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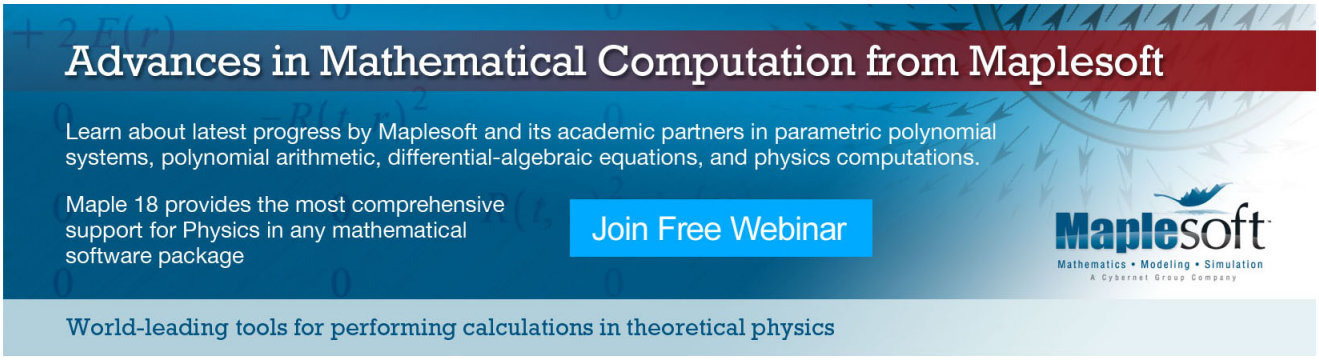

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Heisenberg picture approach to the stability of quantum Markov systems

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Quantum Markovian systems, modeled as unitary dilations in the quantum stochastic calculus of Hudson and Parthasarathy, have become standard in current quantum technological applications. This paper investigates the stability theory of such systems. Lyapunov-type conditions in the Heisenberg picture are derived in order to stabilize the evolution of system operators as well as the underlying dynamics of the quantum states. In particular, using the quantum Markov semigroup associated with this quantum stochastic differential equation, we derive sufficient conditions for the existence and stability of a unique and faithful invariant quantum state. Furthermore, this paper proves the quantum invariance principle, which extends the LaSalle invariance principle to quantum systems in the Heisenberg picture. These results are formulated in terms of algebraic constraints suitable for engineering quantum systems that are used in coherent feedback networks. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4884300]

I. INTRODUCTION

The last two decades have witnessed a rapid development of quantum control technologies, which have proved to be crucial in a large number of quantum systems applications that require a high level of reliability, such as the generation of on-demand quantum states, or the regulation of system performance for quantum information processing.[1,5,23,42,44] Stability is central to these quantum control systems. For example, quantum control tasks may require stabilization of stochastic filtering process,[2,13,28,35] of quantum oscillators in optical systems,[12,37] or of a complex network constructed via the coherent interconnection of quantum components.[11,15,29,43,45]

The traditional approach has been Schrödinger picture Lyapunov techniques, that is, where one defines a Lyapunov function as a positive function over the set of states. This has led to several important results on the stability of quantum states in different control settings.[4,28,32,39,41] In general, the main concern in these control problems is the asymptotic behavior of quantum systems in the Schrödinger picture.

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In contrast, we wish to develop a Heisenberg picture Lyapunov approach which exploits the fact that the Heisenberg picture more readily captures the physical dynamics, and that this allows for a more direct extension of classical stability techniques. We are interested primarily in open quantum systems and, in particular, the quantum stochastic calculus of Hudson and Parthasarathy[14] gives the appropriate mathematical description. This moreover turns out to be a very convenient setup for modeling open quantum coherent networks. It has been shown in a number of references[10,44] that evolution of such networks in the Heisenberg picture can be associated with what is now sometimes referred to as the (S, L, H) representation. (Here, S is a scattering matrix and L, H are coupling and Hamiltonian operators of the system, respectively, and appear as coefficients in the quantum stochastic differential equation for the unitary evolution process.[14,30] As we are interested in the average dynamics in the vacuum state for the environment, we shall take $S = I$ for simplicity as only L and H appear in the Lindbladian.) Ideally, conclusions about stability properties of these operators should be made according to their dynamics in the Heisenberg picture, particularly in the form of conditioning on the infinitesimal generator of a Lyapunov operator which will be introduced in Sec. III. It is worth mentioning that the Lyapunov operator is a generalization of the existing results about quantum stability based on operator inequalities, which are closely related to the dissipativity of the system.[16,31] To fix ideas, suppose that the evolution of an observable $X \in \mathfrak{A}$ in the Heisenberg picture is given by : $t \mapsto j_t(X)$ and that this may be described by a Langevin equation

$$dj_t(X) = j_t(\mathcal{G}(X))dt + \text{Noise},$$

where $\mathcal{G}$ is a Lindblad generator and the "Noise" terms are martingale increments for the environment state. Mirroring the approach to the classical stochastic stability theory, the dynamics of the expectation of an operator can then be established from the properties of the corresponding infinitesimal generator of a Lyapunov operator V and, for instance, an exponential stability criterion would be that

$$\mathcal{G}(V) \leq -cV + dI, \quad c, d > 0, \tag{1}$$

see, e.g., Refs. 16, 17, and 31. Here, I is the identity operator, and we include a dissipation rate d. We point out a similarity between (1) and conditions that arise in the Lyapunov stability theory of classical stochastic differential equations.[18]

Our aim in this paper is to further develop the Lyapunov stability theory of quantum stochastic evolutions in the Heisenberg picture. First, we will study an interplay between stability of states and stability of operators. For example, we will employ the Lyapunov condition (1) to infer the asymptotic behavior of quantum states from the corresponding properties of the operator V. In particular, we will present a stability analysis of the invariant state of an open quantum system based on the Lyapunov method and the quantum semigroup theory. In a sense, this contribution is in parallel with the results in the classical stochastic stability theory, such as the Foster-Lyapunov theory,[27] concerned with the existence and stability of invariant probability measures of Markov processes.

In addition to invariant states, we are also interested in invariant sets of states or operators to which quantum evolutions converge. To characterize the invariance property of the system, we develop a quantum version of the LaSalle invariance principle in the Heisenberg picture. Using this result, we are able to determine a limit set of state trajectories, as well as explore the possibility of stabilizing two non-commuting operators simultaneously. Similar conditions that employ Lyapunov techniques have been developed for classical stochastic systems.[18,19,25] The objective of our development is to pave the way to the design of coherent feedback networks, as operators of a quantum system form a non-commutative algebra. An alternative approach of analyzing stability of state trajectories in the Schrödinger picture is often too difficult in this case.

The main results of this paper are formulated in the way such that the ground states of an operator V or W are stabilized, given that certain Lyapunov conditions are satisfied. This is directly relevant to a recent quantum information processing scheme,[40] where the task is either stabilizing the ground states of a Hamiltonian which encode the solution to a quantum computation problem, or robustly preparing entangled quantum states[21,40] through engineering the dissipative property of

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the systems. The results obtained in this paper thus provide tools to design proper environmental couplings for these applications.

This paper is organized as follows: In Sec. II, we briefly review some facts about Markov dynamics of open quantum systems. In Sec. III, the definition of the Lyapunov operator is given. Then in Sec. IV, we propose a Lyapunov condition to ensure the existence of an invariant state. In Sec. V, we prove the faithfulness and uniqueness of an invariant state using certain non-degeneracy conditions imposed on the diffusion coefficients of the quantum stochastic differential equation. In Sec. VI, we derive the quantum invariance principle and discuss its implications. In Sec. VII, we investigate the stability within the invariant set and provide a sufficient condition to stabilize the system to the ground state. Section VIII is devoted to conclusions.

Notations. In this paper, $\mathfrak{H}$ is a separable Hilbert space, and $\mathfrak{A}$ denotes the von Neumann algebra of operators acting on the Hilbert space $\mathfrak{H}$. The commutator of two operators A and B is written as $[A, B] = AB - BA$, while $X^\dagger$ is the adjoint of an operator X. $\mathfrak{M}' = \{A \in \mathfrak{A} : [A, X] = 0, X \in \mathfrak{M}\}$ denotes the commutant of a von Neumann algebra $\mathfrak{M}$ of operators. In particular, we write $\mathbb{C}I$ for the trivial von Neumann algebra consisting of multiples of identity. $\mathbb{C}$ is the set of complex numbers. A positive-semidefinite operator X is indicated as $X \geq 0$. We shall only work with normal states, and typically write ρ for the corresponding density operator so that the expectation of an operator X will be denoted as $\langle X\rangle_\rho = \mathrm{Tr}\,(\rho X)$, e.g., Ref. 34.

II. QUANTUM MARKOV SYSTEMS

Consider the system defined on a Hilbert space $\mathfrak{H}_S$, and the environment on a Fock space $\mathfrak{H}_B$ over $L^2(\mathbb{R}_+, dt)$ corresponding to a single Boson field mode. The composite system can be regarded as a single closed system whose dynamics are characterized by a unitary evolution $U(t)$ on $\mathfrak{H}_S \otimes \mathfrak{H}_B$ obeying a quantum stochastic differential equation[14,16]

$$dU(t) = \{dB^\dagger L - L^\dagger dB - \frac{1}{2}L^\dagger L dt - iHdt\}U(t).$$

Here, H is the Hamiltonian of the system, and L describes the coupling between the system and environment; $B(\cdot)$ and $B^\dagger(\cdot)$ are the annihilation and creation process defined on $\mathfrak{H}_B$. In the Heisenberg picture, an operator $X(t) = j_t(X)$ of the system evolves as $j_t(X) = U(t)^\dagger(X\otimes I)U(t)$. Given the interaction Hamiltonian of the combined system, the explicit dynamical equation for $X(t)$ can be written as

$$dj_t(X) = j_t(\mathcal{G}(X))dt + j_t(\mathcal{B}(X))dW_1(t) + j_t(\mathcal{C}(X))dW_2(t). \tag{2}$$

Here, the following notation is used:

$$\begin{aligned}\mathcal{G}(X) &= -i[X, H] + \mathfrak{L}(X), \\ \mathcal{B}(X) &= \frac{1}{2}([X, L] + [L(t)^\dagger, X]), \\ \mathcal{C}(X) &= \frac{i}{2}(-[X, L] + [L^\dagger, X]),\end{aligned} \tag{3}$$

with

$$\mathfrak{L}(X) = L^\dagger X L - \frac{1}{2}L^\dagger L X - \frac{1}{2}X L^\dagger L. \tag{4}$$

We have written the noise increments in quadrature form, that is, in terms of

$$dW_1(t) = dB(t) + dB^\dagger(t), \quad dW_2(t) = i(dB(t)^\dagger - dB(t)),$$

with all increments understood in the Itō sense.

Alternatively, we can characterize the average evolution of $X(t)$ by the semigroup T_t acting as

$$T_t(X) = \mathbb{E}_0(U(t)^\dagger(X \otimes I)U(t)).$$

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$\mathbb{E}_0$ is a conditional expectation on the given initial algebra $\mathfrak{A}$ and the initial vacuum state $|0\rangle\langle 0|$ of $\mathfrak{H}_B$. The infinitesimal generator of this Markov semigroup is then given by $\mathcal{G}$. The dissipation functional of the semigroup is defined as[8,22]

$$\mathfrak{D}(X) = \mathcal{G}(X^\dagger X) - \mathcal{G}(X^\dagger)X - X^\dagger \mathcal{G}(X). \tag{5}$$

For a completely positive semigroup T_t, we have $\mathfrak{D}(X) \geq 0$, $X \in \mathfrak{A}$. The dissipation functional characterizes the irreversible nature of the quantum Markov process, and consequently the system is dissipative if $\mathfrak{D} \neq 0$.

The corresponding semigroup in the predual space of the trace class operators (the state space) is denoted as $T_{*t}(\rho) = \rho_t$. The support projection P_{sup} of a state ρ is defined as the smallest projection (in the sense that $P_{\text{sup}} \leq P$) to which the state assigns probability 1.

Definition 1 (Ref. 7). *A state ρ_I is an invariant state, if it satisfies the condition $T_{*t}(\rho_I) = \rho_I$ $\forall t \geq 0$. A state ρ is faithful in $\mathfrak{A}$ if* $\text{Tr}(\rho A) = 0$ *implies $A = 0$ for any positive operator $A \in \mathfrak{A}$. In other words, ρ is faithful if the support projection of ρ is the identity operator in the space of bounded operators on the underlying Hilbert space.*

The faithfulness of an invariant state is essential to the analysis of the asymptotic behavior of a quantum Markov system. For example, if the system possesses a faithful invariant state, then its ergodic properties and the problem of convergence to equilibrium can be studied for this system.[9] Moreover, if this faithful invariant state is unique, then it is the only equilibrium state of the system.[6]

III. QUANTUM LYAPUNOV OPERATORS AND STABILITY IN THE HEISENBERG PICTURE

In classical theory, stability refers to the property that the trajectories of the dynamical systems will remain near an equilibrium point x_e if the initial states x_0 are near x_e. A stronger notion is asymptotic stability which additionally requires that the trajectories that start near the equilibrium state x_e will converge to x_e. In practice, it is often too complicated for nonlinear systems to solve the dynamical equations directly, and the main tool for proving stability of these systems is Lyapunov theory[18,19,38] without finding the trajectories. Generally speaking, if a given system possesses a Lyapunov function $V(x)$, with certain conditions on $V(x)$ and its convective derivative $\dot{V}(x)$, then the trajectories of the system state x_t will be stable in some sense. For example, any continuous scalar function $V : \mathbb{R}^n \to \mathbb{R}$ having the property

$$V(0) = 0, \quad V(x) > 0, \quad \dot{V}(x) < 0, \quad x \in \mathbb{R}^n \backslash \{0\}$$

can be chosen as a Lyapunov function for the purpose of establishing asymptotic stability of the zero equilibrium state of the system. Alternatively, the Lyapunov function can be defined as a continuous function $V(x)$ satisfying[18,24]

$$a(|x|) \leq V(x) \leq b(|x|), \quad \dot{V}(x) \leq 0,$$

where $a(\cdot)$, $b(\cdot)$ are strictly increasing functions with $a(0) = b(0) = 0$ and $a(\infty) = b(\infty) = \infty$. In both cases, if such a V exists, then any trajectory x_t will converge to 0.

As noted in the Introduction, stability of quantum systems may be considered within either the Schrödinger or Heisenberg pictures. In this paper, our intention is to develop the Heisenberg picture approach and tools for studying stability of state trajectories ρ_t within the underlying Schrödinger picture. To this end, we define the Lyapunov operator in the Heisenberg picture.

Definition 2. A quantum Lyapunov operator V is an observable (self-adjoint operator) on a Hilbert space $\mathfrak{H}$ for which the following properties hold:

1. $V \in D(\mathcal{G})$,
2. $V \geq 0$,
3. $\mathcal{G}(V) \leq 0$.

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$D(\mathcal{G})$ is the domain of the generator. Note that $\langle V\rangle_\rho \geq 0$ for all states ρ. In Secs. IV and V, we will show that the existence and stability of the invariant state of the system can be proved using a Lyapunov operator.

One advantage of the Heisenberg approach is that the stability of operators in the Heisenberg picture may be studied, and not just stability of states. This is of practical importance, especially within the framework of quantum coherent networks. Although the stability of operators has been studied using operator semigroup theory, to the best of our knowledge the first approach to stabilization of quantum systems via Lyapunov methods is in Ref. 16.

We now introduce our concept of stability of operators in the Heisenberg picture.

Definition 3. Given a set of positive operators $\mathfrak{S}$, *the system is* $\mathfrak{S}$*-stable if* $\langle j_t(A)\rangle_\rho$ *is bounded for each* $A \in \mathfrak{S}$ *and any initial state* ρ.

In our development of Lyapunov quantum stability, we will make use of the notion of quantum coercivity defined in terms of the spectral decomposition of a Lyapunov operator V.

Definition 4. Consider a positive operator V *with the spectral decomposition* $V = \sum_i v_i P_i$, v_i *being the eigenvalues of* V. V *is coercive if there exists a strictly increasing function* $k(\cdot)$ *with* $\lim_{i\to\infty} k(i) = \infty$ *such that* $v_i \geq k(i)$, $i > i_0$, *for some* i_0.

Note that the definition of quantum coercivity is analogous to its classical counterpart[27,33]

$$V(x) \geq c(|x|)$$

with $c(|x|)$ being a strictly increasing function that goes to infinity as $|x| \to \infty$.

From Definition 4, it follows that if the system possesses a coercive Lyapunov operator $V = \sum_i v_i P_i$, then the set of operators $\mathfrak{S} = \{A \geq 0 : \mathrm{Tr}(AP_i) \leq \epsilon k(i), \epsilon > 0\}$ are bounded in expectation, hence the system is $\mathfrak{S}$-stable. Also, in Sec. VI we will prove that when given a Lyapunov operator V, the set $\mathfrak{S}$ of operators defined by $\mathfrak{S} = \{W \geq 0 : \mathcal{G}(V) \leq -W\}$ are bounded in expectation. Indeed, the expectation $\langle W(t)\rangle_\rho$ will converge to zero according to quantum LaSalle invariance principle. Hence, a conclusion about stability of the system can be made.

As in the classical case, one may have a number of variations on the definition of a Lyapunov function, depending on the context of stability property which one is interested in. The definition of Lyapunov operator can be relaxed for quantum stability analysis in different contexts. Therefore, we still call V a Lyapunov operator when the property $\mathcal{G}(V) \leq 0$ is replaced by a weaker condition (1), as in the following example.

Example 1. Consider a quantum oscillator with the Hamiltonian given by $H = \omega a^\dagger a$, and the coupling operator $L = \alpha a + \beta a^\dagger$, a and $a^\dagger$ are annihilation and creation operators, respectively, and they satisfy the commutation relation $[a, a^\dagger] = 1$. Choose the candidate Lyapunov operator as the photon number operator $V = a^\dagger a$ which represents the energy of the system. By calculation we find $\mathcal{G}(V) = -(|\alpha|^2 - |\beta|^2)V + |\beta|^2 I$. If $|\alpha|^2 > |\beta|^2$, $\mathcal{G}(V)$ satisfies the condition (1) and V becomes a Lyapunov operator in this problem. Furthermore, the system is $\mathfrak{S}$-stable with $\mathfrak{S}$ being the von Neumann algebra generated by V since $\langle V(t)\rangle$ is bounded.[16] If $|\alpha|^2 < |\beta|^2$, $\langle V(t)\rangle$ is unbounded and the system is unstable in energy.

In the sequel, a Lyapunov operator V for which condition (1) holds is referred to as a quantum Lyapunov operator in the weak sense.

IV. QUANTUM TIGHTNESS AND THE EXISTENCE OF INVARIANT STATES

As a first step to study the stability of quantum states, we derive certain conditions to guarantee the existence of invariant state. First, we present the definition of quantum tightness.[26]

Definition 5. A sequence $(\rho_n)_{n\geq 1}$ *in the Banach space of trace-class operators on a Hilbert space* $\mathfrak{H}$ *is tight if for every* $\epsilon > 0$, *there exists a finite rank projection* P *and* $n_0 > 0$ *such that* $\mathrm{Tr}(\rho_n P) > 1 - \epsilon$ *for all* $n \geq n_0$.

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Obviously, trajectories of states corresponding to finite-dimensional systems are tight. We will refer to the following lemma.[26]

Lemma 1. A tight sequence $(\rho_n)_{n\geq 1}$ *of quantum states admits a subsequence converging to a quantum state.*

Theorem 1 (Ref. 7). *If the system possesses a tight family of quantum states,* $(\rho_t, t > 0)$, *then the system possesses at least one invariant state.*

Proof. As ρ_t is tight, any sequence of states $\rho_{t_n} = \frac{1}{t_n}\int_0^{t_n} \rho_{t'} dt'$ is also tight and therefore has normalized sequential limit points. These states are invariant because any sequential limit point of $\frac{1}{t_n}\int_0^{t_n} \rho_{t'} dt'$ is invariant, according to Proposition 2.3 in Ref. 7. □

Based on these properties, we can develop the condition on the tightness of general quantum systems. Recall the inequality (1) involving the (Lindblad) generator $\mathcal{G}$ of a quantum Markov process defined by Eq. (3). Suppose V is a Lyapunov operator in the weak sense, i.e., $\mathcal{G}(V) \leq -cV + dI$, $c > 0$. By integrating (1) we obtain the following inequality:[16]

$$\langle V(t)\rangle \leq e^{-ct}\langle V(0)\rangle + \frac{d}{c},$$

which means $\langle V(t)\rangle \leq \lambda$ for any $t \geq 0$ and some positive λ. Next, we will show that the condition (1) not only gives us the mean stability of V but also implies tightness of the corresponding collection of quantum states $\{\rho_t, t \geq 0\}$.

First, let us consider the following example.

Example 2. The photon number operator for a quantum oscillator can be written as $V = \sum_0^\infty i|i\rangle\langle i|$ where $|i\rangle$ is the photon number state. If $\langle V(t)\rangle \leq c$, $c \geq 0$, then we have $\sum_{i=0}^\infty i\rho_t^{ii} \leq c$ for an arbitrary sequence of states ρ_t; here, $\rho_t^{ii} = \mathrm{Tr}\,(\rho_t|i\rangle\langle i|)$. For an arbitrary $\epsilon > 0$, choose m such that $m = \left[\frac{c}{\epsilon}\right]$, where $[x]$ is the nearest integer to x that is greater than x. Through $m\sum_{i=m}^\infty \rho_t^{ii} \leq \sum_{i=m}^\infty i\rho_t^{ii} \leq c$, we conclude $\sum_m^\infty \rho_t^{ii} \leq \epsilon$ for any t. The finite rank projection $P = \sum_{i=0}^m |i\rangle\langle i|$ then satisfies the condition $\mathrm{Tr}\,(\rho_t P) > 1 - \epsilon$, which indicates that the sequence ρ_t is tight. Hence, the corresponding state trajectory of the quantum oscillator gives rise to an invariant state for the oscillator.

The example shows that under certain conditions, the stability of an operator in the mean sense may imply tightness of a corresponding state trajectory. The inequality

$$m\sum_m^\infty \rho_t^{ii} \leq \sum_m^\infty i\rho_t^{ii} \leq c$$

is essential in this example. In fact, the spectral property of the above operator is the key element connecting tightness and stability. We generalize this idea in the following theorem.

Theorem 2. *Suppose the evolution of a positive observable* V *on a separable Hilbert space* $\mathfrak{H}$, *with spectral decomposition as* $V = \sum_{i=0}^\infty v_i P_i$, *is stable in the mean, that is, there exists a constant* $c \geq 0$ *such that* $\langle V(t)\rangle_\rho \leq c$ *with* ρ *as the initial state. If* V *is coercive, then any sequence* ρ_t *is tight which implies the existence of an invariant state.*

Proof. The proof is similar to the proof used in example 2 to show tightness. The condition that $\langle V(t)\rangle_\rho \leq c$ means $\sum_{i=0}^\infty v_i\rho_t^{ii} \leq c$ for $t \geq 0$. Here, $\rho^{ii} = \mathrm{Tr}\,(\rho P_i)$ denotes the projection P_i on the state ρ. Since V is coercive, there exists some N_0 such that v_i is increasing for $i \geq N_0$ and $v_i \to \infty$ as $i \to \infty$. Choose $m = \max\{N_0, \inf\{i : v_i \geq \frac{c}{\epsilon}\}\}$. Then $v_m\sum_m^\infty \rho_t^{ii} \leq \sum_m^\infty v_i\rho_t^{ii} \leq c$, so we find that $\sum_{i=m}^\infty \rho_t^{ii} \leq \epsilon$. Letting $P = \sum_{i=0}^{m-1} P_i$, we obtain $\mathrm{Tr}\,(\rho_t P) > 1 - \epsilon$; i.e., ρ_t is tight. The result of the theorem then follows from Theorem 1. □

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It follows from Theorem 2 that the existence of a coercive Lyapunov operator in the weak sense (1) guarantees the existence of an invariant state. This prompts the question as to under what condition such an invariant state is unique and/or faithful. This question is addressed in Sec. V.

V. STABILITY OF INVARIANT STATES

In this section, we obtain some conditions to guarantee the faithfulness and uniqueness of an invariant state.

For a particular invariant state ρ_I, its support projection is denoted as P_I. We shall need the following proposition.

Proposition 1 (see, e.g., Ref. 7). *The support projection of an invariant state is subharmonic. That is,* $T_t(P_I) \geq P_I$.

The above property of the support projection can be expressed in terms of the generator $\mathcal{G}$ of the semigroup T_t as $\mathcal{G}(P_I) \geq 0$.

A. Stability of invariant states of finite-dimensional systems

For a finite-dimensional system with the underlying Hilbert space $\mathfrak{H} = \mathbb{C}^n$, the following theorem determines faithfulness and uniqueness of an invariant state.

Definition 6. A state ρ is said to be globally attractive if all system trajectories asymptotically converge to ρ for any initial state.

Theorem 3. *Suppose $\mathfrak{H} = \mathbb{C}^n$. If $PL^\dagger(I - P)LP \neq 0$ for any non-trivial projection P, then the invariant state ρ_I is faithful and unique.*

Proof. A finite dimensional system is tight by Definition 5 and therefore, according to Theorem 1, it admits an invariant state ρ_I. Let P_I be the support projection of ρ_I.

If we take any orthogonal projection P, then we have

$$\mathcal{G}(P) = \mathcal{G}(P^2) = P\mathcal{G}(P) + \mathcal{G}(P)P + \mathfrak{D}(P),$$

where $\mathfrak{D}$ is the dissipation functional defined in (5), and so

$$P\mathcal{G}(P)P = -P\mathfrak{D}(P)P.$$

However, we note that $\mathfrak{D}(X) = [X, L]^\dagger[X, L] \geq 0$, and in particular,

$$P\mathfrak{D}(P)P = PL^\dagger(I - P)LP.$$

Now take the invariant state ρ_I with support projection P_I, then from Proposition 1 we will have $\mathcal{G}(P_I) \geq 0$, and therefore $P_I\mathcal{G}(P_I)P_I \geq 0$. But we then must have $P_I\mathfrak{D}(P_I)P_I = 0$, as $\mathfrak{D} \geq 0$. We thereby deduce that for the invariant state support

$$P_I\mathfrak{D}(P_I)P_I = P_I L^\dagger(I - P_I)LP_I = 0. \tag{6}$$

This is automatically satisfied if ρ_I is faithful, since here $I - P_I \equiv 0$.

Suppose the hypothesis of the theorem is true, namely, that $PL^\dagger(I - P)LP \neq 0$ for any non-trivial orthogonal projection P. If we also now suppose that ρ_I is not faithful, then P_I is non-trivial, then setting $P = P_I$ in (6) leads to a contradiction. Therefore, under the hypothesis, we see that any invariant state must be faithful.

Suppose the invariant state is not unique, then there exist non-trivial orthogonal invariant subspaces by Refs. 3 and 36. This leads to a contradiction since there will exist non-faithful invariant states in the subspace. □

Remark 1. Condition $PL^\dagger(I - P)LP \neq 0$ means that any non-trivial projection P is connected with its orthogonal complement by L. This property can be easily verified when the system has

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reduced dynamics. For example, if the quantum states maintain a diagonal form $\rho(t) = \sum_i \rho^{ii}(t) P_i$, $P_i = |i\rangle\langle i|$ *during evolution, we only need to verify* $P_i L^\dagger (I - P_i) L P_i \neq 0$ *for all* P_i. *To generalize, if there exists a family of projections* $\{P_i\}$ such that $\sum_i P_i = I$ and $P_i L^\dagger (I - P_i) L P_i \neq 0$, *the marginal distribution of the invariant state will have non-vanishing probability on each projector* P_i.

It is worth mentioning that for finite dimensional system, uniqueness of invariant state directly leads to global convergence.[36]

Example 3. Consider the quantum two-level system with a basis denoted as $\{|0\rangle, |1\rangle\}$. $H = \omega\sigma_z = \omega(|0\rangle\langle 0| - |1\rangle\langle 1|)$ and $L = \sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$. The quantum state evolves according to the master equation

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + L\rho(t)L^\dagger - \frac{1}{2}L^\dagger L\rho(t) - \frac{1}{2}\rho(t)L^\dagger L.$$

Obviously, the density matrix of the state will remain diagonal if the initial state is $\alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1|$ with arbitrary α and β satisfying $|\alpha|^2 + |\beta|^2 = 1$. As a result, the system will possess a diagonal invariant state. We only need to consider the projections $\{|1\rangle\langle 1|, |0\rangle\langle 0|\}$ in order to conclude faithfulness of this invariant state. In fact, we have $|1\rangle\langle 1|\sigma_x|0\rangle\langle 0|\sigma_x|1\rangle\langle 1| = |1\rangle\langle 1| \neq 0$ and $|0\rangle\langle 0|\sigma_x|1\rangle\langle 1|\sigma_x|0\rangle\langle 0| = |0\rangle\langle 0| \neq 0$, so the two-level system has a unique faithful invariant state which is globally attractive.

B. Stability of invariant states of infinite-dimensional systems

Now we can prove the main result in this section for the quantum system defined on a separable Hilbert space $\mathfrak{H}$.

Theorem 4. *Suppose there exists a coercive Lyapunov operator in the weak sense (1). If* $PL^\dagger(I - P)LP \neq 0$ *for any non-trivial projection P, then any invariant state* ρ_I *is faithful and unique. Furthermore, this faithful state* ρ_I *is globally attractive.*

Proof. The proof follows along the same lines as the proof of Theorem 3. However, the existence of invariant state comes from condition (1) and coercivity, and is a direct result of Theorem 2. The Lyapunov operator inequality (1) and the algebraic condition $PL^\dagger(I - P)LP \neq 0$ are then combined to guarantee the uniqueness and faithfulness of this invariant state which is also the equilibrium point of the system. In addition, the unique invariant state is also globally attractive due to its faithfulness.[6] □

We also note a certain analogy between Theorem 4 and the corresponding results from the classical theory of stochastic Markov processes; e.g., see Ref. 27. In particular, our condition (1) is analogous to the positive recurrence condition (CD2) in Ref. 27.

Example 4. Consider again a quantum oscillator with the Hamiltonian $H = \omega a^\dagger a$, and the coupling operator $L = \alpha a + \beta a^\dagger$.

Consider the observable $V = a^\dagger a$ which has a strictly increasing and unbounded spectrum. $\mathcal{G}(V) = -(|\alpha|^2 - |\beta|^2)V + |\beta|^2 I$. In order to satisfy the Lyapunov condition in Theorem 4, we need to set $|\alpha| > |\beta|$. In this case, $\langle V(t)\rangle$ is bounded with respect to any initial state. Hence, according to Theorem 2, this system admits an invariant state. Now we want to study the set of projections $\{P_i = |i\rangle\langle i|\}$. Note that since $|i\rangle$ is the photon number state, then $\sum_i P_i = I$. We have $|i\rangle\langle i|L^\dagger|i+1\rangle\langle i+1|L|i\rangle\langle i| = (i+1)|\beta|^2|i\rangle\langle i| \neq 0$ and $|i\rangle\langle i|L^\dagger|i-1\rangle\langle i-1|L|i\rangle\langle i| = i|\alpha|^2|i\rangle\langle i| \neq 0$. Therefore, any photon number state is connected to its two neighboring states, so by induction any non-trivial projection $P = \sum_j P_j \neq I$ is not a support projection of an invariant state ρ_I. Consequently, the photon-number distribution of the invariant state has non-vanishing probability on the entire Fock basis. However, this does not imply that the invariant state is faithful because there may exist other set of projections that does not satisfy the algebraic condition of Theorem 4.

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VI. QUANTUM LASALLE INVARIANCE PRINCIPLE

In Sec. V, we studied the stability property of convergence to faithful invariant states. Other classes of stabilization problems of interest are concerned with stability of non-commuting operators, or require convergence to an invariant set for any state trajectories. Similar to the classical LaSalle's invariance principle[20,25] that is used to identify the asymptotic stability of system trajectories, the invariance theorems which we will derive here pave the way for analyzing the underlying dynamics of general quantum states which may not be faithful in the Heisenberg picture.

The classical LaSalle theorem states the following fact:[20] If a positive and uniformly continuous function $V(x)$ can be found on a compact space such that $\dot{V}(x) \leq 0$, then the limit points of any trajectory x_t are contained in the largest invariant subset of $\{x : \dot{V}(x) = 0\}$.

First, we will derive the direct analogue of the classical LaSalle invariance theorem in the Heisenberg picture.

Definition 7. A quantum state ρ is said to be the zero solution of an operator X if ρ solves $\langle X\rangle_\rho = 0$.

Theorem 5. *If there exists a coercive Lyapunov operator V and a positive operator W with $\mathcal{G}(W)$ bounded in the operator norm such that*

$$\mathcal{G}(V) \leq -W, \tag{7}$$

then $\lim_{t\to\infty}\langle V(t)\rangle_{\rho_0} = \lim_{t\to\infty}\langle V\rangle_{\rho_t}$ *exists for any initial state* ρ_0 *and*

$$\begin{aligned} \textstyle\int_0^\infty \langle W(t')\rangle_{\rho_0} dt' &= \textstyle\int_0^\infty \langle W\rangle_{\rho_{t'}} dt' < +\infty, \\ \lim_{t\to\infty}\langle W(t)\rangle_{\rho_0} &= \lim_{t\to\infty}\langle W\rangle_{\rho_t} = 0. \end{aligned} \tag{8}$$

Proof. Referring to Theorem 2, tightness of ρ_t ensures the existence of a limit point (or an accumulation point) of the system evolutions. The function $\langle V(t)\rangle_{\rho_0}$ is decreasing with t since $\mathcal{G}(V) \leq 0$. Therefore, $\lim_{t\to\infty}\langle V(t)\rangle_{\rho_0} = \lim_{t\to\infty}\langle V\rangle_{\rho_t}$ exists because any decreasing sequence with a lower bound will converge to a limit. Moreover, $\langle V(t)\rangle_{\rho_0}$ evolves according to

$$\langle V(t)\rangle_{\rho_0} - \langle V\rangle_{\rho_0} = \int_0^t \langle \mathcal{G}(V(t'))\rangle_{\rho_0} dt' \leq \int_0^t \langle -W(t')\rangle_{\rho_0} dt'. \tag{9}$$

It follows from (9) that $\int_0^t \langle W(t')\rangle_{\rho_0} dt' \leq \langle V\rangle_{\rho_0}$, which implies

$$\int_0^\infty \langle W(t')\rangle_{\rho_0} dt' = \int_0^\infty \langle W\rangle_{\rho_{t'}} dt' < +\infty.$$

ρ_t is tight, so the positive sequence $\langle W\rangle_{\rho_t}$ must have convergent subsequence. Suppose there exists a subsequence ρ_{t_k} such that $\lim_{k\to\infty}\langle W\rangle_{\rho_{t_k}} = \epsilon > 0$. Now we show that this leads to a contradiction. Since $\mathcal{G}(W)$ is bounded in operator norm by R, we have

$$|\langle W(t_1)\rangle_{\rho_0} - \langle W(t_2)\rangle_{\rho_0}| = |\int_{t_2}^{t_1} \mathrm{Tr}\,(\mathcal{G}(W)\rho_{t'})\, dt'| \leq \int_{t_2}^{t_1} \|\mathcal{G}(W)\| |\rho_{t'}| dt' \leq R|t_1 - t_2|,$$

which means $\langle W\rangle_{\rho_t}$ is uniformly continuous in t. Here, $|\rho_{t'}| = 1$ denotes the trace-norm of the density state $\rho_{t'}$. According to the uniform continuity, we are able to find a $\delta > 0$ such that the following inequality:

$$\langle W\rangle_{\rho_{t'}} > \frac{\epsilon}{2}$$

holds if $|t' - t_k| < \frac{\delta}{2}$ for any t_k. This further implies

$$\int_0^\infty \langle W\rangle_{\rho_t} dt \geq \sum_k^\infty \delta \frac{\epsilon}{2} = +\infty,$$

which is a contradiction.

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The above contradiction implies that every converging subsequence of $\langle W\rangle_{\rho_t}$ converges to 0. Then we conclude that $\lim_{t\to\infty}\langle W\rangle_{\rho_t} = 0$. □

Remark 2. For a Lyapunov operator V with $\mathcal{G}(V) \le 0$, we can always let $W = -\mathcal{G}(V)$ and thus the trajectories will converge to $\{\rho : \langle\mathcal{G}(V)\rangle_\rho = 0\}$ if $\mathcal{G}(\mathcal{G}(V))$ is bounded, according to Theorem 5. The states from the invariant set $\{\rho : \langle\mathcal{G}(V)\rangle_\rho = 0\}$ are zero solutions of W. This conclusion is similar to the statement of the classical LaSalle theorem.

Corollary 1. If inequality (7) in Theorem 5 is replaced by

$$\mathcal{G}(V) \le U - W,$$

where U is a positive operator satisfying

$$\int_0^\infty \langle U(t)\rangle_{\rho_0} dt < \infty$$

for any initial state ρ_0, the conclusions of Theorem 5 still hold.

Proof. The proof is similar to the proof of Theorem 5. □

The question is how we can characterize the pairs of operators V and W for which (7) holds. Note that $\langle\mathcal{G}(V)\rangle_\rho \ge 0$ for any ground state ρ of V and consequently $\langle W\rangle_\rho = 0$. More specifically, W must have the ground states of V as its zero solutions. This observation will limit the set of W we can choose from. For example, if $V = a^\dagger a$ is the energy operator of a quantum oscillator, we will not be able to establish $\mathcal{G}(V) \le -W$ for the position operator $W = (a + a^\dagger)^2$ because the ground state $|0\rangle\langle 0|$ of V has nonzero variance in position. In other words, it is impossible to generate states with zero variance in position by stabilizing the energy of the system. This example reveals the fundamental difficulty in stabilizing non-commuting operators, which is also the implication of the Heisenberg uncertainty principle. Nevertheless, through the stability of V we can still infer the information about the non-commuting operators that are restricted in a subspace. In addition, W can also be used to characterize other invariant limit sets of ρ_t besides the set of the ground states of V. We illustrate these ideas in the following example:

Example 5. Consider a single-qubit system with energy operator $V = \frac{1}{2}(1 + \sigma_z)$. The aim is to make the expectation of the coherence operator $W = \frac{1}{2}(1 + \sigma_x)$ zero and in the same time stabilize the energy of the system. However, the non-commuting observables V and W cannot be stabilized simultaneously via $\mathcal{G}(V) \le -W$ since the ground state $|1\rangle\langle 1|$ of V is not the zero solution of W. An alternative solution to this problem is to consider the augmented system with an ancillary qubit and define $W = |0\rangle\langle 0| \otimes \frac{1}{2}(1 + \sigma_{x_2})$. The energy of the two-qubit system is characterized by the operator V as $V = \sigma_{z_1} + \sigma_{z_2}$. σ_{z_i} is the Pauli operator σ_z acting on the ith qubit. The basis of the bipartite system is chosen as the four eigenstates $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, leading to the following expression of V and W:

$$V = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, W = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Although V is not positive, Theorem 5 still applies to this example by shifting V with a constant. We can engineer $\mathcal{G}(V)$ (see the Appendix) through engineering the couplings between the eigenstates.

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By introducing the couplings $l|01\rangle\langle 00|$ and $l|11\rangle\langle 01|$ with $|l|^2 = \frac{1}{2}$, $\mathcal{G}(V)$ will become

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Set the Hamiltonian control H as $-\frac{1}{2}i|00\rangle\langle 01| + \frac{1}{2}i|01\rangle\langle 00| = |0\rangle\langle 0| \otimes \sigma_{y_2}$, the new $\mathcal{G}(V)$ is

$$\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which satisfies the required inequality $\mathcal{G}(V) \leq -W$. The system will converge to the zero solutions of $W = |0\rangle\langle 0| \otimes \frac{1}{2}(1 + \sigma_{x_2})$ while the energy operator is stabilized (the energy of the two-qubit system is decreasing).

Given the density matrix of ρ as

$$\begin{pmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \rho_{03} \\ \rho_{10} & \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{20} & \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{30} & \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix},$$

the limit states will satisfy $\langle |0\rangle\langle 0| \otimes \frac{1}{2}(1 + \sigma_{x_2})\rangle_\rho = 0$ and hence $\rho_{00} + \rho_{01} + \rho_{10} + \rho_{11} = 0$. In this example, we are able to infer the information about the coherence $\rho_{01} + \rho_{10}$ between $|00\rangle$ and $|01\rangle$ within the two-level subspace through the generator of the energy operator V. Note that V and W do not commute. One particular state satisfying $\rho_{00} + \rho_{01} + \rho_{10} + \rho_{11} = 0$ is $\frac{1}{2}(|00\rangle\langle 00| - |00\rangle\langle 01| - |01\rangle\langle 00| + |01\rangle\langle 01|)$, which is an invariant state of the system. Note that the space spanned by $\{|10\rangle, |11\rangle\}$ also satisfies $\rho_{00} + \rho_{01} + \rho_{10} + \rho_{11} = 0$. We can further narrow down the set of limit points by making G_{22} negative via the methods introduced in the Appendix such that the invariant set will only contain states that are either in the space spanned by $\{|00\rangle, |01\rangle\}$ with stabilized coherence, or in the ground state $|11\rangle\langle 11|$.

Moreover, if we make a projection $|0\rangle\langle 0|$ on the 1st qubit via a quantum measurement, the reduced quantum state of the 2nd qubit will satisfy $\langle \frac{1}{2}(1 + \sigma_{x_2})\rangle_{\rho_2} = 0$. Interestingly, by stabilizing the energy operator $\sigma_{z_1} + \sigma_{z_2}$ of the augmented system and then making a projective measurement $|0\rangle\langle 0|$, we are able to stabilize the coherence operator $\frac{1}{2}(1 + \sigma_x)$ of the qubit in the end.

The interpretation of these results is as follows: Extra space is needed to store the excess noises introduced by the Heisenberg uncertainty principle. This idea is similar to the design of non-degenerate parametric amplifier, where additional channel of noise input is introduced in order to amplify the amplitude and phase quadratures simultaneously. □

For a positive operator W with unbounded $\mathcal{G}(W)$, we have the following theorem.

Theorem 6. *If there exists a Lyapunov operator V and a positive operator W such that*

$$\mathcal{G}(V) \leq -W, \quad \mathcal{G}(W) \leq 0,$$

then $\lim_{t\to\infty}\langle V(t)\rangle_{\rho_0} = \lim_{t\to\infty}\langle V\rangle_{\rho_t}$ *exists for any state trajectory ρ_t and (8) holds.*

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Proof. Following the same reasoning as in the proof of Theorem 5, we can conclude $\int_0^\infty \langle W\rangle_{\rho_t} dt < +\infty$. The conditions $\mathcal{G}(W) \le 0$ and $\langle W\rangle_{\rho_t}$ is bounded from below guarantee that $\langle W\rangle_{\rho_t}$ is convergent. The limit of $\langle W\rangle_{\rho_t}$ can only be 0 because $\int_0^\infty \langle W\rangle_{\rho_t} dt$ is finite. □

Remark 3. Suppose $\mathcal{G}(V) \le -cV$, $c > 0$, and V is a Lyapunov operator. Let $W = cV$ and we have $\mathcal{G}(W) = c\mathcal{G}(V) \le -c^2 V \le 0$. The system will converge to the zero solutions of V, or equivalently speaking, to the set of ground states $Z_V = \{\rho : \langle V\rangle_\rho = 0\}$.

Theorem 6 can be extended to treat a general Hermitian operator W.

Theorem 7. *If there exists a Lyapunov operator V satisfying $\langle V(t)\rangle \le c$ for $t > 0$ and*

$$\mathcal{G}(V) = W,$$

where the generator of W satisfies $\mathcal{G}(W) \le 0$, then

$$\lim_{t\to\infty} \langle W(t)\rangle_{\rho_0} = \lim_{t\to\infty} \langle W\rangle_{\rho_t} = 0.$$

Proof. $\langle V(t)\rangle$ is bounded for all $t > 0$. From (9) we know that $-\infty < \int_0^\infty \langle W\rangle_{\rho_t} dt < +\infty$. If $\mathcal{G}(W) \le 0$ by assumption, then the monotonic sequence $\langle W\rangle_{\rho_t}$ is bounded from below and hence will converge to a limit. The limit is exactly 0 since the integral $\int_0^\infty \langle W\rangle_{\rho_t} dt$ is bounded. □

VII. STABILITY WITHIN THE INVARIANT SET

We have used multiple Lyapunov conditions in Theorem 6 and Theorem 7. Similarly, we can use additional Lyapunov conditions to further engineer the dynamics of the trajectories within the invariant set. For example, we can make use of the Lyapunov operator $W = V^2$ to drive the system states to the zero solutions of V, where in general the system will converge only to the zero solutions of $\mathcal{G}(V)$ by LaSalle invariance principle.

As we have known from classical stochastic stability and quantum semigroup theory, the asymptotic dynamics of the trajectories are determined by the diffusion terms[18] or the dissipation functional $\mathfrak{D}(\cdot)$.[7–9] As shown in the proof of Theorem 3, we can make explicit connection between the dissipation functional and the diffusion terms $j_t(\mathcal{B}(X))$, $j_t(\mathcal{C}(X))$ by calculating $\mathcal{G}(V^2)$.

Theorem 8. *Suppose $\mathcal{G}(V) \le 0$ for the Lyapunov operator V of a finite-dimensional system. The state trajectory ρ_t will converge to the set of zero solutions $Z_V = \{\rho : \langle V\rangle_\rho = 0\}$ if $\langle [L^\dagger, V][V, L]\rangle_\rho > 0$ for $\rho \notin Z_V$ and $[\mathcal{G}(V), V] = 0$.*

Proof. Since $\mathcal{G}(V) \le 0$, $\lim_{t\to\infty}\langle V\rangle_{\rho_t}$ exists and Z_V is an invariant set. We only need to prove that ρ_t will exit the domain $\{\rho : \langle V\rangle_\rho \ge \epsilon\}$ for arbitrary $\epsilon > 0$. Consider the positive operator $W(V) = V^2$. Similar to the derivations in Theorem 3, the generator for $W(V)$ can be calculated using the quantum Itō formula

$$\mathcal{G}(W) = V\mathcal{G}(V) + \mathcal{G}(V)V + \mathfrak{D}(V)$$

with

$$\mathfrak{D}(V) = \mathcal{B}(V)^2 + \mathcal{C}(V)^2 + i\mathcal{B}(V)\mathcal{C}(V) - i\mathcal{C}(V)\mathcal{B}(V) = [L^\dagger, V][V, L].$$

For finite-dimensional system, any state trajectory is tight. Suppose the trajectory ρ_t is restricted to a domain $\{\rho : \langle V\rangle_\rho \ge \epsilon\}$ for some $\epsilon > 0$. Then by Theorem 1 there exists an invariant state ρ_I which is the limit point of the tight sequence $\frac{1}{t}\int_0^t \rho_{t'} dt'$. Note that $\frac{1}{t}\int_0^t \rho_{t'} dt'$ is the mean of the sequence ρ_t, so ρ_I is in the same domain $\{\rho : \langle V\rangle_\rho \ge \epsilon\}$ as ρ_t which means $\rho_I \notin Z_V$.

Let the initial state be exactly the invariant state ρ_I. First, we prove $\langle V\mathcal{G}(V) + \mathcal{G}(V)V\rangle_{\rho_I} = 0$. Since V is positive and $[\mathcal{G}(V), V] = 0$, $V\mathcal{G}(V)$ and $\mathcal{G}(V)V$ are negative hermitian operators which make $\langle V\mathcal{G}(V) + \mathcal{G}(V)V\rangle_{\rho_I} \le 0$. Furthermore, we have $\langle V\mathcal{G}(V) + \mathcal{G}(V)V\rangle_{\rho_I} = \langle (V+\beta)\mathcal{G}(V) + \mathcal{G}(V)(V+\beta)\rangle_{\rho_I}$ due to the fact that $\langle \mathcal{G}(V)\rangle_{\rho_I} = 0$. V is bounded, so we can choose $\beta < 0$ such that

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$V+\beta$ is negative. Given this β, we can conclude $\langle (V+\beta)\mathcal{G}(V)+\mathcal{G}(V)(V+\beta)\rangle_{\rho_I} \geq 0$ which gives us $\langle V\mathcal{G}(V)+\mathcal{G}(V)V\rangle_{\rho_I} \geq 0$. So $\langle V\mathcal{G}(V)+\mathcal{G}(V)V\rangle_{\rho_I} = 0$. Next, we have the following relation by integrating $\mathcal{G}(W)$:

$$\begin{aligned}\langle W(V)\rangle_{\rho_I} - \langle W(V)\rangle_{\rho_I} &= \int_0^t \langle V\mathcal{G}(V)+\mathcal{G}(V)V+\mathfrak{D}(V)\rangle_{\rho_I} dt' \\ &= \int_0^t \langle \mathfrak{D}(V)\rangle_{\rho_I} dt'. \end{aligned} \tag{10}$$

The LHS of the equality is zero, however the RHS of the equality is strictly positive, since $\langle \mathfrak{D}(V)\rangle_{\rho_I} = \langle [L^\dagger, V][V, L]\rangle_{\rho_I} > 0$ by assumption. So we arrive at a contradiction. The contradiction shows that a trajectory ρ_t cannot be confined to the domain $\{\rho : \langle V\rangle_\rho \geq \epsilon\}$. Hence, ρ_t will approach Z_V asymptotically. □

Corollary 2. Assume in a finite-dimensional system the Lyapunov operator V has the decomposition $V = M^\dagger M$. If M solves $M = [V, L]$ and then $\mathcal{G}(V) \leq 0$, $[\mathcal{G}(V), V] = 0$, the state trajectory ρ_t will converge to the zero solutions of V.

Example 6. Consider a qubit with $V = \frac{1}{2}(1+\sigma_z)$, or in matrix expression

$$V = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The decomposition is found to be $V = \sigma_+\sigma_-$ with

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$\sigma_+^\dagger = \sigma_- = M$. The solution to $M = [V, L]$ is

$$L = \begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix}$$

with a and b being arbitrary constants. With this L, the dissipation part $\mathfrak{L}(V)$ equals

$$\begin{pmatrix} -1 & -\frac{b}{2} \\ -\frac{b}{2} & 0 \end{pmatrix}.$$

Let $H = 0$ and $b = 0$, then $\mathcal{G}(V) = \mathfrak{L}(V) \leq 0$. The system will converge to the ground state $|1\rangle\langle 1|$.

VIII. CONCLUSION

Many theorems concerning asymptotic properties of quantum Markov semigroups have the existence of a faithful invariant state as an essential assumption. We have derived sufficient conditions to verify this assumption. If these sufficient conditions hold, the unique and faithful state is an equilibrium point which is also globally attractive. Our approach makes use of the Lyapunov method complemented by additional algebraic conditions. Our result exhibits some analogy with the classical Foster-Lyapunov theory concerning the existence of invariant measures of Markov processes. Beyond invariant states, we have introduced the quantum invariance principle to characterize the set of limit states of the system dynamics. More specifically, the system will asymptotically converge to the ground state of an operator W if we are able to engineer the generator of a Lyapunov operator V. These invariance theorems are established via a Lyapunov inequality between these two operators, which has potential to provide useful tools for stability analysis of the non-commutative algebra associated with general quantum coherent control systems. Moreover, the system can be driven further to the ground state of V within the invariant set if additional conditions on the Lyapunov operator can be engineered. These results may also find essential applications in quantum information

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processing, since the outcomes of quantum computations can be encoded in the ground state of a particular operator.[40]

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APPENDIX: CONSTRUCTIVE METHODS FOR ENGINEERING A NEGATIVE GENERATOR

In this appendix, we introduce a constructive method to engineer a negative generator for the Lyapunov operator V.

First, we will focus on engineering the dissipation part $\mathfrak{L}(V) = \frac{1}{2}(2L^\dagger VL - L^\dagger LV - VL^\dagger L)$ of the generator $\mathcal{G}(V)$ by assuming $[V, H] = 0$. In a separable space, we can decompose L and V as

$$L = \begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}, V = \begin{pmatrix} V_{00} & V_{01} \\ V_{10} & V_{11} \end{pmatrix}.$$

V is a positive hermitian operator, so we can always make $V_{01} = V_{10} = 0$ through spectral decomposition. The generator $\mathcal{G}(V)$ is calculated to be

$$\mathcal{G} = \begin{pmatrix} G_{00} & G_{01} \\ G_{10} & G_{11} \end{pmatrix}$$

with

$$G_{00} = (L_{00}^\dagger V_{00} L_{00} + L_{10}^\dagger V_{11} L_{10}) - \frac{1}{2}\{L_{00}^\dagger L_{00} + L_{10}^\dagger L_{10}, V_{00}\},$$

$$G_{01} = (L_{00}^\dagger V_{00} L_{01} + L_{10}^\dagger V_{11} L_{11}) - \frac{1}{2}(L_{00}^\dagger L_{01} + L_{10}^\dagger L_{11})V_{11} - \frac{1}{2}V_{00}(L_{00}^\dagger L_{01} + L_{10}^\dagger L_{11}),$$

$$G_{10} = G_{01}^\dagger,$$

$$G_{11} = (L_{01}^\dagger V_{00} L_{01} + L_{11}^\dagger V_{11} L_{11}) - \frac{1}{2}\{V_{11}, L_{01}^\dagger L_{01} + L_{11}^\dagger L_{11}\}.$$

Here, we only present the calculations for two-level system. For higher dimensional systems, the blocks of G, L, and H will be matrices or operators. However, we can still do analysis for arbitrary dimensional systems by carefully engineering the two-dimensional subsystems and compensating the interactions between different subsystems. This approach is possible because of the linearity of the generator $\mathcal{G}$. For example, G_{00} can be divided into the internal dynamics $2L_{00}^\dagger V_{00} L_{00} - \{L_{00}^\dagger L_{00}, V_{00}\}$ and the interaction with other dimensions $2L_{10}^\dagger V_{11} L_{10} - \{L_{10}^\dagger L_{10}, V_{00}\}$.

1. $V_{00} = V_{11}$

We can set $L_{00} = 0$ and $L_{11} = 0$ to make $G_{01} = G_{10} = 0$. However, due to the degeneracy of V, we also have $G_{00} = G_{11} = 0$. Therefore, $\mathcal{G} = 0$ and the entire two-dimensional space is irreducible. The off-diagonal elements of L will not affect $\mathcal{G}$.

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2. $V_{00} \neq V_{11}$

Without loss of generality we assume $V_{11} - V_{00} = v > 0$. Again assuming $L_{00} = 0$ and $L_{11} = 0$, the generator becomes

$$\mathcal{G} = \begin{pmatrix} vL_{10}^{\dagger}L_{10} & 0 \\ 0 & -vL_{01}^{\dagger}L_{01} \end{pmatrix}.$$

Now we can set $L_{10} = 0$, $L_{01} = l \neq 0$ so that $G_{00} = 0$, $G_{11} < 0$. The coupling operator L for engineering negative $\mathcal{G}(V)$ with non-degenerate spectrum could be

$$L = \begin{pmatrix} 0 & l \\ 0 & 0 \end{pmatrix}.$$

3. $[V, H] \neq 0$

$[V, H] \neq 0$ happens when V is not representing the energy of the system or additional Hamiltonian control H_c is needed for stabilization. Since $V_{01} = V_{10} = 0$, the commutator $C = -i[V, H]$ can be calculated as

$$C = -i \begin{pmatrix} [V_{00}, H_{00}] & V_{00}H_{01} - H_{01}V_{11} \\ V_{11}H_{10} - H_{10}V_{00} & [V_{11}, H_{11}] \end{pmatrix}.$$

C_{00} is the internal unitary dynamics within the subspace X_{00}. For two-dimensional system, V_{ij} and H_{ij} are complex numbers, which gives $C_{00} = C_{11} = 0$. If $V_{00} = V_{11}$, then $C_{01} = C_{10} = 0$ and the unitary dynamics induced by H will not affect $\mathcal{G}$. If $V_{00} \neq V_{11}$, we set $V_{11} - V_{00} = v > 0$ and $L_{10} = 0$. In this case, the generator will still satisfy the relations $G_{00} = 0$ and $G_{11} < 0$ after adding C to $\mathcal{G}$. However, G_{01} cannot be made vanish if the diagonal entries of L are all zero. In fact, we have

$$G_{01} = L_{00}^{\dagger}L_{01}(V_{00} - V_{11}) - iH_{01}(V_{00} - V_{11}),$$

so $L_{00}^{\dagger}L_{01} = iH_{01}$ must be satisfied. If we choose $L_{01} = l \neq 0$, the coupling operator L should be in the following form:

$$L = \begin{pmatrix} -\frac{iH_{01}^{*}}{l^{*}} & l \\ 0 & L_{11} \end{pmatrix}$$

to completely eliminate the influence of H.

[1] C. Altafini and F. Ticozzi, "Modelling and control of quantum systems: An introduction," IEEE Trans. Autom. Control **57**(8), 1898–1917 (2012).
[2] H. Amini, M. Mirrahimi, and P. Rouchon, "Stabilization of a delayed quantum system: The photon box case-study," IEEE Trans. Autom. Control **57**(8), 1918–1930 (2012).
[3] B. Baumgartner and H. Narnhofer, "The structures of state space concerning quantum dynamical semigroups," Rev. Math. Phys. **24**, 1250001 (2012).
[4] D. Burgarth and V. Giovannetti, "The generalized Lyapunov theorem and its application to quantum channels," New J. Phys. **9**(3), 150 (2007).
[5] D. Dong and I. Petersen, "Quantum control theory and applications: A survey," IET Control Theory Appl. **4**, 2651–2671 (2010).
[6] F. Fagnola and R. Rebolledo, "Subharmonic projections for a quantum Markov semigroup," J. Math. Phys. **43**, 1074 (2002).
[7] F. Fagnola and R. Rebolledo, "Quantum Markov semigroups and their stationary states," *Stochastic Analysis and Mathematical Physics II* (Springer, 2003), pp. 77–128.
[8] A. Frigerio, "Stationary states of quantum dynamical semigroups," Commun. Math. Phys. **63**(3), 269–276 (1978).
[9] A. Frigerio and M. Verri, "Long-time asymptotic properties of dynamical semigroups on w*-algebras," Math. Z. **180**(3), 275–286 (1982).
[10] J. Gough and M. James, "Quantum feedback networks: Hamiltonian formulation," Commun. Math. Phys. **287**, 1109–1132 (2009).
[11] J. Gough, M. James, and H. Nurdin, "Squeezing components in linear quantum feedback networks," Phys. Rev. A **81**, 023804 (2010).

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[12] R. Hamerly and H. Mabuchi, "Advantages of coherent feedback for cooling quantum oscillators," Phys. Rev. Lett. **109**, 173602 (2012).
[13] R. van Handel, J. Stockton, and H. Mabuchi, "Feedback control of quantum state reduction," IEEE Trans. Autom. Control **50**(6), 768–780 (2005).
[14] R. L. Hudson and K. R. Parthasarathy, "Quantum Ito's formula and stochastic evolutions," Commun. Math. Phys. **93**(3), 301–323 (1984).
[15] M. James, H. Nurdin, and I. Petersen, "H-infinity control of linear quantum stochastic systems," IEEE Trans. Autom. Control **53**(8), 1787–1803 (2008).
[16] M. R. James and J. Gough, "Quantum dissipative systems and feedback control design by interconnection," IEEE Trans. Autom. Control **55**(8), 1806–1821 (2010).
[17] M. R. James, I. R. Petersen, and V. Ugrinovskii, "A Popov stability condition for uncertain linear quantum systems," in *Proceedings of the American Control Conference*, Washington, DC, 2013.
[18] R. Khasminskii, *Stochastic Stability of Differential Equations* (Springer-Verlag, Berlin, 2012), Vol. 66.
[19] H. J. Kushner, *Stochastic Stability and Control* (Academic Press, New York, 1967).
[20] J. P. Lasalle, "Stability theory for ordinary differential equations," J. Differ. Equ. **4**, 57–65 (1968).
[21] Y. Lin, J. P. Gaebler, F. Reiter, T. R. Tan, R. Bowler, A. S. Sorensen, D. Leibfried, and D. J. Wineland, "Dissipative production of a maximally entangled steady state of two quantum bits," Nature (London) **504**, 415–418 (2013).
[22] G. Lindblad, "On the generators of quantum dynamical semigroups," Commun. Math. Phys. **48**, 119–130 (1976).
[23] H. Mabuchi and N. Khaneja, "Principles and applications of control in quantum systems," Int. J. Robust Nonlin. Control **15**, 647–667 (2005).
[24] M. Malisoff and F. Mazenc, *Constructions of Strict Lyapunov Functions*, Communications and Control Engineering Vol. 70 (Springer, London, 2009).
[25] X. Mao, "Stochastic versions of the LaSalle theorem," J. Differ. Equ. **153**, 175–195 (1999).
[26] P. A. Meyer, *Quantum Probability for Probabilists* (Springer Verlag, 1995), Vol. 1538.
[27] S. P. Meyn and R. L. Tweedie, "Stability of markovian processes III: Foster-Lyapunov criteria for continuous-time processes," Adv. Appl. Probab. **25**, 518–548 (1993).
[28] M. Mirrahimi and R. Van Handel, "Stabilizing feedback controls for quantum systems," SIAM J. Control Optim. **46**(2), 445–467 (2007).
[29] H. Nurdin, M. James, and I. Petersen, "Coherent quantum LQG control," Automatica **45**, 1837–1846 (2009).
[30] K. R. Parthasarathy, *An Introduction to Quantum Stochastic Calculus* (Springer, 1992), Vol. 85.
[31] I. R. Petersen, V. Ugrinovskii, and M. R. James, "Robust stability of uncertain linear quantum systems," Philos. Trans. R. Soc. A **370**(1979), 5354–5363 (2012).
[32] B. Qi, H. Pan, and L. Guo, "Further results on stabilizing control of quantum systems," IEEE Trans. Autom. Control **58**(5), 1349–1354 (2013).
[33] M. Renardy and R. C. Rogers, *An Introduction to Partial Differential Equations* (Springer-Verlag, New York, 2004).
[34] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, 1985).
[35] C. Sayrin, I. Dotsenko, X. Zhou, B. Peaudecerf, T. Rybarczyk, S. Gleyzes, P. Rouchon, M. Mirrahimi, H. Amini, M. Brune, J. M. Raimond, and S. Haroche, "Real-time quantum feedback prepares and stabilizes photon number states," Nature (London) **477**, 73–77 (2011).
[36] S. G. Schirmer and X. Wang, "Stabilizing open quantum systems by markovian reservoir engineering," Phys. Rev. A **81**(6), 062306 (2010).
[37] A. Serafini, "Feedback control in quantum optics: An overview of experimental breakthroughs and areas of application," ISRN Opt. **2012**, 275016 (2012).
[38] U. H. Thygesen, *Technical Report: A Survey of Lyapunov Techniques for Stochastic Differential Equations* (IMM, Department of Mathematical Modelling, Technical University of Denmark, Lyngby, 1997).
[39] F. Ticozzi, K. Nishio, and C. Altafini, "Stabilization of stochastic quantum dynamics via open and closed loop control," IEEE Trans. Autom. Control **58**(1), 74–85 (2013).
[40] F. Verstraete, M. M. Wolf, and J. Ignacio Cirac, "Quantum computation and quantum-state engineering driven by dissipation," Nat. Phys. **5**(9), 633–636 (2009).
[41] X. Wang and S. Schirmer, "Analysis of Lyapunov method for control of quantum states," IEEE Trans. Autom. Control **55**(10), 2259–2270 (2010).
[42] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2009).
[43] M. Yanagisawa and H. Kimura, "Transfer function approach to quantum control-part I: Dynamics of quantum feedback systems," IEEE Trans. Autom. Control **48**, 2107–2120 (2003).
[44] G. Zhang and M. James, "Quantum feedback networks and control: A brief survey," Chin. Sci. Bull. **57**(18), 2200–2214 (2012).
[45] J. Zhang, R. B. Wu, Y. Liu, C. Li, and T. J. Tarn, "Quantum coherent nonlinear feedback with applications to quantum optics on chip," IEEE Trans. Autom. Control **57**(8), 1997–2008 (2012).

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