(\mathbf{y}), \quad (45)$$

which completely agrees with the asymptotic model developed in [4,5,6] (see also [10,18]).

Second, in the case of very stiff (inextensible) coating we will have $\mathbf{v}_0(\mathbf{y}) \equiv \mathbf{0}$, and formula (36) reduces to

$$w_0(\mathbf{y}) \simeq -\frac{h^3}{12a_{44}}\Delta_y p(\mathbf{y}). \quad (46)$$

In other words, comparing (45) and (46), we conclude that the inextensible membrane coating attached to the surface of a thin incompressible elastic layer reduces the out-of-plane shear compliance of the bonded layer by four times. **Therefore, the reinforcing effect of an elastic membrane on the surface of an incompressible elastic layer, which is governed by the dimensionless ratio $\hat{h}\hat{A}_{11}/(ha_{44})$, will be less than this value.**

7 Discussion and conclusion

We note that the employed above perturbation technique was widely used for deriving the transmission conditions for thin elastic interfaces (see, e.g., [11,22]). By using FEM, the accuracy of the transmission conditions derived by asymptotic analysis was analyzed in detail in [23,24,25].

The obtained results can shed light on the experimentally established fact [9] that the indentation stiffness of the articular cartilage (see also [7]) is sensitive to mild degeneration at the articular surface (which reduces the stiffness of the superficial tissue layer), while being insensitive to changes associated with normal aging or to slight variations in cartilage thickness.

Note also that the problem examined in this paper can be of potential interest for its application in geophysics and foundation mechanics, where reinforcing layers are used to enhance the stiffness characteristics of geomaterials (see, e.g., [31] and [1], where the cases of inextensible and elastic membrane utilized for reinforcing an elastic half-space are considered).

The main result of the paper, presented in Section 4, is the leading-order asymptotic formulas for the normal displacement (called the local indentation) of the surface points for compressible and incompressible thin bonded elastic layers reinforced with an elastic membrane.

8 Acknowledgment

The authors are grateful for support from the FP7 IRSES Marie Curie grant TAMER No 610547.

References

- [1] S.F. Ahmadi, M. Eskandari, Axisymmetric circular indentation of a half-space reinforced by a buried elastic thin film, *Math. Mech. Solids* 19 (2014) 703–712.
- [2] V.M. Alexandrov, S.M. Mkhitarian, Contact Problems for Solids with Thin Coatings and Layers [in Russian], Nauka, Moscow, 1985.
- [3] I.I. Argatov, Pressure of a paraboloidal die on a thin elastic layer, *Doklady Phys.* 50 (2005) 524–528.
- [4] I.I. Argatov, Development of an asymptotic modeling methodology for tibio-femoral contact in multibody dynamic simulations of the human knee joint, *Multibody Syst. Dyn.* 28 (2012) 3–20.
- [5] I. Argatov, G. Mishuris, Elliptical contact of thin biphasic cartilage layers: Exact solution for monotonic loading, *J. Biomech.* 44 (2011) 759–761.
- [6] I. Argatov, G. Mishuris, Frictionless elliptical contact of thin viscoelastic layers bonded to rigid substrates, *Appl. Math. Model.* 35 (2011) 3201–3212.
- [7] I.I. Argatov, F.J. Sabina, Spherical indentation of a transversely isotropic elastic half-space reinforced with a thin layer, *Int. J. Eng. Sci.* 50 (2012) 132–143.
- [8] V.I. Avilkin, V.M. Alexandrov, E.V. Kovalenko, On using the more-accurate equations of thin coatings in the theory of axisymmetric contact problems for composite foundations, *J. Appl. Math. Mech.* 49 (1985) 770–777.
- [9] W.C. Bae, M.M. Temple, D. Amiel, R.D. Coutts, G.G. Niederauer, R.L. Sah, Indentation testing of human cartilage: Sensitivity to articular surface degeneration, *Arthritis Rheum.* 48 (2003) 3382–3394.
- [10] J.R. Barber, Contact problems for the thin elastic layer. *Int. J. Mech. Sci.* 32 (1990) 129–132.
- [11] Y. Benveniste, T. Miloh, Imperfect soft and stiff interfaces in two-dimensional elasticity, *Mech. Mater.* 33 (2001) 309–323.
- [12] W.Q. Chen, Ch. Zhang, Anti-plane shear Green's functions for an isotropic elastic half-space with a material surface, *Int. J. Solids Struct.* 47 (2010) 1641–1650.
- [13] I.C. Clarke, Articular cartilage: a review and scanning electron microscope study. 1. The interterritorial fibrillar architecture, *J. Bone Joint Surg. Br.* 53-B (1971) 732–750.
- [14] A.L. Gol'denveizer, Derivation of an approximate theory of bending of a plate by the method of asymptotic integration of the equations of the theory of elasticity, *J. Appl. Math. Mech.* 26 (1962) 1000–1025.
- [15] M.E. Gurtin, A.I. Murdoch, A continuum theory of elastic material surfaces, *Arch. Rat. Mech. Anal.* 57 (1975) 291–323.

- [16] L.C. Hughes, C.W. Archer, I. ap Gwynn, The ultrastructure of mouse articular cartilage: collagen orientation and implications for tissue functionality. A polarised light and scanning electron microscope study and review, *Europ. Cells Materials* 9 (2005) 68–84.
- [17] M. Itskov, N. Aksel, Elastic constants and their admissible values for incompressible and slightly compressible anisotropic materials. *Acta Mech.* 157 (2002) 81–96.
- [18] M.J. Jaffar, Asymptotic behaviour of thin elastic layers bonded and unbonded to a rigid foundation. *Int. J. Mech. Sci.* 31 (1989) 229–235.
- [19] K.L. Johnson, *Contact Mechanics*. Cambridge Univ. Press, Cambridge, UK, 1985.
- [20] R.K. Korhonen, M. Wong, J. Arokoski, R. Lindgren, H.J. Helminen, E.B. Hunziker, J.S. Jurvelin, Importance of the superficial tissue layer for the indentation stiffness of articular cartilage, *Med. Eng. Phys.* 24 (2002) 99–108.
- [21] J.M. Long, G.F. Wang, Effects of surface tension on axisymmetric Hertzian contact problem, *Mech. Mater.* 56 (2013) 65–70.
- [22] G. Mishuris, Imperfect transmission conditions for a thin weakly compressible interface. 2D problems, *Arch. Mech.* 56 (2004) 103–115.
- [23] G. Mishuris, A. Öchsner, Edge effects connected with thin interphases in composite materials, *Compos. Struct.* 68 (2005) 409–417.
- [24] G. Mishuris, A. Öchsner, 2D modelling of a thin elasto-plastic interphase between two different materials: Plane strain case, *Compos. Struct.* 80 (2007) 361–372.
- [25] A. Öchsner, G. Mishuris, A new finite element formulation for thin non-homogeneous heat-conducting adhesive layers, *J. Adhes. Sci. Technol.* 22 (2008) 1365–1378.
- [26] R.V. Patel, J.J. Mao, Microstructural and elastic properties of the extracellular matrices of the superficial zone of neonatal articular cartilage by atomic force microscopy, *Front. Biosci.* 8 (2003) a18–25.
- [27] V.L. Popov, *Contact Mechanics and Friction*. Springer, New York, 2010.
- [28] V.L. Popov, M. Heß, *Method of Dimensionality Reduction in Contact Mechanics and Friction*. Springer-Verlag, Berlin, Heidelberg, 2015.
- [29] M. Rahman, G. Newaz, Elastostatic surface displacements of a half-space reinforced by a thin film due to an axial ring load, *Int. J. Eng. Sci.* 35 (1997) 603–611.
- [30] M. Rahman, G. Newaz, Boussinesq type solution for a transversely isotropic half-space coated with a thin film, *Int. J. Eng. Sci.* 38 (2000) 807–822.
- [31] A.P.S. Selvadurai, Boussinesq indentation of an isotropic elastic halfspace reinforced with an inextensible membrane, *Int. J. Eng. Sci.* 47 (2009) 1339–1345.
- [32] S. Stupkiewicz, *Micromechanics of contact and interface layers. Lecture Notes in Applied and Computational Mechanics*, vol. 30. Springer, Berlin, 2007.
- [33] S. Zhou, X.-L. Gao, Solutions of half-space and half-plane contact problems based on surface elasticity, *Z. Angew. Math. Phys.* 64 (2013) 145–166.