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Hele-Shaw Flow with a small obstacle

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Abstract Asymptotic analysis of the flow passing over a small obstacle in the Hele-Shaw cell is performed. The results are based on the asymptotic formulas for Green's and Neumann functions recently obtained by V. Maz'ya and A. Movchan. Theoretical results are illustrated by the numerical simulations.

Keywords: Hele-Shaw flow, obstacle, Green's function, Neumann function, asymptotic analysis

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1 Introduction

The paper is devoted to an asymptotic analysis of the Hele-Shaw moving boundary value problem with a small obstacle in the flow. Classical Hele-Shaw problem ([22]) deals with the description of the free boundary encircling the domain occupied by incompressible fluid in so called Hele-Shaw cell (see, e.g. [20]), i.e. in a narrow space between two closely related plates. Different driving mechanisms can be considered for the fluid flow, e.g. presence of a source/sink in the fluid domain.

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Various physical assumptions lead to different boundary value problems either on the free boundary or on permeable walls and obstacles. A comprehensive discussion on these conditions and other features of the flow in the Hele-Shaw cell is presented in the recent book by Gustafsson and Vasil'ev [20].

There exist two basis mathematical models for the flow in the Hele-Shaw cell. The *complex-analytic model* is formulated as a nonlinear mixed boundary value problem with respect to a family of conformal mappings of the canonical domain onto domain occupied by the fluid. This approach goes back to the work by Polubarinova-Kochina [39] and Galin [15]. The proof of the existence (locally in time) and uniqueness of analytic solution to this model was done by Kufarev & Vinogradov [26] (rediscovered later by Richardson [47]) on the base of the method of successive approximations. Simplified proof of existence and uniqueness of an analytic solution was given by Reissig & von Wolfersdorf [41]. In this work the model was interpreted as a special case of an abstract Cauchy-Kovalevsky problem, which was solved by a variant of the Cauchy-Kovalevsky theorem ([37], [34], [35]).

In the *real-variable model* proposed by Gustafsson [17] the flow is described by a family of parametrizations of the boundary of fluid domain. This model was generalized to multi-dimensional case by Begehr & Gilbert [4]. Among variants of the proof of existence and uniqueness for this model we have to point out the article by Reissig [42], Escher & Simonett [13]. In the most general form, the proof of the existence and uniqueness of the classical solution was given by Antontsev, Gonçalves and Meirmanov [1].

Variational formulation of the Hele-Shaw model was proposed by Gustafsson [18], who proved the weak solvability of the problem (see also [3], [20]).

The classical (real-variable) Hele-Shaw model can be reinterpreted as a mixed boundary value problem for Laplace equation with respect to unknown parametrization of the boundary and corresponding Green's function of this problem in the reference domain. The aim of our work is to perform an asymptotic analysis for such variant of the model and discover on the base of this analysis new features of the flow in the Hele-Shaw cell with a small obstacle.

Application of asymptotic methods for approximation of Green's function goes back to the classical paper by J. Hadamard [21], where the method of regular perturbation was performed. Recently, V. Maz'ya and A. Movchan obtain a number of asymptotic formulas for Green's function related to different boundary value problems for a number of differential operators in the case of singular perturbations of the domains (see [27], [29] and references therein).

The paper is organized as follows. Sec. 2 describes the geometry problem under consideration and presents the (real-variable) Hele-Shaw model in a domain with an obstacle. The model is further reduced to the form containing an unknown parametrization of the boundary of the fluid domain and an unknown Green's function of the corresponding mixed boundary value problem for the Laplace operator.

In Sec. 3 we rewrite the original problem in a form suitable for direct application of Maz'ya & Movchan asymptotic representation of Green's function.

Sec. 4 is devoted to the asymptotic study of the solution to the Hele-Shaw boundary value problem in a domain with an obstacle. Numerical example is presented in Sec. 5. The final conclusions are summarised in Sec. 6.

2 Problem formulation

Let us consider the Hele-Shaw problem with an internal obstacle located in the flow region. Viscous incompressible fluid occupies the doubly connected domain $D_1(t)$ at a time instant $t \geq 0$. Internal domain F is a fixed small obstacle. The simply connected domain without hole will be denoted $D(t)$. We assume that $F \subset D(0)$ has a nonempty interior, and a diameter of the obstacle $2\delta := diam\, F$ is positive. $D(0)$ is supposed to be open bounded set with a smooth boundary such that

$$c \leq \min dist\,\{0, D(0)\} \leq \max dist\,\{0, D(0)\} \leq 1$$

and

$$dist\,\{\partial F, \partial D(0)\} = d > 0. \tag{1}$$

Without loss of generality we can assume that δ and d ($\delta < d$) are dimensionless parameters, and $d + \delta < c$. To avoid technical difficulties, we accept a circular shape of the the obstacle of the radius δ.

Following [11], [13], the initial free boundary $\partial D(0)$ is to satisfy the smoothness assumptions

$$\partial D(0) \in \mathcal{C}^{1,\alpha},\ 0 < \alpha < 1. \tag{2}$$

We consider a two-dimensional potential flow of incompressible fluid in the Hele-Shaw cell, i.e. in a gap between two parallel plates of distance h (see [20], [22]). The flow is described by a stationary velocity field $\mathbf{V}(V_1, V_2)$, which is proportional to the pressure gradient (see, e.g. [17])

$$\mathbf{V} = -\frac{h^2}{12\mu}\nabla p, \tag{3}$$

where μ is the dynamic viscosity of the fluid. Equation (3) is known as the Hele-Shaw equation.

For h and μ being constant, Hele-Shaw equation together with continuity (or divergence-free) property yield

$$\triangle p = 0, \tag{4}$$

in the area of the flow without of sources/sinks.

Different driving mechanisms of the flow are considered (see, e.g. [17], [24], [47]). Hereinafter, we restrict our attention a single source/sink situated, without loss of generality, at a fixed point $z_0 = (x_0, y_0)$, while the origin $O = (0, 0)$ belongs to the interior of the fixed domain F, i.e.

$$z_0 \in D_1(0) = D(0) \setminus cl\, F, \quad O \in int\, F.$$

In this case, the pressure p exhibits a logarithmic singularity at the origin, i.e. satisfies the following asymptotic relation:

$$p(z,t) \sim -\frac{Q(t)}{2\pi}\log|z - z_0|, \quad z \to z_0. \tag{5}$$

The real-valued function $Q(t)$ models injection ($Q(t) > 0$) or suction ($Q(t) < 0$). By introducing new time-variable $\tau = \int\limits_0^t Q(\xi)d\xi$, one reduces the original problem to the case of constant influx $Q(t) = Q = const$.

The so called impermeability condition is imposed on ∂F

$$\frac{\partial p}{\partial n} = 0,\ z \in \partial F, \tag{6}$$

where n is an inward normal vector on each component of ∂F.

We assume also, that two boundary conditions on the free boundary $\Gamma(t) = \partial D(t)$ have to be satisfied, namely, zero surface tension dynamic boundary condition [20]

$$p(z) = 0, \quad z \in \Gamma(t), \tag{7}$$

and kinematic boundary condition

$$\frac{d\Gamma}{dt} = \mathbf{V}, \quad z \in \Gamma(t). \tag{8}$$

On substitution of (3) to (8) one has:

$$\frac{d\Gamma}{dt} = -\frac{h^2}{12\mu}\nabla p. \tag{9}$$

Let us now introduce a new unknown function (see [17], [23]), namely, a one-parametric family of $\mathcal{C}^2$-diffeomorphisms

$$w(s,t) = (u(s,t), v(s,t)) : \partial\,\mathbb{U} \times I \to \Gamma(t), \tag{10}$$

$$\mathbb{U} = \{s = (s_1, s_2) \in \mathbb{R}^2 : |z| < 1\}.$$

From the computational argument, we restrict our interest to a small time-interval, i.e. $I = (-\eta, \eta)$.

The function $w(s,t)$ in (10) determines an unknown parametrization of the free boundary $\Gamma(t)$ [23]:

(i) $w(s,t) \in \Gamma(t)$ for all $(s,t) \in \partial\,\mathbb{U} \times I$,

(ii) $w(\cdot,t) : \partial\mathbb{U} \to \Gamma(t)$ is a $\mathcal{C}^2$-diffeomorphism for each fixed $t \in I$,

(iii) $w(\cdot,\cdot) \in \mathcal{C}^2\left(\partial\mathbb{U} \times I; \mathbb{R}^2\right)$.

As it follows from the relations (4) – (7), the unknown pressure p coincides up to constant factor with Green's function of the operator $-\triangle$ in the doubly connected domain $D_1(t)$ with the homogeneous Neumann condition on the fixed boundary ∂F and the homogeneous Dirichlet condition on the free boundary $\Gamma(t)$:

$$p = Q \cdot \mathcal{G}_{D_1(t)}, \tag{11}$$

and for each $t \in I$ the function $\mathcal{G}_{D_1(t)} = \mathcal{G}_{D_1(t)}(z, z_0; t)$ is the solution of the following mixed boundary value problem

$$\triangle\mathcal{G}_{D_1(t)}(z, z_0; t) + \delta_0(z - z_0) = 0, \quad z \in D_1(t), \tag{12}$$

$$\mathcal{G}_{D_1(t)}(z, z_0; t) = 0, \quad z \in \Gamma(t), \tag{13}$$

$$\frac{\partial\,\mathcal{G}_{D_1(t)}}{\partial\,n}(z, z_0; t) = 0, \quad z \in \partial F. \tag{14}$$

Finally, the problem is formulated in terms of the pair $\{w(s,t); \mathcal{G}_{D_1(t)}(z, z_0)\}$, where the unknown function $\mathcal{G}_{D_1(t)}(z, z_0)$ depends on the spatial variable z, on a position of the source/sink z_0 and on the time variable t. Taking this into account we will use hereinafter the notation $\mathcal{G}(z, z_0; t) = \mathcal{G}_{D_1(t)}(z, z_0)$.

Problem ($\mathbf{HS_0}$). Find a pair $\{w(s,t); \mathcal{G}(z, z_0; t)\}$, such that $w(s,t) : \partial\,\mathbb{U} \times I \to \mathbb{R}^2$ is a $\mathcal{C}^2$-diffeomorphism satisfying

(i) $w(s,t) \in \Gamma(t)$ for all $(s,t) \in \partial\,\mathbb{U} \times I$;

(ii) $w(\cdot,t) : \partial\,\mathbb{U} \to \Gamma(t)$ is a $\mathcal{C}^2$-diffeomorphism for each fixed $t \in I$;

(iii) $w^{(0)}(s) = w(s,0)$ is a given $\mathcal{C}^2$-diffeomorphism of the unit circle $\partial\,\mathbb{U}$, which describes the boundary $\Gamma(0)$ of initial domain $D_1(0)$;

(iv) $\mathcal{G}(z, z_0; t)$ is Green's function of the operator $-\triangle$ in the doubly connected domain $D_1(t)$ with the homogeneous Neumann condition on the fixed boundary $\partial\,F$ and the homogeneous Dirichlet condition on the free boundary $\Gamma(t)$, i.e. satisfies conditions (12)–(14) for each fixed $t \in I$;

(v) $\partial_t\, w(s,t) = -\frac{Qh^2}{12\mu} \cdot \nabla\mathcal{G}(w(s,t), z_0; t)$ for all $(s,t) \in \partial\,\mathbb{U} \times I$.

Existence and uniqueness of the local in time classical solution to the Problem (HS_0) was proved in [42] in the case of injection/suction trough the unique source/sink. The result was obtained by using the reduction of the considered problem to an abstract Cauchy-Kovalevsky problem, which is handled in an appropriate scale of Banach spaces of real analytic functions ([37]). Another proof was given in [41] for so called complex-variable Hele-Shaw model (see [20], [36], [51]).

It follows from [12] (see also [18], [20]) that the necessary condition for existence of classical solution is the analyticity of the boundary of the initial fluid domain. Thus, the unique classical solution generates a smooth one-parametric family of analytic diffeomorphisms of the unit circle onto the moving boundary. For more general driving mechanisms (in particular, for Hele-Shaw flow in a domain with obstacles) the prove of existence and uniqueness was obtained in [1] (see discussion of classical solvability of the Hele-Shaw problem in [20], [51]).

3 Reformulation of the problem HS_0

The method of uniform asymptotic approximation of Green's function related to different boundary value problems for a number of differential operators in singularly and regularly perturbed domains was created and developed in a series of articles by V. Maz'ya and A. Movchan (see [27], [29], [30]). One of these results can be useful for our analysis and thus we reformulate the problem of HS_0 in such a way to apply directly respective formulae by Maz'ya and Movchan. For simplicity we assume that the obstacle F is a disc of radius δ centered at the origin $(F = B(0, \delta))$. For further convenience, we introduce here a small parameter ε related to the diameter of the inclusion $(\varepsilon = \delta)$. Another variant of the problem is discussed in the conclusion.

Let us make the transformation

$$\zeta = (\zeta_1, \zeta_2), \quad \zeta_1 = \frac{\varepsilon x}{|z|^2}, \zeta_2 = -\frac{\varepsilon y}{|z|^2} \quad (z = (x, y)). \tag{15}$$

In this notation $\zeta_0 = (\zeta_{0,1}, \zeta_{0,2})$, $\zeta_{0,1} = \frac{\varepsilon x_0}{|z_0|^2}, \zeta_{0,2} = -\frac{\varepsilon y_0}{|z_0|^2}$ is the image of the source/sink point, $\omega(s,t) = (\omega_1(s,t), \omega_2(s,t)), \omega_1(s,t) = \frac{\varepsilon w_1(s,t)}{|w(s,t)|^2}, \omega_2(s,t) = -\frac{\varepsilon w_2(s,t)}{|w(s,t)|^2}$ is the image $\Gamma_{tr}(t)$ of the moving boundary $\Gamma(t)$, where $\omega(s,0)$ defines its initial shape.

Denote by $F_{tr} = \{\zeta \in \mathbb{R}^2 : |\zeta| < 1\}$ and $\Omega_{tr}(t) = ext\Gamma_{tr}(t)$ the images of the domains $ext\{clF\}$ and $D(t)$, respectively, under transformation (15). The domain of our interest is now $\Omega(t) = F_{tr} \cap \Omega_{tr}(t)$.

Simple calculations show that the problem HS_0 can be rewritten in the form

Problem ($\mathbf{HS_{tr}}$). *Find a pair $\{\omega(s,t); \mathcal{G}_{tr}(\zeta, \zeta_0; t)\}$, such that $\omega(s,t) : \partial\mathbb{U} \times I \to \mathbb{R}^2$ is a $\mathcal{C}^2$-diffeomorphism satisfying*

(i) $\omega(s,t) \in \Gamma_{tr}(t)$ for all $(s,t) \in \partial\mathbb{U} \times I$;

(ii) $\omega(\cdot,t) : \partial\mathbb{U} \to \Gamma_{tr}(t)$ is a $\mathcal{C}^2$-diffeomorphism for each fixed $t \in I$;

(iii) $\omega^{(0)}(s) = \omega(s,0)$ is a given $\mathcal{C}^2$-diffeomorphism of the unit circle $\partial\mathbb{U}$, which describes the boundary $\Gamma_{tr}(0)$ of initial domain $\Omega_{tr}(0)$;

(iv) $\mathcal{G}_{tr}(\zeta, \zeta_0; t)$ is Green's function of the operator $-\triangle$ in the doubly connected domain $\Omega(t)$ with the homogeneous Neumann condition on the fixed boundary ∂F_{tr} and the homogeneous Dirichlet condition on $\Gamma_{tr}(t)$;

(v) $\partial_t\, \omega(s,t) = -\frac{Qh^2|\omega|^4}{12\mu\varepsilon^2} \cdot \nabla\mathcal{G}_{tr}(\omega(s,t), \zeta_0; t)$ for all $(s,t) \in \partial\mathbb{U} \times I$.

Direct calculations show that the new function $\mathcal{G}_{tr}(\zeta, \zeta_0; t)$ is indeed the Green's functions for the operator $-\triangle$, satisfying the defined above homogenous mixed boundary value problem.

According to [31, Thm. 2.2] Green's function for exterior domain $\Omega_{tr}(t)$ encircled by the curve $\Gamma_{tr}(t)$ can be rewritten as:

$$\mathcal{G}_{tr}(\mathbf{X}; \mathbf{Y}; t) = \mathcal{G}_{tr}(X_1, X_2; Y_1, Y_2; t) = \tag{16}$$

$$= G\left(\frac{1}{\varepsilon}\mathbf{X}; \frac{1}{\varepsilon}\mathbf{Y}; t\right) + N(\mathbf{X}; \mathbf{Y}) - \frac{1}{2\pi}\log\frac{1}{|\mathbf{X} - \mathbf{Y}|} +$$

$$+R(\mathbf{0},\mathbf{0}) + \varepsilon\mathbf{D}\left(\frac{1}{\varepsilon}\mathbf{Y}; t\right) \cdot \nabla_{\mathbf{Y}} R(\mathbf{X}, \mathbf{0}) +$$

$$+\varepsilon\mathbf{D}\left(\frac{1}{\varepsilon}\mathbf{X}; t\right) \cdot \nabla_{\mathbf{X}} R(\mathbf{0}, \mathbf{Y}) + r_\varepsilon(\mathbf{X}; \mathbf{Y}; t).$$

This function satisfies the following conditions

$$\begin{cases} \triangle_\xi G(\xi;\eta;t) + \delta(\xi - \eta) = 0, \quad \xi, \eta \in \frac{1}{\varepsilon}\Omega_{tr}(t), \\ G(\xi;\eta;t) = 0, \quad \xi \in \frac{1}{\varepsilon}\Gamma_{tr}(t), \eta \in \frac{1}{\varepsilon}\Omega_{tr}(t), \\ G(\xi;\eta;t) \quad \text{is bounded as } |\xi| \to \infty; \end{cases} \tag{17}$$

N is the Neumann function for the interior of the "fixed" domain ($|\mathbf{Y}| < R$)

$$\begin{cases} \triangle_{\mathbf{X}} N(\mathbf{X}, \mathbf{Y}) + \delta(\mathbf{X} - \mathbf{Y}) = 0, \quad |\mathbf{X}|, \\ \frac{\partial}{\partial n_{\mathbf{X}}} N(\mathbf{X}, \mathbf{Y}) = -\frac{\partial}{\partial n_{\mathbf{X}}}\frac{1}{2\pi}\log|\mathbf{X}|, \quad |\mathbf{X}| = R, \\ \int\limits_{|\mathbf{X}|=R} N(\mathbf{X}, \mathbf{Y}) \frac{\partial}{\partial n_{\mathbf{X}}}\log|\mathbf{X}| dS_{\mathbf{X}} = 0; \end{cases} \tag{18}$$

$R(\mathbf{X}, \mathbf{Y})$ is the regular part of the Neumann function (18), namely,

$$R(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2\pi}\log|\mathbf{X} - \mathbf{Y}| - N(\mathbf{X}, \mathbf{Y}); \tag{19}$$

$\mathbf{D}(\xi; t)$ is a vector whose components $D_j(\xi; t)$, $(j = 1, 2)$, are solutions of the Dirichlet problems for the Laplace equation in the exterior domain $\Omega_{tr}(t)$:

$$\begin{cases} \triangle_\xi D_j(\xi, t) = 0, \ \xi \in \frac{1}{\varepsilon}\Omega_{tr}(t), \\ D_j(\xi, t) = \xi_j, \quad \xi = (\xi_1, \xi_2) \in \frac{1}{\varepsilon}\Gamma_{tr}(t), \\ D_j(\xi, t) \qquad \text{is bounded as } |\xi| \to \infty. \end{cases} \tag{20}$$

In our case $\mathbf{X} = \zeta, \mathbf{Y} = \zeta_0$. Since the domain $\Omega_{tr}(t)$ is small, we re-scale it by an auxiliary parameter ε

$$\Omega_\varepsilon = \frac{1}{\varepsilon}\Omega_{tr}(t)$$

where $\xi = \frac{1}{\varepsilon}\mathbf{X} = \frac{1}{\varepsilon}\zeta$, $\eta = \frac{1}{\varepsilon}\mathbf{Y} = \frac{1}{\varepsilon}\zeta_0 \in \Omega_\varepsilon$.

4 Green's function for the problem HS_{tr}

In this section we analyze the components of the representation (16). Let us first consider the Neumann function N having in this case an explicit representation:

$$N(\mathbf{X}, \mathbf{Y}) = -\frac{1}{4\pi}\log|\mathbf{X} - \mathbf{Y}|^2 + \frac{1}{2\pi}\log R^2 - \tag{21}$$

$$-\frac{1}{4\pi}\log\left|(R^2 - |\mathbf{X}|^2)(R^2 - |\mathbf{Y}|^2) + R^2|\mathbf{X} - \mathbf{Y}|^2\right|,$$

satisfying the conditions (18) and symmetric $N(\mathbf{X}, \mathbf{Y}) = N(\mathbf{Y}, \mathbf{X})$. Its regular part $R(\mathbf{X}, \mathbf{Y})$ is also symmetric and calculated explicitly too

$$R(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2\pi}\log R^2 + \tag{22}$$

$$+\frac{1}{4\pi}\log\left|(R^2 - |\mathbf{X}|^2)(R^2 - |\mathbf{Y}|^2) + R^2|\mathbf{X} - \mathbf{Y}|^2\right|.$$

Green's function $G(\xi; \eta; t)$ for the exterior domain Ω_ε is related to a normalised conformal mapping of Ω_ε onto the unit disc $\mathbb{U}$:

$$G(\xi; \eta; t) = \frac{1}{2\pi}\log\frac{1}{|g(\xi, \eta)|}, \tag{23}$$

where $g(\xi,\eta) = (g_1(\xi,\eta), g_2(\xi,\eta)) : \Omega_\varepsilon \to \mathbb{U}$ satisfies the following normalizing conditions $g(\xi,\eta)\big|_{\xi=\eta_0} = 0$, and $g'(\xi,\eta)\big|_{\xi=\eta_0} > 0$. In our case, η_0 stands for the image of the source point, i.e. $\eta_0 = \frac{1}{\varepsilon}\zeta_0$. It is customary to start with an arbitrary conformal mapping $g_0(\xi) : \Omega_\varepsilon \to \mathbb{U}$ and determine the normalised one:

$$g(\xi,\eta) = g(\xi,\eta_0) = \frac{g_0(\xi) - g_0(\eta_0)}{1 - \overline{g_0(\eta_0)}g_0(\xi)}.$$

Function (23) is also used to determine the components of the vector $\mathbf{D}$ via the following formulas

$$D_j(\xi,t) = -\int\limits_{\partial\Omega_\varepsilon} \eta_j \frac{\partial}{\partial n_\eta} G(\eta;\xi;t)dS_\eta, \quad j = 1,2. \tag{24}$$

4.1 Calculation of Green's function derivatives

Let us start with the simplest terms corresponding to the Neumann function $N(\mathbf{X},\mathbf{Y})$. Since the relation (19) holds, we have to differentiate only regular part:

$$(R(\mathbf{X},\mathbf{Y}))'_{X_k} = \frac{1}{2\pi}\times$$

$$\times\frac{-X_k(R^2 - |\mathbf{Y}|^2) + R^2(X_k - Y_k)}{(R^2 - |\mathbf{X}|^2)(R^2 - |\mathbf{Y}|^2) + R^2|\mathbf{X} - \mathbf{Y}|^2}. \tag{25}$$

It follows, in particular, that

$$\nabla_{\mathbf{X}} R(\mathbf{0},\mathbf{Y}) = -\frac{1}{2\pi}\left(Y_1, Y_2\right), \tag{26}$$

$$\nabla_{\mathbf{Y}} R(\mathbf{X},\mathbf{0}) = -\frac{1}{2\pi}\left(X_1, X_2\right). \tag{27}$$

In our notations, equation (25) takes the form

$$(R(\omega,\zeta_{\mathbf{0}}))'_{X_k} = \frac{1}{2\pi}\frac{\omega_k(\zeta_{0,1}^2 + \zeta_{0,2}^2) - \zeta_{0,k}}{A(\omega,\zeta_0)}, \tag{28}$$

$$A(\omega,\zeta_0) = (1 - \omega_1^2 - \omega_2^2)(1 - \zeta_{0,1}^2 - \zeta_{0,2}^2) +$$
$$+1[(\omega_1 - \zeta_{0,1})^2 + (\omega_2 - \zeta_{0,2})^2].$$

The following notation will be used in the final system (41):

$$\widetilde{\mathcal{R}}_k = (R(\omega,\zeta_{\mathbf{0}}))'_{X_k}, k = 1,2. \tag{29}$$

The derivatives of Green's function for external domain are

$$(G(\xi;\eta;t))'_{x_k} =$$

$$= -\frac{g_1(\xi;\eta)\,(g_1(\xi;\eta))'_{\xi_k} + g_2(\xi;\eta)\,(g_2(\xi;\eta))'_{\xi_k}}{2\pi\varepsilon(g_1^2(\xi;\eta) + g_2^2(\xi;\eta))}. \tag{30}$$

To compute the leading term of Green's function in (16), we compute

$$\frac{1}{\varepsilon}\widetilde{\mathcal{G}}_k \equiv \left(G\left(\frac{1}{\varepsilon}\omega;\frac{1}{\varepsilon}\zeta_{\mathbf{0}};t\right)\right)'_{x_k} = -\frac{1}{2\pi\varepsilon}\times \tag{31}$$

$$\times\left[g_1\left(\frac{1}{\varepsilon}\omega;\frac{1}{\varepsilon}\zeta_{\mathbf{0}}\right)\left(g_1\left(\frac{1}{\varepsilon}\omega;\frac{1}{\varepsilon}\zeta_{\mathbf{0}}\right)\right)'_{\xi_k} + \right.$$

$$\left. + g_2\left(\frac{1}{\varepsilon}\omega;\frac{1}{\varepsilon}\zeta_{\mathbf{0}}\right)\left(g_2\left(\frac{1}{\varepsilon}\omega;\frac{1}{\varepsilon}\zeta_{\mathbf{0}}\right)\right)'_{\xi_k}\right].$$

Analogously,

$$\left(\mathbf{D}\left(\frac{1}{\varepsilon}\zeta_0;t\right)\cdot\nabla_{\mathbf{Y}} R(\zeta,\mathbf{0})\right)'_{X_k} = \tag{32}$$

$$= D_k\left(\frac{1}{\varepsilon}\zeta_0;t\right)\left(-\frac{1}{2\pi R^2}\right) = D_k\left(\frac{1}{\varepsilon}\zeta_0;t\right)\left(-\frac{1}{2\pi R^2}\right).$$

Similarly

$$\left(\mathbf{D}\left(\frac{1}{\varepsilon}\mathbf{X};t\right)\cdot\nabla_{\mathbf{X}} R(\mathbf{0},\mathbf{Y})\right)'_{X_k} = \tag{33}$$

$$= \left(\mathbf{D}\left(\frac{1}{\varepsilon}\zeta_0;t\right)\cdot\nabla_{\mathbf{Y}} R(\zeta,\mathbf{0})\right)'_{X_k} =$$

$$= -\frac{1}{2\pi\varepsilon R^2}\left\{\zeta_{0,1}\left(D_1(\xi;t)\right)'_{\xi_k} + \zeta_{0,2}\left(D_2(\xi;t)\right)'_{\xi_k}\right\}.$$

Let us calculate the component:

$$D_1(\eta_0;t) = \frac{1}{4\pi}\int\limits_{\partial\Omega_\varepsilon} \eta_1 \frac{\partial}{\partial n_\eta}\log|g(\eta;\eta_0)|^2\, dS_\eta.$$

Since both functions η_1 and $\log|g(\eta;\eta_0)|^2$ are harmonic in $\Omega_\varepsilon \setminus \{\eta_0\}$ then, by Green's formula, we have

$$D_1(\eta_0;t) = \frac{1}{4\pi}\int\limits_{|\eta-\eta_0|=a} \eta_1 \frac{\partial}{\partial n_\eta}\log|g(\eta;\eta_0)|^2\, dS_\eta.$$

As the function $g(\eta;\eta_0)$ has a simple zero at $\eta = \eta_0$,

$$D_1(\eta_0;t) = \frac{1}{4\pi}\int\limits_{|\eta-\eta_0|=a} \eta_1 \frac{\partial}{\partial n_\eta}\log|\eta - \eta_0|^2\, dS_\eta +$$

$$+\frac{1}{4\pi}\int\limits_{|\eta-\eta_0|=a} \eta_1 \frac{\partial}{\partial n_\eta}\log|g_0(\eta;\eta_0)|^2\, dS_\eta,$$

where $\log|g_0(\eta;\eta_0)|^2$ is harmonic (and thus continuous) in the disc $B(\eta_0,a)$. Direct calculations show that the first integral tends to 0 as $a \to 0$. By Green's formula the second integral is equal

$$\frac{1}{4\pi}\int\limits_{|\eta-\eta_0|=a} \frac{\partial}{\partial n_\eta}(\eta_1 - \eta_{0,1})\cdot\log|g_0(\eta;\eta_0)|^2\, dS_\eta =$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi}\cos\theta\cdot\log|g_0(\eta_{0,1} + a\cos\theta, \eta_{0,2} + a\sin\theta;\eta_0)|\, a d\theta.$$

By properties of g_0, the integral tends to 0 as $a \to 0$ too. Similar analysis can be done for D_2. Therefore,

$$\left(\mathbf{D}\left(\frac{1}{\varepsilon}\zeta_0; t\right) \cdot \nabla_{\mathbf{Y}} R(\zeta, \mathbf{0})\right)'_{X_k} = 0. \tag{34}$$

In order to calculate respective derivatives in (33), we rewrite the representation of the functions D_j:

$$D_j(\xi; t) = \frac{1}{4\pi} \int\limits_{\partial D} \eta_j \frac{\partial}{\partial n_\eta} \log |g(\eta; \xi)|^2 \, dS_\eta = \tag{35}$$

$$= I_{1,j}(\xi) + I_{2,j}(\xi),$$

where

$$I_{1,j}(\xi) = \frac{1}{4\pi} \int\limits_{\partial \Omega_\varepsilon} \eta_j \frac{\partial}{\partial n_\eta} \log \left| \frac{g(\eta; \xi)}{1 - \frac{\xi \cdot \eta}{|\eta|^2}} \right|^2 dS_\eta,$$

$$I_{2,j}(\xi) = \frac{1}{4\pi} \int\limits_{\partial \Omega_\varepsilon} \eta_j \frac{\partial}{\partial n_\eta} \log \left| 1 - \frac{\xi \cdot \eta}{|\eta|^2} \right|^2 dS_\eta,$$

and $\xi \cdot \eta$ denotes the scalar product of corresponding vectors. In $I_{1,j}(\xi)$ both functions η_j and $\log \left| \frac{g(\eta;\xi)}{1 - \frac{\xi \cdot \eta}{|\eta|^2}} \right|^2$ are harmonic in the exterior domain Ω_ε. Therefore, by applying Green's formula, we can replace the contour of integration by the circle $\partial B(0, A)$ of radius A and interchange the normal derivative. Thus

$$(I_{1,j}(\xi))'_{\xi_k} = \frac{1}{4\pi} \int\limits_{\partial B(0,A)} \frac{\partial}{\partial n_\eta} \eta_j \cdot \left(\log \left| \frac{g(\eta; \xi)}{1 - \frac{\xi \cdot \eta}{|\eta|^2}} \right|^2 \right)'_{\xi_k} dS_\eta.$$

Denote by F_k the values of the derivatives of $\log |g|^2$ with respect to ξ_k

$$F_k(\eta; \xi) = \left(\log |g(\eta; \xi)|^2 \right)'_{\xi_k}.$$

Expanding this function in Taylor formula at $\eta = \infty$, we calculate a part of the integral in $(I_{1,j}(\xi))'_{\xi_k}$ by setting A to ∞. Analogously, we can calculate explicitly the remaining part in $(I_{1,j}(\xi))'_{\xi_k}$.

For the integrals in $(I_{2,j}(\xi))'_{\xi_k}$, we apply Green's formula for a doubly connected domain to have

$$(I_{2,j}(\xi))'_{\xi_k} = \frac{1}{4\pi} \int\limits_{\partial B(0,a)} \frac{\partial}{\partial n_\eta} \eta_j \cdot \left(\log \left| 1 - \frac{\xi \cdot \eta}{|\eta|^2} \right|^2 \right)'_{\xi_k} dS_\eta,$$

where $B(0, a)$ is a disc of a small radius a. Determining explicitly the values of these integrals and setting a to 0 we make these terms vanishing. Finally, we obtain:

$$\left(\mathbf{D}\left(\frac{1}{\varepsilon}\mathbf{X}; t\right) \cdot \nabla_{\mathbf{X}} R(\mathbf{0}, \mathbf{Y})\right)'_{X_1} = \frac{1}{4\pi^2 \varepsilon R^2} \times \tag{36}$$

$$\times \left[\zeta_{0,1} \left(\frac{(F_1)'_{\eta_1}(\infty; \xi)}{2} + 1 \right) - \zeta_{0,2} \left(\frac{(F_1)'_{\eta_2}(\infty; \xi)}{2} \right) \right],$$

$$\left(\mathbf{D}\left(\frac{1}{\varepsilon}\mathbf{X}; t\right) \cdot \nabla_{\mathbf{X}} R(\mathbf{0}, \mathbf{Y})\right)'_{X_2} = \frac{1}{4\pi^2 \varepsilon R^2} \times \tag{37}$$

$$\times \left[\zeta_{0,1} \left(\frac{(F_2)'_{\eta_1}(\infty; \xi)}{2} \right) - \zeta_{0,2} \left(\frac{(F_2)'_{\eta_2}(\infty; \xi)}{2} - 1 \right) \right].$$

By relation (33) in our notation we have

$$\frac{1}{\varepsilon}\widetilde{\mathcal{D}}_1 \equiv \left(\mathbf{D}\left(\frac{1}{\varepsilon}\omega; t\right) \cdot \nabla_{\mathbf{X}} R(\mathbf{0}, \zeta_0)\right)'_{X_1} = \frac{1}{4\pi^2 \varepsilon R^2} \times \tag{38}$$

$$\left[\zeta_{0,1} \left(\frac{(F_1)'_{\eta_1}(\infty; \frac{1}{\varepsilon}\omega)}{2} + 1 \right) - \zeta_{0,2} \left(\frac{(F_1)'_{\eta_2}(\infty; \frac{1}{\varepsilon}\omega)}{2} \right) \right]$$

$$\frac{1}{\varepsilon}\widetilde{\mathcal{D}}_2 \equiv \left(\mathbf{D}\left(\frac{1}{\varepsilon}\omega; t\right) \cdot \nabla_{\mathbf{X}} R(\mathbf{0}, \zeta_0)\right)'_{X_2} = \frac{1}{4\pi^2 \varepsilon R^2} \times \tag{39}$$

$$\left[\zeta_{0,1} \left(\frac{(F_2)'_{\eta_1}(\infty; \frac{1}{\varepsilon}\omega)}{2} \right) - \zeta_{0,2} \left(\frac{(F_2)'_{\eta_2}(\infty; \frac{1}{\varepsilon}\omega)}{2} - 1 \right) \right]$$

4.2 Final system of differential equations

The final system of equations to determine the unknown boundary has the following form

$$\begin{cases} \partial_t \omega_1(s,t) = -\frac{Qh^2|\omega|^4}{12\mu\varepsilon^2} \left(J_{1,1} - J_{1,2} + \varepsilon J_{1,3} + r_1 \right), \\ \partial_t \omega_2(s,t) = -\frac{Qh^2|\omega|^4}{12\mu\varepsilon^2} \left(J_{2,1} - J_{2,2} + \varepsilon J_{2,3} + r_2 \right), \end{cases} \tag{40}$$

where

$$J_{k,1} = \left(G\left(\frac{1}{\varepsilon}\omega; \frac{1}{\varepsilon}\zeta_{\mathbf{0}}; t \right) \right)'_{X_k},$$

$$J_{k,2} = \left(R\left(\omega, \zeta_{\mathbf{0}}\right) \right)'_{X_k},$$

$$J_{1,3} = \left(\mathbf{D}\left(\frac{1}{\varepsilon}\omega; t\right) \cdot \nabla_{\mathbf{X}} R(\mathbf{0}, \zeta_0)\right)'_{X_1},$$

$$J_{2,3} = \left(\mathbf{D}\left(\frac{1}{\varepsilon}\omega; t\right) \cdot \nabla_{\mathbf{X}} R(\mathbf{0}, \zeta_0)\right)'_{X_2}$$

are given respectively by the formulas (31), (28), (38) and (39), and

$$r_1 = \left(r_\varepsilon \left(\omega; \zeta_0; t \right) \right)'_{X_1}, \; r_2 = \left(r_\varepsilon \left(\omega; \zeta_0; t \right) \right)'_{X_2}.$$

Another representation of the system (40) is

$$\begin{cases} \partial_T \widetilde{\omega}_1(s,T) = -\frac{Qh^2|\widetilde{\omega}|^4}{12\mu} \left(\widetilde{\mathcal{G}}_1 - \varepsilon\widetilde{\mathcal{R}}_1 + \varepsilon\widetilde{\mathcal{D}}_1 + \varepsilon r_1 \right), \\ \partial_T \widetilde{\omega}_2(s,T) = -\frac{Qh^2|\widetilde{\omega}|^4}{12\mu} \left(\widetilde{\mathcal{G}}_2 - \varepsilon\widetilde{\mathcal{R}}_2 + \varepsilon\widetilde{\mathcal{D}}_2 + \varepsilon r_2 \right), \end{cases} \tag{41}$$

where new time-variable T is introduced

$$T = \varepsilon^{-3} t,$$

$$\widetilde{\omega}_k(s,T) = \omega_k(s,\varepsilon^3 T),\ k=1,2.$$

The third representation of the system (40) is due to the rescaling of the spatial variables:

$$\mathbf{Z} = \frac{1}{\varepsilon}\omega, \quad \mathbf{Z}_0 = \frac{1}{\varepsilon}\zeta_0. \tag{42}$$

In this notations, the system (40) has the form:

$$\begin{cases} \partial_t \widetilde{Z}_1(s,t) = -\frac{Qh^2|\mathbf{Z}|^4}{12\mu}\Big(\widetilde{\mathcal{G}}_1(\mathbf{Z};\mathbf{Z}_0;t) - \varepsilon\widetilde{\mathcal{R}}_1(\varepsilon\mathbf{Z};\varepsilon\mathbf{Z}_0;t) \\ \qquad +\varepsilon\widetilde{\mathcal{D}}_1(\mathbf{Z};\mathbf{Z}_0;t) + \varepsilon r_1(\varepsilon\mathbf{Z};\varepsilon\mathbf{Z}_0;t)\Big), \\ \partial_t \widetilde{Z}_2(s,t) = -\frac{Qh^2|\omega|^4}{12\mu}\Big(\widetilde{\mathcal{G}}_2(\mathbf{Z};\mathbf{Z}_0;t) - \varepsilon\widetilde{\mathcal{R}}_2(\varepsilon\mathbf{Z};\varepsilon\mathbf{Z}_0;t) \\ \qquad +\varepsilon\widetilde{\mathcal{D}}_2(\mathbf{Z};\mathbf{Z}_0;t) + \varepsilon r_2(\varepsilon\mathbf{Z};\varepsilon\mathbf{Z}_0;t)\Big). \end{cases}$$

5 Numerical examples

Numerical computations were done for the classical benchmark from Polubarinova-Kochina [39, p. 29]. We started from two different shapes shown in Fig. 1 by dashed lines. Bearing in mind the limitations of solution continuity for the PDEs [20], we simulate the process only for small time (as a result, we have small expansion of the fluid domain). The final shapes of the domains are depicted by solid lines in the same figure. Position of the source is marked by the star, while two alternative locations of the small inclusion (1 or 2 on the figure) are given by the filled green dots.

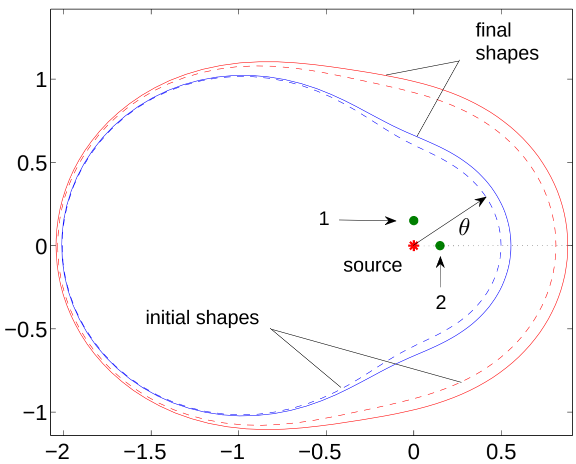

Fig. 1 Two different shapes of the fluid domain before and after the injection. The source is situated at the origin of the coordinate system. Filled green dots mark two possible locations of the small inclusion under consideration.

The numerical computations of the Green's function (23) were done by means of the Schwarz-Christoffel Toolbox for MATLAB (see [5], [6]). The shapes of the domain were approximated by polygons built on $N =$ 100 vertices. The derivatives of the conformal mapping along the boundary were approximated accordingly. The results were verified in various ways. First, we estimated the accuracy of computations by comparing the numerical solution for the problem without inclusion. Also the fluid global balance was controlled. This analysis proved that in case of the smaller initial domain (blue dashed line in Fig. 1), the relative error was less than 0.1% for the maximal time increment, while for the lager domain (red dashed line in the same figure), it was near 0.01%.

A number of simulations for the flow with inclusion were performed. Since the inclusion is small, the final shapes of the domains (with and without the inclusion) are hardly distinguishable in the scale of the pictures. Therefore, in Fig. 2 – Fig. 4 we present the relative deformation, $\delta\rho$, of the fluid domain, computed in the standard manner: $\delta\rho(\theta) = (\rho_{in}(\theta) - \rho_0(\theta))/\rho_0(\theta)$. Here $\rho_{in}(\theta)$ and $\rho_0(\theta)$ are the radii describing the boundary curves, with and without inclusion, while θ is the angle coordinate shown in Fig. 1.

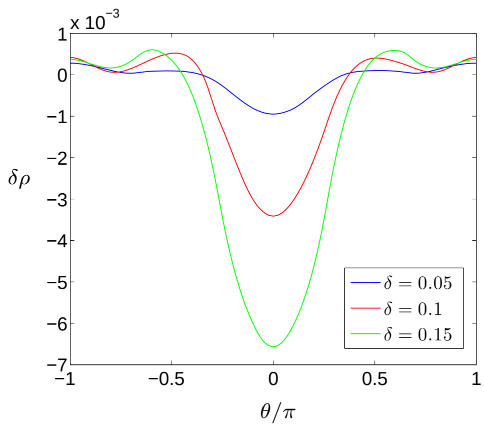

Fig. 2 The relative deformation of the initial domain (denoted by the blue lines in Fig. 1). The center of the inclusion is situated in the position 2 on the Fig. 1. Three different sizes of the inclusion defined by their radii $\delta = 0.05$, 0.1 and 0.15 are considered and the corresponding results are shown by the different colors (see the legend).

6 Outline and discussion

The solution obtained in this paper is applicable also for another geometric configuration. Namely, if one considers the case when the diameter of obstacle is finite (say $\mathit{diam}\, F = 2, 0 \in F$) while the domain $D(0)$ is large enough ($cR \le \min \mathit{dist}\{0, D(0)\} \le \max \mathit{dist}\{0, D(0)\} \le R$), then by performing transformation (15) with $\varepsilon = 1$ we arrive at the same as our geometry in ζ-plane.

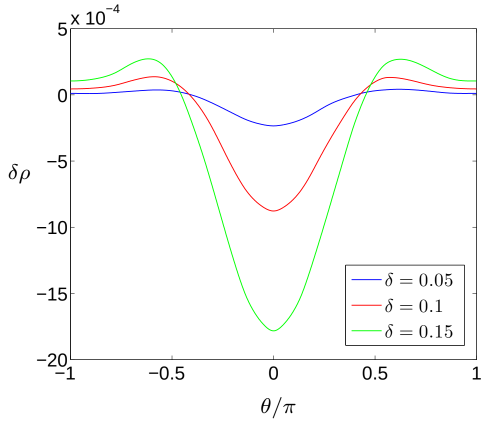

Fig. 3 The relative deformation of the lager initial domain (denoted by the red lines in Fig. 1). The center of the inclusion is situated in position 2 – see Fig. 1. All other notations are the same as in Fig. 2.

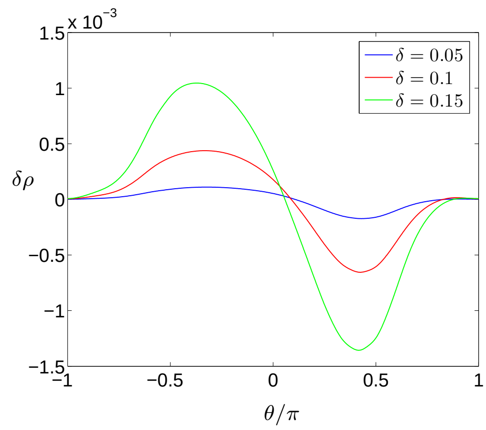

Fig. 4 The relative deformation of the lager initial domain. The center of the inclusion is situated in position 1 – see Fig. 1. All other notations are the same as in Fig. 2.

In such a case, the role of the small parameter will be attributed to $\delta = 1/R$.

As we have shown, the asymptotic approach by Maz'ya, Movchan and Nieves [31] is an efficient tool to deal with the local deformations of the fluid domain with a small inclusion. Since the asymptotic formula for Green's function is uniform, the only difficulties with its application result from the numerics. However, to study the process at large time scale, one needs to analyse additionally the asymptotic properties of the solution at infinity.

As one could expect, a small inclusion has only local impact on the process and its overall influence is rather negligible. In our examples it caused no more than 3% deviation measured as the relative difference between the increments $\delta\rho_{in}(\theta)$ and $\delta\rho_0(\theta)$ of the shape position defined in the following manner $\delta\rho_{in,0}(\theta) = \rho_{in,0}(t,\theta) - \rho_{in,0}(0,\theta)$. As anticipated, the greatest distortion is observed approximately along the line source-inclusion.

However, in the case of a set of small inclusions, the qualitative and quantitative results of the process may change dramatically. To consider such situation, one can use an approach proposed in [29], developed to find a uniform asymptotic expansion for Green's function in a finite domain with a cloud of inclusions.

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