Impact of the tangential traction for radial hydraulic fracture

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Impact of the tangential traction for radial hydraulic fracture
Supplementary material

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A Derivation of the updated elasticity equation

Updateroo2

The derivation of the elasticity equation accounting for tangential traction was previously provided in [3] (a similar form was also derived independently in [7]), but is included here for completeness. We consider a 3D penny-shaped crack, defined in polar coordinates by the system $\{r, \theta, z\}$, with associated crack dimensions $\{l(t), w(t)\}$ as the fracture radius and aperture respectively. As the flow is axisymmetric, all variables will be independent of the angle θ.

The equation for the net fluid pressure on the fracture walls (i.e. $p = p_f - \sigma_0$, σ_0 is the confining stress), including the tangential stress term, is given in Cartesian coordinates (x_1, x_2, x_3) by [5]:

$$\begin{aligned} p(r,t) = {} & \frac{E}{8\pi(1-\nu^2)} \int_\Omega \frac{1}{\sqrt{(x_1-\xi_1)^2+(x_3-\xi_3)^2}} \left[\frac{\partial^2 w}{\partial \xi_1^2} + \frac{\partial^2 w}{\partial \xi_3^2} \right] d\xi_1 d\xi_3 \\ & - \frac{1-2\nu}{8\pi(1-\nu)} \int_\Omega \frac{1}{\sqrt{(x_1-\xi_1)^2+(x_3-\xi_3)^2}} \left[\frac{\partial [[p_{\xi_1}]]}{\partial \xi_1} + \frac{\partial [[p_{\xi_3}]]}{\partial \xi_3} \right] d\xi_1 d\xi_3. \end{aligned} \tag{A.1}$$

Here $[[x]]$ indicates the jump in x (i.e. $[[p]] = p_+ - p_-$), $\Omega = \left\{ (x_1, x_3) : \sqrt{x_1^2 + x_3^2} \leq l(t) \right\}$ is the fracture domain, while E and ν are the Young's modulus and Poisson ratio respectively.

As the problem is invariant of the angle θ, the pressure term can be obtained by transforming this into radial coordinates (r, θ), integrated with respect to the corresponding variables (η_1, η_2). We obtain the relationship:

$$\begin{aligned} p(r,t) = {} & \frac{E}{8\pi(1-\nu^2)} \int_0^{l(t)} \frac{\partial}{\partial \eta_1} \left(\eta_1 \frac{\partial w(\eta_1, t)}{\partial \eta_1} \right) \int_0^{2\pi} \frac{1}{\sqrt{r^2 + \eta_1^2 - 2r\eta_1 \cos(\theta - \eta_2)}} d\eta_2 d\eta_1 \\ & - \frac{1-2\nu}{4\pi\,(1-\nu)} \int_0^{l(t)} \frac{\partial}{\partial \eta_1} \left(\eta_1 \tau\left(\eta_1, t\right) \right) \int_0^{2\pi} \frac{1}{\sqrt{r^2 + \eta_1^2 - 2r\eta_1 \cos(\theta - \eta_2)}} d\eta_2 \, d\eta_1. \end{aligned} \tag{A.2}$$

It can be shown that:

$$\int_0^{2\pi} \frac{1}{\sqrt{r^2 + \eta_1^2 - 2r\eta_1 \cos(\theta - \eta_2)}} d\eta_2 = \frac{4K\left(\frac{4r\eta_1}{(\eta_1 + r)^2} \right)}{|\eta_1 + r|}, \tag{A.3}$$

int3

where $K(x)$ is the complete elliptic integral of the first kind [1].

Inserting this, before using integration by parts, gives:

$$p(r,t) = -\int_0^{l(t)} \left[k_2 \frac{\partial w}{\partial \eta_1} - k_1 \tau(\eta_1)\right] \mathcal{M}\left(r, \eta_1\right)\, d\eta_1, \tag{A.4}$$

newPre

substituting the dimensionless variable $\rho = \eta_1 / l(t)$, we have:

$$p(r,t) = -\frac{1}{l(t)} \int_0^1 \left[k_2 \frac{\partial w(\rho l(t))}{\partial \rho} - k_1 l(t) \tau(\rho l(t))\right] \mathcal{M}\left(\frac{r}{l(t)}, \rho\right)\, d\rho, \tag{A.5}$$

newPre

where:

$$\begin{aligned} \mathcal{M}(\tilde{r}, \rho) &= \frac{1}{2(\tilde{r}+\rho)} K\left(\frac{4\tilde{r}\rho}{(\rho+\tilde{r})^2}\right) - \frac{1}{2(\tilde{r}-\rho)} E\left(\frac{4\tilde{r}\rho}{(\rho+\tilde{r})^2}\right) \\ &= \frac{1}{2(\tilde{r}+\rho)} K\left(1 - \left(\frac{\rho-\tilde{r}}{\rho+\tilde{r}}\right)^2\right) - \frac{1}{2(\tilde{r}-\rho)} E\left(1 - \left(\frac{\rho-\tilde{r}}{\rho+\tilde{r}}\right)^2\right), \end{aligned} \tag{A.6}$$

KerNew

$$k_1 = \frac{1-2\nu}{\pi(1-\nu)}, \quad k_2 = \frac{E}{2\pi(1-\nu^2)}. \tag{A.7}$$

Here $E(x)$ is the complete elliptic integral of the second kind [1]. It can be shown numerically that, within the corresponding domains ($0 \le \tilde{r} \le 1$, $0 \le \rho \le 1$), this kernel function $\mathcal{M}$ is merely an alternative representation of the standard kernel for this problem [2]:

$$\mathcal{M}\left[\tilde{r}, \rho\right] = \begin{cases} \frac{1}{\tilde{r}} K\left(\frac{\rho^2}{\tilde{r}^2}\right) + \frac{\tilde{r}}{\rho^2 - \tilde{r}^2} E\left(\frac{\rho^2}{\tilde{r}^2}\right), & \tilde{r} > \rho \\ \frac{\rho}{\rho^2 - \tilde{r}^2} E\left(\frac{\tilde{r}^2}{\rho^2}\right), & \rho > \tilde{r} \end{cases} \tag{A.8}$$

M1b

Additionally, it is worth noting that the constants k_1, k_2, are identical to those obtained in the KGD case [9].

With the elasticity equation with tangential stresses obtained, it is clear that the next step is to invert the operator and obtain the inverse relation. This is achieved by noting that we can place (A.5) in the form:

$$p(r,t) = -\int_0^{l(t)} g'(x) \mathcal{M}\left(r, x\right)\, dx, \quad 0 < r < 1, \tag{A.9}$$

where:

$$g(r) = \int_r^{l(t)} \left(k_1 \tau(s,t) - k_2 \frac{\partial w}{\partial s}\right) ds = k_2 w(r,t) + k_1 \int_r^{l(t)} \tau(s,t)\, ds. \tag{A.10}$$

From this, it immediately follows that the inverse relation must be (compare with the classical result, see e.g. [6]):

$$k_2 w(r,t) + k_1 \int_r^{l(t)} \tau(s,t)\, ds = \frac{4}{\pi^2} l(t) \int_{r/l(t)}^1 \frac{\xi}{\sqrt{\xi^2 - (r/l(t))^2}} \int_0^1 \frac{\eta p(\eta \xi l(t), t)}{\sqrt{1-\eta^2}}\, d\eta d\xi, \tag{A.11}$$

Nobel_InvElast

Following the steps previously outlined in [4], this can alternatively be written in the form:

$$k_2 w(r,t) + k_1 \int_r^{l(t)} \tau\left(s,t\right)\, ds = \tag{A.12}$$

Nobel_InvElast

$$\frac{4}{\pi^2} l(t) \left[\int_0^1 \frac{\partial p(y l(t), t)}{\partial y} \mathcal{K}\left(y, \frac{r}{l(t)}\right)\, dy + \sqrt{1 - \left(\frac{r}{l(t)}\right)^2} \int_0^1 \frac{\eta p(\eta l(t), t)}{\sqrt{1-\eta^2}}\, d\eta\right],$$

where:

$$\mathcal{K}(y,\tilde{r}) = y\left[E\left(\arcsin(y)\,\middle|\,\frac{\tilde{r}^2}{y^2}\right) - E\left(\arcsin(\psi)\,\middle|\,\frac{\tilde{r}^2}{y^2}\right)\right], \quad \psi = \min\left(\frac{y}{\tilde{r}},1\right), \tag{A.13}$$

with $E\,(\phi\,|\,m)$ denoting the incomplete elliptic integral of the second kind.

B Normalized form of the governing equations

Append_Norm

B.1 Normalization

We introduce the following normalization scheme:

$$\begin{gathered}
\tilde{r} = \frac{r}{l(t)}, \quad \tilde{t} = \frac{t}{t_n}, \quad \tilde{w}(\tilde{r},\tilde{t}) = \frac{w(r,t)}{l_*}, \quad L(\tilde{t}) = \frac{l(t)}{l_*}, \quad \tilde{q}_l(\tilde{r},\tilde{t}) = \frac{t_n}{l_*}q_l(r,t),\\
\tilde{q}(\tilde{r},\tilde{t}) = \frac{t_n}{l_*^2}q(r,t), \quad \tilde{Q}_0(\tilde{t}) = \frac{t_n}{2\pi l_*^2 l(t)}Q_0(t), \quad \tilde{v}(\tilde{r},\tilde{t}) = \frac{t_n}{l_*}v(r,t), \quad \tilde{\tau}(\tilde{r},\tilde{t}) = \frac{t_n}{M}\tau(r,t),\\
\tilde{p}(\tilde{r},\tilde{t}) = \frac{t_n}{M}p(r,t), \quad \tilde{K}_{\{Ic,I,f\}} = \frac{\gamma}{\sqrt{l_*}}K_{\{Ic,I,f\}}, \quad \tilde{\chi}(\tilde{r}) = \frac{\chi(r,t)}{l(t)}, \quad t_n = \frac{M}{k_2},
\end{gathered} \tag{B.1}$$

NobelNormaliza

where $\tilde{r}\in[0,1]$ and l_* is chosen for convenience.

Under the normalization scheme provided in (B.1), the Poiseuille equation provides the following relation for the fluid velocity (2.14):

$$\tilde{v} = -\frac{\tilde{w}^2}{L(\tilde{t})}\frac{\partial\tilde{p}}{\partial\tilde{r}}, \tag{B.2}$$

Npv

while the tangential (sheer) stress is now given by (2.18):

$$\tilde{\tau} = -\frac{1}{2}\frac{\tilde{\chi}}{L(\tilde{t})}\tilde{w}\frac{\partial\tilde{p}}{\partial\tilde{r}}, \tag{B.3}$$

Nobel_taudefNo

where $\tilde{\chi}$ (2.20) is given by

$$\tilde{\chi}(\tilde{r}) = 1-(1-\tilde{r})^{\beta}, \quad \beta\geq 1. \tag{B.4}$$

final_ss_rstar

As such the fluid mass balance equation (2.1), alongside the global balance equation (2.13), become:

$$\frac{\partial\tilde{w}}{\partial\tilde{t}} - \frac{L'(\tilde{t})}{L(\tilde{t})}\tilde{r}\frac{\partial\tilde{w}}{\partial\tilde{r}} + \frac{1}{\tilde{r}L(\tilde{t})}\frac{\partial}{\partial\tilde{r}}\left(\tilde{r}\tilde{w}\tilde{v}\right) + \tilde{q}_l = 0, \tag{B.5}$$

Nobel_fluidmas

$$\int_0^1 \tilde{r}\left[L^2(\tilde{t})\tilde{w}(\tilde{r},\tilde{t}) - L^2(0)\tilde{w}_*(\tilde{r})\right]\,d\tilde{r} + \int_0^{\tilde{t}}\int_0^1 \tilde{r}L^2(s)\tilde{q}_l(\tilde{r},s)\,d\tilde{r}\,ds = \int_0^{\tilde{t}} L(s)\tilde{Q}_0(s)\,ds. \tag{B.6}$$

Nobel_globalba

The elasticity equation (2.3) takes the form:

$$\tilde{p}(\tilde{r},\tilde{t}) = -\frac{1}{L(\tilde{t})}\int_0^1\left[\frac{\partial\tilde{w}}{\partial\eta} - k_1L(\tilde{t})\tilde{\tau}(\eta,\tilde{t})\right]\mathcal{M}\left[\tilde{r},\eta\right]\,d\eta, \tag{B.7}$$

Nobel_ElastNor

alongside associated inverse (2.4):

$$\tilde{w}(\tilde{r},\tilde{t}) + k_1L(\tilde{t})\int_{\tilde{r}}^1 \tilde{\tau}(s,\tilde{t})\,ds = \frac{4}{\pi^2}L(\tilde{t})\left[\int_0^1\frac{\partial\tilde{p}(y,\tilde{t})}{\partial y}\mathcal{K}(y,\tilde{r})\,dy + \sqrt{1-\tilde{r}^2}\int_0^1\frac{\eta\tilde{p}(\eta,\tilde{t})}{\sqrt{1-\eta^2}}\,d\eta\right], \tag{B.8}$$

NormElast2

where the kernel $\mathcal{K}$ is given by (2.6). By evaluating the asymptotic limit of (B.8) at the crack tip, it can be shown that:

$$\tilde{w}_0 + k_1\tilde{w}_0\tilde{p}_0 = \frac{4\sqrt{2}}{\pi^2}L(\tilde{t})\int_0^1 \frac{\eta\tilde{p}(\eta,\tilde{t})}{\sqrt{1-\eta^2}}\,d\eta, \tag{B.9}$$

NewElast

which replaces the standard integral definition of the stress intensity factor.

The boundary conditions for the problem (2.10)-(2.11) are now given by:

$$\lim_{\tilde{r}\to 0}\tilde{r}\tilde{w}\tilde{v} = \tilde{Q}_0, \tag{B.10}$$

NQ0

$$\tilde{w}(1,\tilde{t}) = 0, \quad \tilde{q}(1,\tilde{t}) = 0, \tag{B.11}$$

where the system has initial conditions (2.12):

$$\tilde{w}(\tilde{r},0) = \tilde{w}_*(r), \quad L(0) = L_0, \tag{B.12}$$

NQ1

The crack tip asymptotics (2.22)-(2.24) now take the form:

$$\tilde{w}(\tilde{r},\tilde{t}) = \tilde{w}_0(\tilde{t})\sqrt{1-\tilde{r}} + \tilde{w}_1(\tilde{t})\,(1-\tilde{r}) + \tilde{w}_2(\tilde{t})\,(1-\tilde{r})^{\frac{3}{2}}\log(1-\tilde{r}) + \dots, \quad \tilde{r}\to 1, \tag{B.13}$$

wasym2

$$\tilde{p}(\tilde{r},\tilde{t}) = \tilde{p}_0(\tilde{t})\log(1-\tilde{r}) + \tilde{p}_1(\tilde{t}) + \tilde{p}_2(\tilde{t})\sqrt{1-\tilde{r}} + \tilde{p}_3\,(1-\tilde{r})\log(1-\tilde{r}) + \dots, \quad \tilde{r}\to 1, \tag{B.14}$$

pasym2

$$\tilde{v}(\tilde{r},\tilde{t}) = \tilde{v}_0(\tilde{t}) + \tilde{v}_1(\tilde{t})\sqrt{1-\tilde{r}} + \dots, \tilde{r}\to 1, \tag{B.15}$$

vasym2

We note that we can rewrite the parameter $\bar{\omega}$ $(2.35)_3$ as:

$$\tilde{\omega} = \frac{\tilde{p}_0}{\pi(1-\nu)-\tilde{p}_0}, \quad 0 < \tilde{p}_0 < \pi(1-\nu), \tag{B.16}$$

Nobel_OmegaNor

as such, the first term of the aperture asymptotics at the fracture tip are given by (2.33), (2.36):

$$\tilde{w}_0(\tilde{t}) = \sqrt{L(\tilde{t})}\frac{1+\tilde{\omega}}{\sqrt{1+4(1-\nu)\tilde{\omega}}}\tilde{K}_{Ic} = \sqrt{L(\tilde{t})}\left[\tilde{K}_I + \tilde{K}_f\right], \tag{B.17}$$

w024

while the stress intensity factors (2.34) are described by:

$$\tilde{K}_I = \frac{\tilde{K}_{Ic}}{\sqrt{1+4(1-\nu)\tilde{\omega}}}, \quad \tilde{K}_f = \frac{\tilde{K}_{Ic}\tilde{\omega}}{\sqrt{1+4(1-\nu)\tilde{\omega}}}, \tag{B.18}$$

Nobel_StressIn

The Steffan condition (2.16), utilizing the Poiseuille equation (B.2) and terms from the asymptotic representation (B.13)-(B.14), can be expressed as:

$$\frac{dL}{d\tilde{t}} = \tilde{v}_0(\tilde{t}) = -\frac{1}{L(\tilde{t})}\lim_{\tilde{r}\to 1}\tilde{w}^2\frac{\partial\tilde{p}}{\partial\tilde{r}} = \frac{\tilde{w}_0^2\tilde{p}_0}{L(\tilde{t})}, \tag{B.19}$$

Steffan1

Utilizing (B.17), we can rewrite this condition as follows:

$$\frac{1}{\tilde{K}_{Ic}^2}\tilde{v}_0 = \tilde{p}_0 F(\tilde{p}_0), \tag{B.20}$$

NewRel00k1

where:

$$F(\tilde{p}_0) = \frac{\pi^2(1-\nu)^2}{\left[\pi(1-\nu)+(3-4\nu)\tilde{p}_0\right]\left[\pi(1-\nu)-\tilde{p}_0\right]}. \tag{B.21}$$

Fdef2

Noting the above definition, we can rewrite (B.17) in the form:

$$\tilde{w}_0(\tilde{t}) = \tilde{K}_{Ic}\sqrt{L(\tilde{t})F(\tilde{p}_0)}, \tag{B.22}$$

NewRel00k2

Further, by integrating (B.19), we can obtain a formula for the crack length:

$$L(\tilde{t}) = \sqrt{L^2(0) + 2\int_0^{\tilde{t}}\tilde{w}_0^2(s)\tilde{p}_0(s)\,ds}. \tag{B.23}$$

Norm_CrackLeng

B.2 The self-similar formulation

As we are incorporating the effect of tangential traction, it is not possible to obtain a self-similar solution of power-law type. Instead, an exponential variant must be obtained, similar to that utilized in [8]. We formulate utilize the following separation of variables:

$$\tilde{Q}_0(\tilde{t}) = \hat{Q}_0 e^{2\alpha\tilde{t}}. \tag{B.24}$$

SSdef1Nobel

This can be obtained by assuming the parameters take the form:

$$\begin{gathered}
\tilde{w}(\tilde{r},\tilde{t}) = \sqrt{L_0\hat{Q}_0}\, e^{\alpha\tilde{t}}\hat{w}(\tilde{r}), \quad L(t) = L_0^{\frac{3}{2}}\sqrt{\hat{Q}_0}\, e^{\alpha\tilde{t}}, \quad \tilde{p}(\tilde{r},\tilde{t}) = \hat{p}(\tilde{r}), \\
\tilde{q}_l(\tilde{r},\tilde{t}) = \alpha\sqrt{L_0\hat{Q}_0}\, e^{\alpha\tilde{t}}\hat{q}_l(\tilde{r}), \quad \tilde{v}(\tilde{r},\tilde{t}) = \sqrt{\frac{\hat{Q}_0}{L_0}}\, e^{\alpha\tilde{t}}\hat{v}(\tilde{r}), \quad \tilde{q}(\tilde{r},\tilde{t}) = \hat{Q}_0 e^{2\alpha\tilde{t}}\hat{q}(\tilde{r}), \\
\tilde{K}_{\{Ic,I,f\}} = \left(L_0\hat{Q}_0\right)^{\frac{1}{4}} e^{\frac{\alpha\tilde{t}}{2}}\hat{K}_{\{Ic,I,f\}}, \quad \tilde{\tau}(\tilde{r},\tilde{t}) = \hat{\tau}(\tilde{r}),
\end{gathered} \tag{B.25}$$

where:

$$\hat{v}_0 = \hat{v}(1), \quad L_0 = \sqrt{\frac{\hat{v}_0}{\alpha}}. \tag{B.26}$$

L0def

Under this scheme, the Poiseuille equation provides the following relation for the fluid velocity (B.2):

$$\hat{v}(\tilde{r}) = -\hat{w}^2(\tilde{r})\frac{d\hat{p}(\tilde{r})}{d\tilde{r}}, \tag{B.27}$$

PvSS5

The tangential traction (B.3) is now given as

$$\hat{\tau}(\tilde{r}) = -\frac{\tilde{\chi}(\tilde{r})}{2L_0}\hat{w}(\tilde{r})\frac{d\hat{p}}{d\tilde{r}}, \tag{B.28}$$

where the term $\tilde{\chi}$ is given by (B.4).

As such the fluid mass balance equation (B.5), alongside the global balance equation (B.6), become:

$$\hat{w} - \tilde{r}\frac{d\hat{w}}{d\tilde{r}} + \frac{1}{\hat{v}_0\tilde{r}}\frac{d}{d\tilde{r}}\left(\tilde{r}\hat{w}\hat{v}\right) + \hat{q}_l = 0, \tag{B.29}$$

Nobel_SSFluidM

$$3\int_0^1 \tilde{r}\hat{w}(\tilde{r})\, d\tilde{r} + \int_0^1 \tilde{r}\hat{q}_l(\tilde{r})\, d\tilde{r} = \frac{1}{\hat{v}_0}, \tag{B.30}$$

SSbalance5

The elasticity equation (B.7) takes the form:

$$\hat{p}(\tilde{r}) = -\frac{1}{L_0}\int_0^1 \left[\frac{d\hat{w}}{d\eta} + \frac{k_1}{2}\tilde{r}(\eta)\hat{w}(\eta)\frac{d\hat{p}}{d\eta}\right]\mathcal{M}\left[\tilde{r},\eta\right]\, d\eta, \tag{B.31}$$

the associated inverse (B.8) is simplified following the approach from [4], to become:

$$\hat{w}(\tilde{r}) = \frac{4}{\pi^2}L_0\left[\int_0^1 \frac{d\hat{p}}{dy}\mathcal{K}(y,\tilde{r})\, dy + \sqrt{1-\tilde{r}^2}\int_0^1 \frac{\eta\hat{p}(\eta)}{\sqrt{1-\eta^2}}\, d\eta\right] + \frac{k_1}{2}\int_{\tilde{r}}^1 \tilde{\chi}(s)\hat{w}(s)\frac{d\hat{p}}{ds}\, ds, \tag{B.32}$$

Nobel_SSElasti

By evaluating the asymptotic limit of (B.32) at the crack tip, it can be shown that:

$$\hat{w}_0 + k_1\hat{w}_0\hat{p}_0 = \frac{4\sqrt{2}}{\pi^2}L_0\int_0^1 \frac{\eta\hat{p}(\eta)}{\sqrt{1-\eta^2}}\, d\eta, \tag{B.33}$$

Nobel_SSElasti

which replaces the standard integral definition of the stress intensity factor.

Meanwhile, the source intensity and boundary conditions (B.10) - (B.12) are given by:

$$\lim_{\tilde{r}\to 0} \tilde{r}\hat{w}\hat{v} = 1, \tag{B.34}$$

$$\hat{w}(1) = 0, \quad \hat{q}(1) = 0, \tag{B.35}$$

The crack tip asymptotics (B.13), (B.14), (B.15) now take the form:

$$\hat{w}(\tilde{r}) = \hat{w}_0\sqrt{1-\tilde{r}} + \hat{w}_1\left(1-\tilde{r}\right) + \hat{w}_2\left(1-\tilde{r}\right)^{\frac{3}{2}}\log\left(1-\tilde{r}\right) + \ldots, \quad \tilde{r}\to 1, \tag{B.36} \qquad \texttt{wasym8}$$

$$\hat{p}(\tilde{r}) = \hat{p}_0\log\left(1-\tilde{r}\right) + \hat{p}_1 + \hat{p}_2\sqrt{1-\tilde{r}} + \hat{p}_3\left(1-\tilde{r}\right)\log\left(1-\tilde{r}\right) + \ldots, \quad \tilde{r}\to 1, \tag{B.37} \qquad \texttt{pasym8}$$

$$\hat{v}(\tilde{r}) = \hat{v}_0 + \hat{v}_1\sqrt{1-\tilde{r}} + \ldots, \quad \tilde{r}\to 1, \tag{B.38} \qquad \texttt{vasym8}$$

where:

$$\hat{v}_0 = \hat{w}_0^2\hat{p}_0, \quad \hat{v}_1 = \frac{\hat{w}_0^2\hat{p}_2 + 4\hat{w}_0\hat{w}_1\hat{p}_0}{2}, \quad L_0 = \hat{w}_0\sqrt{\frac{\hat{p}_0}{\alpha}}. \tag{B.39} \qquad \texttt{L0rel}$$

We note that we can rewrite the parameter $\tilde{\omega}$ (B.16) as:

$$\hat{\omega} = \frac{\hat{p}_0}{\pi(1-\nu) - \hat{p}_0}. \tag{B.40} \qquad \texttt{SSomega5}$$

As such, the first term of the aperture asymptotics at the fracture tip (B.17), (B.22) is given by:

$$\hat{w}_0 = \sqrt{L_0}\frac{1+\hat{\omega}}{\sqrt{1+4(1-\nu)\hat{\omega}}}\hat{K}_{Ic} = \sqrt{L_0}\left[\hat{K}_I + \hat{K}_f\right] = \hat{K}_{Ic}\sqrt{L_0\hat{F}(\hat{p}_0)}, \tag{B.41} \qquad \texttt{w0ss5}$$

with $\hat{F}$ following from (B.21):

$$\hat{F}\left(\hat{p}_0\right) = \frac{\pi^2\left(1-\nu\right)^2}{\left[\pi\left(1-\nu\right) + \left(3-4\nu\right)\hat{p}_0\right]\left[\pi\left(1-\nu\right) - \hat{p}_0\right]}. \tag{B.42} \qquad \texttt{Fdef}$$

while the stress intensity factors (B.18) are described by:

$$\hat{K}_I = \frac{\hat{K}_{Ic}}{\sqrt{1+4(1-\nu)\hat{\omega}}}, \quad \hat{K}_f = \frac{\hat{K}_{Ic}\hat{\omega}}{\sqrt{1+4(1-\nu)\hat{\omega}}}, \tag{B.43} \qquad \texttt{SSKI5}$$

Noting the definition of $\hat{F}(\hat{p}_0)$ from (B.42), relationship (B.41) immediately yields:

$$(4\nu-3)\hat{p}_0^2 + 2\pi(1-\nu)(1-2\nu)\hat{p}_0 + \pi^2(1-\nu)^2\left[1 - \frac{L_0\hat{K}_{Ic}^2}{\hat{w}_0^2}\right] = 0. \tag{B.44} \qquad \texttt{SSp05}$$

Finally, asymptotic analysis of (B.29) and (B.32) reveals that, provided the fluid leak-off at the crack tip is finite $\hat{q}_l(1) < \infty$, the second asymptotic coefficients can be obtained using the relations:

$$\hat{w}_1 = \frac{2\hat{p}_0}{2-k_1\hat{p}_0}\left[L_0 - \frac{k_1}{4}\hat{q}_l(1)\right], \quad \hat{p}_2 = \frac{\hat{p}_0}{\hat{w}_0}\left[\hat{q}_l(1) - 4\hat{w}_1\right]. \tag{B.45}$$

Author Contributions

D.P. derived the initial problem formulation and constructed the self-similar solver. G.D.F. developed the time-dependent solver and performed the quantitative analysis. The final paper was prepared collaboratively between the authors.

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