

Aberystwyth University

An asymptotic method of factorization of a class of matrix-functions

Mishuris, Gennady; Rogosin, Sergei

Published in:

Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences

DOI:

[10.1098/rspa.2014.0109](https://doi.org/10.1098/rspa.2014.0109)

Publication date:

2014

Citation for published version (APA):

Mishuris, G., & Rogosin, S. (2014). An asymptotic method of factorization of a class of matrix-functions. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 470(2166), 1-15. Article 20140109. <https://doi.org/10.1098/rspa.2014.0109>

General rights

Copyright and moral rights for the publications made accessible in the Aberystwyth Research Portal (the Institutional Repository) are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the Aberystwyth Research Portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the Aberystwyth Research Portal

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

tel: +44 1970 62 2400
email: is@aber.ac.uk

An asymptotic method of factorization of a class of matrix-functions

Gennady MISHURIS¹, Sergei ROGOSIN^{1,2}

¹*Aberystwyth University, Penlais, SY23 2BZ Aberystwyth, UK;*

e-mail: ggm@ber.ac.uk; ser14@aber.ac.uk

²*Belarusian State University, Nezavisimosti Ave., 4, 220030 Minsk, Belarus;*

e-mail: rogosin@bsu.by

Abstract. A novel method of asymptotic factorization of $n \times n$ matrix functions is proposed. Considered class of matrices is motivated by certain problems originated in the elasticity theory. An example is constructed to illustrate effectiveness of the proposed procedure. Further applications of the method is discussed.

Key words: factorization of matrix-functions, asymptotic method

AMS 2010 Classification: Primary: 15A23; Secondary: 15A54, 30E25, 45E10

1 Introduction

We are considering here the problem of factorization of continuous matrix functions of the real variable. It means the representation of a given invertible square matrix $G \in (\mathcal{C}(\mathbb{R}))^{n \times n}$ in the following form

$$G(x) = G^-(x)\Lambda(x)G^+(x), \quad (1.1)$$

where continuous invertible matrices $G^-(x)$, $G^+(x)$ possess an analytic continuation in the lower $\Pi^- = \{z = x + iy : \text{Im} < 0\}$ and upper $\Pi^+ = \{z = x + iy : \text{Im} > 0\}$ half-planes, respectively, and

$$\Lambda(x) = \text{diag} \left(\left(\frac{x-i}{x+i} \right)^{\kappa_1}, \dots, \left(\frac{x-i}{x+i} \right)^{\kappa_n} \right), \quad \kappa_1, \dots, \kappa_n \in \mathbb{Z}. \quad (1.2)$$

The representation (1.1) is called *right (continuous or standard) factorization* and can be considered for any oriented curve Γ of certain classes which divides complex plane into two domain D^- and D^+ with changing of diagonal entries in $\Lambda(x)$ for $\left(\frac{x-t^+}{x-t^-} \right)^{\kappa_j}$, $t^\mp \in D^\mp$, or for x^{κ_j} . Similar representation

$$G(x) = G^+(x)\Lambda(x)G^-(x)$$

is called *left (continuous or standard) factorization*. If right- (left-) factorization exists then the integer numbers $\kappa_1, \dots, \kappa_n$, called *partial indices*, are determined uniquely up to the order. The factors G^-, G^+ are not unique. Relations of pairs of factors are described, e.g., in [28]. In particular, there exist constant transformations of factors, such that $\kappa_1 \geq \dots \geq \kappa_n$. The right- (left-) factorization is called *canonical factorization* if all partial indices are equal to 0, i.e. $\kappa_1 = \dots = \kappa_n = 0$.

Factorization of matrix functions was first studied in relation to the vector-matrix Riemann (or Riemann-Hilbert) boundary value problem (see [17]). The later was formulated by Riemann in his work on construction of complex differential equations with algebraic coefficients having a prescribed monodromy group (see, e.g., [12]). By using Cauchy type integral's method the vector-matrix Riemann boundary value problem was reduced in [31], [43] to a system of the Fredholm integral equations. A part of the theory of the factorization problem is based on the study of such systems (see also [16]) though this approach does not answer, in particular, on the questions when it is possible to get factorization, how to construct factors and how to determine partial indices.

Among other sources of interest to the factorization problems one can point out the vector-valued Wiener-Hopf equations on a half-line (see [19], [39]) and their discrete analogous, namely the block Toeplitz equations (see, e.g., [5], [18]). The developed technique found several applications in diffraction theory, fracture mechanics, geophysics, financial mathematics etc. (see a brief description given, e.g., in [42] and references therein).

Theoretical background for the study of the matrix factorization and its numerous generalizations is presented in [6], [12], [28], [43] (see also [27]).

The theory of the factorization is more or less complete (see [6]), but the above mentioned constructive questions about existence, factors and partial indices (which are very important for practical applications) have been answered only in a number of special cases. Among them one can mention rational matrix functions (see, e.g., [16]), functional commutative matrix functions (those satisfying $G(t)G(s) = G(s)G(t), \forall t, s \in \Gamma$, see [9]), upper- (lower-) triangular matrices with factorizable diagonal elements (see [8], [13]), certain classes of meromorphic matrix functions (see [1], [4], [26]), special cases of 2×2 Daniele-Khrapkov matrix functions (with a small degree of deviator polynomial) (see [10], [11], [22], [23]), special cases of 2×2 matrix functions with three rationally independent entries (see [2], [11] and references therein), special cases of $n \times n$ generalization of the Daniele-Khrapkov

matrix functions (see, e.g., [7], [21], [37], [42]), special classes of matrices possessing certain symmetry property (see [20], [44] and references therein).

In this paper we propose an asymptotic method of construction of factors for a special class of $n \times n$ nonrational matrix functions. To the best of authors' knowledge, this class does not coincide with any of the above mentioned classes. This class contains matrix functions appeared at the study of certain problems in fracture mechanics related to perturbation of the crack propagation ([30], [33] – [35]). The main idea of our study is to reduce determination of factors at each step of approximation to the solution of so-called vector-matrix jump boundary value problem. The paper is organized as follows. In Sec. 2 we introduce necessary notation and formulate the problem. Constructive algorithm is presented in Sec. 3. We also find interesting to present here the realization of the algorithm in a special case of matrices of practical importance. The method is illustrated by an example given in Sec. 4. We conclude our study by showing the quality of the factorization approximation by restricting ourselves only to the first asymptotic term and discuss a role of the chosen small parameter.

2 A class of matrices. Problem formulation

Let us introduce the following class of invertible continuous $n \times n, n \geq 2$, matrix-functions $\mathcal{G}K_n$ depending on a real parameter $\varphi \in \mathbb{R}$, satisfying the following conditions:

(1) $G_\varphi \in (\mathcal{C}(\mathbb{R}))^{n \times n}$ belongs to $\mathcal{G}K_n$ if it can be represented in the form

$$G_\varphi = R_\varphi F R_\varphi^{-1}, \quad (2.1)$$

where bounded locally Hölder-continuous on \mathbb{R} (in general non-rational) invertible matrix $R_\varphi(x)$ is such that

$$(2) \quad R_0 = R_\varphi|_{\varphi=0} = I, \quad (2.2)$$

(3) matrix function F does not depend on parameter φ , has Hölder-continuous entries f_{kl} on the extended real line $\overline{\mathbb{R}}$, i.e. $\forall k, l = 1, \dots, n$,

$$|f_{kl}(x_1) - f_{kl}(x_2)| \leq C \left| \frac{1}{x_1 + i} - \frac{1}{x_2 + i} \right|^\mu, \quad \forall x_1, x_2 \in \overline{\mathbb{R}}, \quad 0 < \mu < 1, \quad (2.3)$$

satisfies the following asymptotic estimate at infinity

$$(4) \quad F(x) \rightarrow I, \quad |x| \rightarrow \infty, \quad (2.4)$$

(5) F admits a right canonical factorization, i.e.

$$F(x) = F^-(x)F^+(x), \quad (2.5)$$

where Hölder-continuous on $\overline{\mathbb{R}}$ matrix-functions $F^-(x), F^+(x)$ possess an analytic continuation in the lower Π^- and the upper Π^+ half-plane, respectively.

The matrices of the following form constitute a simple subclass of class \mathcal{GK}_2 :

$$G_\varphi(x) = \begin{pmatrix} p(x) & q(x)e^{i\varphi x} \\ q(x)e^{-i\varphi x} & p(x) \end{pmatrix}. \quad (2.6)$$

This function appears after Fourier transforms of the Wiener-Hopf equation describing a problem of fracture mechanics.

In this particular case

$$R_\varphi(x) = \begin{pmatrix} e^{-i\varphi x/2} & 0 \\ 0 & e^{i\varphi x/2} \end{pmatrix}.$$

We note that the matrix-functions of this type do not belong to any known class of matrix-functions which admit explicit factorization.

It can be seen below that in our algorithm the factors retain one of the properties of the matrix-function G_φ , namely,

$$G_\varphi^-(z), G_\varphi^+(z) \rightarrow I, \text{ as } z \rightarrow \infty, \quad \mp \text{Im } z > 0. \quad (2.7)$$

3 An algorithm

3.1 General construction

By assumptions any matrix $G_\varphi(x) \in \mathcal{GK}_n$ can be written in the form

$$G_\varphi(x) = F^-(x) (F^-(x))^{-1} G_\varphi (F^+(x))^{-1} F^+(x) = F^-(x) G_{1,\varphi} F^+(x), \quad (3.1)$$

where $F^-(x), F^+(x)$ are component of canonical factorization of the corresponding matrix F , and (see (2.2), (2.4)) the matrix $G_{1,\varphi}(x)$ is represented in the form

$$G_{1,\varphi}(x) = (F^-(x))^{-1} G_\varphi (F^+(x))^{-1} = (F^-(x))^{-1} R_\varphi F R_\varphi^{-1} (F^+(x))^{-1}.$$

(6) There exist a small parameter $\varepsilon = \varepsilon(\varphi)$ (more exactly its value will be described later), such that for all $x \in \mathbb{R}$ and any finite φ

$$G_{1,\varphi}(x) = I + \varepsilon N_\varphi(x), \quad (3.2)$$

and matrix $N_\varphi(x)$ is bounded and locally Hölder continuous on \mathbb{R} .

Note that by assumption each entry of the matrix $N_\varphi(x)$ has a limit when $|x| \rightarrow +\infty$, i.e. there exists the value $N_\varphi(\infty)$. Note also that no commutativity of the involved matrices is assumed.

Let us look for factorization of the matrix $G_{1,\varphi}(x)$ in the form

$$G_{1,\varphi}(x) = I + \varepsilon N_\varphi(x) = (I + \varepsilon N_{1,\varphi}^-(x)) (I + \varepsilon N_{1,\varphi}^+(x)). \quad (3.3)$$

Comparing terms at different powers of ε we get, in particular, the following relation for determination of factors $N_{1,\varphi}^-$, $N_{1,\varphi}^+$:

$$N_{1,\varphi}^-(x) + N_{1,\varphi}^+(x) = N_\varphi(x), \quad x \in \mathbb{R}. \quad (3.4)$$

It is customary to denote

$$M_{0,\varphi}(x) \equiv N_\varphi(x).$$

The jump boundary value problem (3.4) has a solution represented in terms of a slight modification of the matrix-valued Cauchy type integral (see, e.g., [28], [31])

$$N_{1,\varphi}^\mp(z) = \frac{1}{2} M_{0,\varphi}(\infty) \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{M_{0,\varphi}(t) - M_{0,\varphi}(\infty)}{t - z} dt \equiv \frac{1}{2} M_{0,\varphi}(\infty) \mp (\mathbf{C}_0 M_{0,\varphi})(z). \quad (3.5)$$

The above modification is proposed in order to avoid extra discussion of the convergence of the above integrals. In this form the integrals are convergent automatically. Moreover, its boundary values

$$(\mathbf{C}_0 M_{0,\varphi})^\mp(x) = \lim_{\text{Im}z \rightarrow \mp 0} (\mathbf{C}_0 M_{0,\varphi})(z)$$

satisfy Sokhotsky-Plemelj formulas, i.e.

$$(\mathbf{C}_0 M_{0,\varphi})^\mp(x) = \frac{1}{2} M_{0,\varphi}(x) - \frac{1}{2} M_{0,\varphi}(\infty) \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{M_{0,\varphi}(t) - M_{0,\varphi}(\infty)}{t - x} dt =$$

$$= \frac{1}{2}M_{0,\varphi}(x) - \frac{1}{2}M_{0,\varphi}(\infty) \mp \frac{1}{2}(\mathbf{S}_0 M_{0,\varphi})(x), \quad (3.6)$$

or

$$N_{1,\varphi}^\mp(x) = \frac{1}{2}M_{0,\varphi}(x) \mp \frac{1}{2}(\mathbf{S}_0 M_{0,\varphi})(x), \quad (3.7)$$

where \mathbf{S}_0 is the singular integral operator along the real line with density $M_{0,\varphi}(t) - M_{0,\varphi}(\infty)$. It follows from [17, n. 4.6] that both matrices $N_{1,\varphi}^-(x)$, $N_{1,\varphi}^+(x)$ satisfy Hölder conditions on $\overline{\mathbb{R}}$, are bounded there with $N_{1,\varphi}^\mp(\infty) = \frac{1}{2}M_{0,\varphi}(\infty)$, and possess an analytic continuation into lower Π^- and upper Π^+ half-planes, respectively. Surely, its product is also Hölder conditions on $\overline{\mathbb{R}}$ and bounded.

Let us refine the factorization of the matrix $G_{1,\varphi}(x)$, i.e. look for factorization in the form

$$G_{1,\varphi}(x) = I + \varepsilon N_\varphi(x) = (I + \varepsilon N_{1,\varphi}^-(x) + \varepsilon^2 N_{2,\varphi}^-(x)) (I + \varepsilon N_{1,\varphi}^+(x) + \varepsilon^2 N_{2,\varphi}^-(x)), \quad (3.8)$$

where $N_{1,\varphi}^-(x)$, $N_{1,\varphi}^+(x)$ are those found at the previous step.

Comparing terms at different powers of ε we get, in particular, the following relation for determination of factors $N_{2,\varphi}^-, N_{2,\varphi}^+$:

$$N_{1,\varphi}^-(x)N_{1,\varphi}^+(x) + N_{2,\varphi}^-(x) + N_{2,\varphi}^+(x) = 0. \quad (3.9)$$

Denoting

$$M_{1,\varphi}(x) \equiv -N_{1,\varphi}^-(x)N_{1,\varphi}^+(x),$$

we arrive at the following jump boundary value problem

$$N_{2,\varphi}^-(x) + N_{2,\varphi}^+(x) = M_{1,\varphi}(x), \quad x \in \mathbb{R}, \quad (3.10)$$

with already known right hand-side. Solution of this problem is given by the formula similar to (3.5)

$$N_{2,\varphi}^\mp(z) = \frac{1}{2}M_{1,\varphi}(\infty) \mp (\mathbf{C}_0 M_{1,\varphi})(z). \quad (3.11)$$

It has the same properties as the solution of (3.5), in particular, its boundary values satisfy the relation

$$N_{2,\varphi}^\mp(x) = \frac{1}{2}M_{1,\varphi}(x) \mp \frac{1}{2}(\mathbf{S}_0 M_{1,\varphi})(x), \quad (3.12)$$

and $N_{2,\varphi}^\mp(\infty) = -\frac{1}{8}N_\varphi^2(\infty)$.

One can proceed in the same manner. Thus on the k -th step we use the representation

$$\begin{aligned} G_{1,\varphi}(x) &= I + \varepsilon N_\varphi(x) = \\ &= \left(I + \varepsilon N_{1,\varphi}^-(x) + \dots + \varepsilon^k N_{k,\varphi}^-(x) \right) \left(I + \varepsilon N_{1,\varphi}^+(x) + \dots + \varepsilon^k N_{k,\varphi}^-(x) \right), \end{aligned} \quad (3.13)$$

where $N_{1,\varphi}^-(x), \dots, N_{k-1,\varphi}^-(x), N_{1,\varphi}^+(x), \dots, N_{k-1,\varphi}^-(x)$ are found at the previous steps. It leads to the jump boundary value problem

$$N_{k,\varphi}^-(x) + N_{k,\varphi}^+(x) = M_{k-1,\varphi}, \quad x \in \mathbb{R}, \quad (3.14)$$

where

$$M_{k-1,\varphi} = - \left[N_{1,\varphi}^-(x) N_{k-1,\varphi}^+(x) + N_{2,\varphi}^-(x) N_{k-2,\varphi}^+(x) + \dots + N_{k-1,\varphi}^-(x) N_{1,\varphi}^+(x) \right].$$

Solution of this problem is given by the formula similar to (3.5) (or to (3.11)).

Thus the factorization of the matrix function $G_{1,\varphi}(x)$ is given in the form of asymptotic series

$$G_{1,\varphi}(x) = \left(I + \sum_{k=1}^{\infty} \varepsilon^k N_{k,\varphi}^-(x) \right) \left(I + \sum_{k=1}^{\infty} \varepsilon^k N_{k,\varphi}^+(x) \right), \quad (3.15)$$

where the pair $N_{k,\varphi}^-(x), N_{k,\varphi}^+(x)$ is the unique solution to the jump problem (3.14) for any $k \in \mathbb{N}$.

The following Theorem gives conditions when this asymptotic factorization becomes an explicit one, i.e. gives convergence conditions for the asymptotic series involved.

Theorem 3.1. *Let the parameter ε satisfies the inequality*

$$|\varepsilon| \leq 1/A \quad (3.16)$$

with the constant $A = A(\varphi)$ being equal to

$$A = \|N_\varphi(\cdot)\|_\mu (1 + C_\mu)^2, \quad (3.17)$$

$\|N_\varphi(\cdot)\|_\mu$ being the norm of the matrix function $N_\varphi(x)$ in the Hölder space H_μ equal to the maximum of the norms of its entries, and C_μ being the norm of the singular integral operator $\mathbf{S}_0 : H_\mu \rightarrow H_\mu$.

Then both series in the right-hand side of (3.15) converge for all $x \in \mathbb{R}$.

◁ Boundedness of the modified singular integral operator \mathbf{S}_0 in Hölder spaces follows (see, e.g., [17, p. 48]) from the boundedness of the “standard” singular integral operator \mathbf{S} (Hilbert transform on the real line) in these spaces (the later is well-known, see [3] for the exact value of the norm of $\|\mathbf{S}\|_{H_\mu \rightarrow H_\mu}$). Let us denote the norm of \mathbf{S}_0 in Hölder space $H_\mu(\mathbb{R})$ by C_μ , i.e.

$$C_\mu = \|\mathbf{S}_0\|_{H_\mu \rightarrow H_\mu}.$$

Then we have the following series of estimates

$$\|N_{1,\varphi}^\mp(\cdot)\|_\mu \leq \alpha_1 \|N_\varphi(\cdot)\|_\mu (1 + C_\mu), \text{ where } \alpha_1 = \frac{1}{2},$$

$$\|N_{2,\varphi}^\mp(\cdot)\|_\mu \leq \frac{1}{2} \|M_{1,\varphi}(\cdot)\|_\mu (1 + C_\mu),$$

and

$$\|M_{1,\varphi}(\cdot)\|_\mu \leq (\alpha_1 \|N_\varphi(\cdot)\|_\mu (1 + C_\mu))^2,$$

i.e.

$$\|N_{2,\varphi}^\mp(\cdot)\|_\mu \leq \alpha_2 \|N_\varphi(\cdot)\|_\mu^2 (1 + C_\mu)^3, \text{ where } \alpha_2 = \frac{1}{2} \alpha_1^2.$$

Finally, for each $k \geq 2$

$$\|N_{k,\varphi}^\mp(\cdot)\|_\mu \leq \frac{1}{2} \|M_{k-1,\varphi}(\cdot)\|_\mu (1 + C_\mu),$$

and

$$\|M_{k-1,\varphi}(\cdot)\|_\mu \leq (\alpha_1 \alpha_{k-1} + \alpha_2 \alpha_{k-2} + \dots + \alpha_{k-1} \alpha_1) \|N_\varphi(\cdot)\|_\mu^{k-1} (1 + C_\mu)^{2k-2},$$

i.e.

$$\|N_{k,\varphi}^\mp(\cdot)\|_\mu \leq \alpha_k \|N_\varphi(\cdot)\|_\mu^k (1 + C_\mu)^{2k-1}, \text{ where } \alpha_k = \frac{1}{2} (\alpha_1 \alpha_{k-1} + \dots + \alpha_{k-1} \alpha_1).$$

Few first coefficients α_k we can calculate explicitly, namely, $\alpha_1 = 1/2$, $\alpha_2 = 1/8$, $\alpha_3 = 1/16$. As for coefficients with large enough indices we can proof by induction that

$$\alpha_k < \frac{1}{16(k-3)}, \quad \forall k \geq 12.$$

Therefore

$$\|\varepsilon^k N_{k,\varphi}^\mp(\cdot)\|_\mu \leq |\varepsilon|^k \frac{1}{32(k-3)(1+C_\mu)} (\|N_\varphi(\cdot)\|_\mu (1+C_\mu))^k, \quad \forall k \geq 12.$$

Since the sequence $\sqrt[k]{\frac{1}{32(k-3)(1+C_\mu)}} \leq 1$ is increasing for sufficiently large k and

$$\lim_{k \rightarrow \infty} \sqrt[k]{\frac{1}{32(k-3)(1+C_\mu)}} = 1,$$

then the convergence of the series

$$\left(I + \sum_{k=1}^{\infty} \varepsilon^k N_{k,\varphi}^-(x) \right), \left(I + \sum_{k=1}^{\infty} \varepsilon^k N_{k,\varphi}^+(x) \right)$$

for all $x \in \mathbb{R}$ follows from (3.16). \triangleright

Remark 3.1. *It follows from the standard properties of the Cauchy type integral and singular integral with Cauchy kernel that conditions of Theorem 3.1 guarantee convergence of the series in the right-hand side of (3.15) in the half-planes Π^- , Π^+ , respectively.*

Remark 3.2. *In fact, the decay of the second term in the right-hand side of (3.2) at infinity follows from the asymptotic relations (2.7) which in turn follows from the properties of matrices of the considered class and the proposed construction).*

Remark 3.3. *If the number $A = A(\varphi)$ in Theorem 3.1 is small enough, i.e.*

$$A = \|N_\varphi(\cdot)\|_\mu (1 + C_\mu)^2 < 1, \quad (3.18)$$

then the results remains valid for $\varepsilon = 1$ and the described procedure is working then without any changes.

3.2 Special case

Let us consider the problem of factorization of 2×2 invertible matrices from a subclass of \mathcal{GK}_2 , namely

$$G_\varphi(x) = \begin{pmatrix} p(x) & q(x)e^{ix\varphi} \\ q(x)e^{-ix\varphi} & p(x) \end{pmatrix}, \quad (3.19)$$

given on the real line ($x \in \mathbb{R}$) and depending on the real parameter $\varphi \in \mathbb{R}$.

We assume that the following assumptions hold.

- 1) the entries $p(x), q(x)$ are real-valued Hölder continuous functions on $\overline{\mathbb{R}}$, i.e. $p, q \in H_\mu(\overline{\mathbb{R}})$;
 2) the combinations of the functions are positive:

$$p(x) \pm q(x) > 0, x \in \mathbb{R}; \quad (3.20)$$

- 3) the following limits exist

$$\lim_{|x| \rightarrow +\infty} p(x) = 1, \quad \lim_{|x| \rightarrow +\infty} q(x) = 0; \quad (3.21)$$

- 4) the following symmetry condition is valid

$$\overline{G_\varphi(-x)} = G_\varphi(x); \quad (3.22)$$

As in general case the factorization of matrices of type (3.19) is motivated by certain problem of fracture mechanics. The considered matrices are similar to those which are studied and explicitly factorized in [2], [15], but certain conditions of the above cited papers do not satisfy in our case.

Note, that even if one supposes that the functions $p(x), q(x)$ are meromorphically continued into semi-planes Π^-, Π^+ , then it does not mean that these extended functions have finite number of zeroes and poles there.

Remark 3.4. *Under conditions 1)–4) the matrix (3.19) admits the canonical factorization:*

$$G_\varphi(x) = G_\varphi^-(x)G_\varphi^+(x). \quad (3.23)$$

It follows, in particular, from condition 4) and [44, p. 52].

Let us start with factorization of an auxiliary matrix $F(x) = G_0(x)$ having no exponential term in their entries

$$F(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & p(x) \end{pmatrix}. \quad (3.24)$$

Note that

$$F(x) = P \begin{pmatrix} p(x) + q(x) & 0 \\ 0 & p(x) - q(x) \end{pmatrix} P, \quad (3.25)$$

where the projector P is defined

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

It follows from the conditions 1)–3) that both diagonal elements of the middle diagonal matrix have index equal to zero. Hence they admit the representation

$$\begin{aligned} p(x) + q(x) &= (p(x) + q(x))^- \cdot (p(x) + q(x))^+; \\ p(x) - q(x) &= (p(x) - q(x))^- \cdot (p(x) - q(x))^+ \end{aligned} \quad (3.26)$$

with the factors of the form (see, e.g., [17])

$$\begin{aligned} (p(z) + q(z))^\mp &= \exp \left\{ \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log [p(\tau) + q(\tau)]}{\tau - z} d\tau \right\}, \quad z \in \Pi^\mp, \\ (p(z) - q(z))^\mp &= \exp \left\{ \mp \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\log [p(\tau) - q(\tau)]}{\tau - z} d\tau \right\}, \quad z \in \Pi^\mp \end{aligned} \quad (3.27)$$

and boundary values $(p(x) + q(x))^\mp$ of the functions $(p(z) + q(z))^\mp$ (and $(p(x) - q(x))^\mp$ of the functions $(p(z) - q(z))^\mp$ are determined by using Sokhotsky-Plemelj formulas (see, e.g., [17]).

Therefore, we obtain immediately the right canonical factorization of the matrix $G_0(x)$:

$$F(x) = F^-(x)F^+(x), \quad (3.28)$$

where

$$F^-(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} (p(x) + q(x))^- & (p(x) - q(x))^- \\ (p(x) + q(x))^- & -(p(x) - q(x))^- \end{pmatrix}, \quad (3.29)$$

$$F^+(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} (p(x) + q(x))^+ & (p(x) + q(x))^+ \\ (p(x) - q(x))^+ & -(p(x) - q(x))^+ \end{pmatrix}. \quad (3.30)$$

Now we can represent the initial matrix $G_\varphi(k)$ in the form:

$$G_\varphi(x) = F^-(x) (F^-(x))^{-1} G_\varphi(x) (F^+(x))^{-1} F^+(x) \equiv F^-(x) G_{1,\varphi}(x) F^-(x) \quad (3.31)$$

and proceed with factorization of the matrix $G_{1,\varphi}(x)$.

The inverse matrices $(F^-(x))^{-1}$, $(F^+(x))^{-1}$ are equal, respectively

$$(F^-(x))^{-1} = \frac{1}{\Delta_-} \begin{pmatrix} (p(x) - q(x))^- & (p(x) - q(x))^- \\ (p(x) + q(x))^- & -(p(x) + q(x))^- \end{pmatrix}, \quad (3.32)$$

$$\Delta_- = \frac{2}{\sqrt{2}} (p(x) + q(x))^- (p(x) - q(x))^-,$$

$$(F^+(x))^{-1} = \frac{1}{\Delta_+} \begin{pmatrix} (p(x) - q(x))^+ & (p(x) + q(x))^+ \\ (p(x) - q(x))^+ & -(p(x) + q(x))^+ \end{pmatrix}, \quad (3.33)$$

$$\Delta_+ = \frac{2}{\sqrt{2}}(p(x) + q(x))^+(p(x) - q(x))^+.$$

Hence (for shortness we omit further argument x of functions p and q)

$$G_{1,\varphi} = \frac{1}{\Delta} \begin{pmatrix} (p-q)(2p+q)(e^{i\varphi k} + e^{-i\varphi k}) & (p-q)^-(p+q)^+q(e^{-i\varphi k} - e^{i\varphi k}) \\ (p-q)^+(p+q)^-q(e^{i\varphi k} - e^{-i\varphi k}) & (p+q)(2p-q)(e^{i\varphi k} - e^{-i\varphi k}) \end{pmatrix}, \quad (3.34)$$

$$\Delta = 2(p^2 - q^2).$$

It is not hard to see that $G_{1,\varphi}$ is a sum of the unit matrix and the matrix which is “small” for appropriate choice of φ . Hence, following Remark 3.3, we rewrite the right-hand side of (3.34) as the following sum

$$G_{1,\varphi} = I + N_\varphi, \quad (3.35)$$

where

$$N_\varphi = \begin{pmatrix} -\frac{2q \sin^2 \frac{\varphi x}{2}}{(p+q)} & -\frac{iq(p-q)^-(p+q)^+ \sin \varphi x}{(p^2-q^2)} \\ \frac{iq(p-q)^+(p+q)^- \sin \varphi x}{(p^2-q^2)} & \frac{2q \sin^2 \frac{\varphi x}{2}}{(p-q)} \end{pmatrix}. \quad (3.36)$$

Note that the latter matrix can be written as the sum of two diagonal matrices, namely

$$N_\varphi = N_1 + N_2 \quad (3.37)$$

and both matrices N_1, N_2 do not depend on φ

$$N_1 = 2 \sin^2 \frac{\varphi x}{2} \begin{pmatrix} -\frac{q}{(p+q)} & 0 \\ 0 & \frac{q}{(p-q)} \end{pmatrix}, \quad (3.38)$$

$$N_2 = i \sin \varphi x \begin{pmatrix} 0 & -\frac{q(p-q)^-(p+q)^+}{(p^2-q^2)} \\ \frac{q(p-q)^+(p+q)^-}{(p^2-q^2)} & 0 \end{pmatrix}. \quad (3.39)$$

Let us denote

$$\varepsilon_1 = \varepsilon_1(\varphi) \equiv \max_{x \in \mathbb{R}} \left| q(x) \sin \frac{\varphi x}{2} \right|. \quad (3.40)$$

Lemma 3.1. *The parameter ε_1 can be taken smaller than any positive number δ by an appropriate choice of φ .*

◁ Indeed, taking into account the condition (3.21)₂ one concludes that there exists $x_\delta > 0$ such that for any $|x| \geq x_\delta$

$$|q(x) \sin \frac{\varphi x}{2}| \leq |q(x)| \leq \delta.$$

On the other hand, since $xq(x)$ belong to the space $H_\mu[-x_\delta, x_\delta] \subset C[-x_\delta, x_\delta]$ there exists a constant $q_h > 0$ such that

$$|xq(x)| \leq q_h, \quad \text{for all } |x| \leq x_\delta.$$

Finally this means

$$\left| q(x) \sin \frac{\varphi x}{2} \right| \leq |xq(x)| \frac{\varphi}{2} \leq \frac{\varphi}{2} q_h.$$

Choosing $\varphi = 2\delta/q_h$ we finish the proof.▷

From the structure of the matrix N_φ from (3.36), it follows that that parameter $A = A(\varphi)$ discussed in Theorem 3.1 is smaller than 1 for an appropriate choice of φ . It guarantees applicability of the general procedure in this special case.

4 An example

We present here an example of 2×2 matrix function $G_\varphi \in \mathcal{GK}_2$ for which the above discussed factorization does not involve Cauchy type integration for the components and auxiliary matrices.

Let

$$G_\varphi(x) = \begin{pmatrix} \frac{x^2+10}{x^2+1} & \frac{6}{x^2+1} e^{ix\varphi} \\ \frac{6}{x^2+1} e^{-ix\varphi} & \frac{x^2+10}{x^2+1} \end{pmatrix}, \quad x \in \mathbb{R}. \quad (4.1)$$

It is a special case of the matrix functions discussed in Subsec. 3.2. Here $p(x) = \frac{x^2+10}{x^2+1}$, $q(x) = \frac{6}{x^2+1}$ and an auxiliary matrix $F(x) = G_0(x)$ has the form

$$G_0(x) = \begin{pmatrix} \frac{x^2+10}{x^2+1} & \frac{6}{x^2+1} \\ \frac{6}{x^2+1} & \frac{x^2+10}{x^2+1} \end{pmatrix}, \quad x \in \mathbb{R}. \quad (4.2)$$

The above matrix (4.1) satisfies all conditions 1) – 4) of Subsec. 3.2, moreover, the functions p, q are not only Hölder continuous on \mathbb{R} but infinitely differentiable.

It follows from condition 2) that

$$\text{Ind det } G_\varphi(x) = \text{Ind det } G_0(x) = 0,$$

besides, due to condition 4) the matrix $G_\varphi(x)$ admits the canonical factorization (if exists).

Canonical factorization of the matrix $G_0(x)$

$$G_0(x) = G_0^-(x)G_0^+(x)$$

can be found in an explicit form:

$$G_0^-(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{x-4i}{x-i} & \frac{x-2i}{x-i} \\ \frac{x-4i}{x-i} & -\frac{x-2i}{x-i} \end{pmatrix}, \quad (4.3)$$

$$G_0^+(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{x+4i}{x+i} & \frac{x+4i}{x+i} \\ \frac{x+2i}{x+i} & -\frac{x+2i}{x+i} \end{pmatrix}. \quad (4.4)$$

Hence, one can calculate an auxiliary matrix

$$G_{1,\varphi}(x) = (G_0^-(x))^{-1} G_\varphi(x) (G_0^+(x))^{-1}$$

in the following form

$$G_{1,\varphi}(x) = \frac{1}{2(x^2 + 16)(x^2 + 4)} \begin{pmatrix} g_{1,1}(x) & g_{1,2}(x) \\ g_{2,1}(x) & g_{2,2}(x) \end{pmatrix}, \quad (4.5)$$

where

$$g_{1,1}(x) = 2(x^2 + 4)(x^2 + 10) + 6(x^2 + 4) (e^{i\varphi x} + e^{-i\varphi x}), \quad (4.6)$$

$$g_{1,2}(x) = 12(x - 2i)(x + 4i) (-e^{i\varphi x} + e^{-i\varphi x}), \quad (4.7)$$

$$g_{2,1}(x) = 12(x + 2i)(x - 4i) (e^{i\varphi x} - e^{-i\varphi x}), \quad (4.8)$$

$$g_{2,2}(x) = 2(x^2 + 16)(x^2 + 10) + 6(x^2 + 16) (-e^{i\varphi x} - e^{-i\varphi x}). \quad (4.9)$$

It leads to the following representation of the matrix

$$G_{1,\varphi}(x) = I + N_1 + N_2, \quad (4.10)$$

where we can take $\varepsilon = 1$ due to Theorem 3.1 (see also Remark 3.3), and

$$N_1 = 6i \sin^2 \frac{\varphi x}{2} \begin{pmatrix} \frac{1}{x^2+16} & 0 \\ 0 & -\frac{1}{x^2+4} \end{pmatrix},$$

$$N_2 = 12i \sin \varphi x \begin{pmatrix} 0 & -\frac{1}{(x+2i)(x-4i)} \\ \frac{1}{(x-2i)(x+4i)} & 0 \end{pmatrix}.$$

This follows from the consideration presented in the previous Section.

Following general scheme of Subsec. 3.1 we factorize $G_{1,\varphi}(x)$ at first in the form (3.4) where the first terms of factorization should be computed by the formula (3.5). Following the asymptotic procedure described above, it is sufficient at the first stage to factorize the matrix

$$M_{0,\varphi}(x) = 6i \begin{pmatrix} \frac{\sin^2 \frac{\varphi x}{2}}{x^2+16} & -\frac{2 \sin \varphi x}{(x+2i)(x-4i)} \\ \frac{2 \sin \varphi x}{(x-2i)(x+4i)} & -\frac{\sin^2 \frac{\varphi x}{2}}{x^2+4} \end{pmatrix}. \quad (4.11)$$

For this particular case, instead of using the Cauchy integrals, one can factorize each entry $n_{ij}(x)$, $i = 1, 2$, of the matrix by using decomposition in simple fraction and Taylor formula. Combining the obtained results we get the following representation of matrix $N_{1,\varphi}^-(x)$, $N_{1,\varphi}^+(x)$ (first components of asymptotic factorization)

$$N_{1,\varphi}^-(x) = 6i \begin{pmatrix} \frac{-e^{-i\varphi x} - e^{-4\varphi} + 2}{32i(x-4i)} + \frac{e^{-i\varphi x} - e^{-4\varphi}}{32i(x+4i)} & \frac{-ie^{-i\varphi x} + ie^{-4\varphi}}{6i(x-4i)} + \frac{ie^{-i\varphi x} - ie^{-2\varphi}}{6i(x+2i)} \\ \frac{ie^{-i\varphi x} - ie^{-2\varphi}}{6i(x-2i)} - \frac{ie^{-i\varphi x} - ie^{-4\varphi}}{6i(x+4i)} & \frac{e^{-i\varphi x} + e^{-2\varphi} - 2}{16i(x-2i)} - \frac{e^{-i\varphi x} - e^{-2\varphi}}{16i(x+2i)} \end{pmatrix}. \quad (4.12)$$

$$N_{1,\varphi}^+(x) = 6i \begin{pmatrix} \frac{e^{i\varphi x} + e^{-4\varphi} - 2}{32i(x+4i)} - \frac{e^{i\varphi x} - e^{-4\varphi}}{32i(x-4i)} & \frac{-ie^{i\varphi x} + ie^{-2\varphi}}{6i(x+2i)} + \frac{ie^{i\varphi x} - ie^{-4\varphi}}{6i(x-4i)} \\ \frac{ie^{i\varphi x} - ie^{-4\varphi}}{6i(x+4i)} - \frac{ie^{i\varphi x} - ie^{-2\varphi}}{6i(x-2i)} & \frac{-e^{i\varphi x} - e^{-2\varphi} + 2}{16i(x+2i)} + \frac{e^{i\varphi x} - e^{-2\varphi}}{16i(x-2i)} \end{pmatrix}. \quad (4.13)$$

Note that both plus ($N_{1,\varphi}^+$) and minus ($N_{1,\varphi}^-$) matrix-functions vanish at infinity in the corresponding half-plane. The procedure can be performed further as it was described in the general case.

5 Outlook and discussions

To illuminate the efficiency of the proposed procedure, we present here the numerical results related to the previous example (Sec. 4) showing the quality of the factorization if one decides to restrict the approximation to the first asymptotic term only.

In Fig. 1 and Fig. 2 we present the normalized absolute error of the factorization using the only first asymptotic term for different values of the parameter φ . We compute the errors for each component of reminder, ΔK ,

that is the difference between the exact factorization and its asymptotic approximation along the real axis

$$\Delta K(x) = G_{1,\varphi}(x) - (I + N_{1,\varphi}^-) (I + N_{1,\varphi}^+) = N_{1,\varphi}^- N_{1,\varphi}^+.$$

As it follows from the properties of the Cauchy type integral [31], the obtained estimates are valid also into the upper and lower half-planes.

In Fig. 1 we depict the error related to the diagonal elements of the reminder while in Fig. 2 we show the result for the off-diagonal element. Note, that since they are complex conjugate with respect to each other, it is enough to discuss only one of them. Unexpectedly, even for rather large value of the parameter $\varphi = 1$, the error appeared to be not high, while for smaller magnitudes of φ it decays fast with argument and its larger value is concentrated only near the center of the coordinate. Moreover, one can observe that for $j = 1, 2$

$$\Delta k_{jj}(0) = 4\varphi^2 + O(\varphi^3), \quad \varphi \rightarrow 0.$$

This result can be verified also analytically.

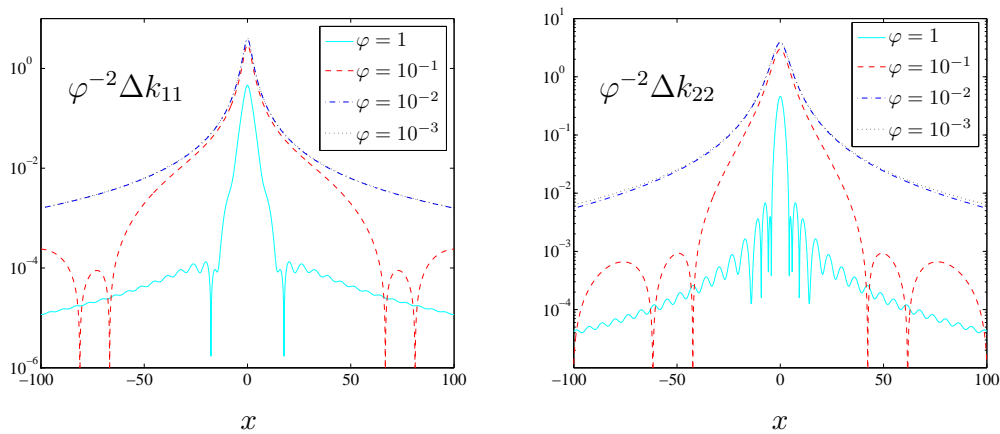


Figure 1: Absolute error in the diagonal elements appeared by replacing the exact factorization with the only first asymptotic terms (4.12) and (4.13) for various values of the parameter φ .

With decrease of the parameter φ , the amplitude of the error oscillations increases but their support moves out from the coordinate center to infinity.

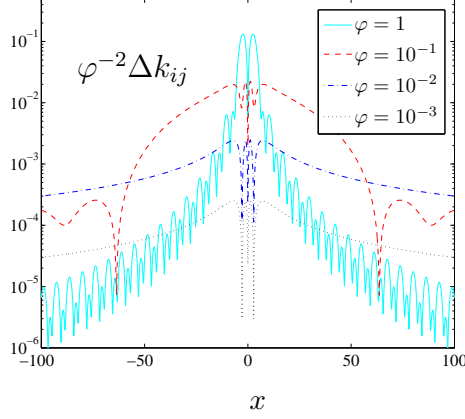


Figure 2: Absolute error in the off-diagonal elements appeared by replacing the exact factorization with the only first terms (4.12) and (4.13) for various values of the parameter φ .

Thus, such oscillations plays minor role and the numerical computation of the Cauchy integrals in the described procedure would not affect the accuracy of the computations.

It is not correct to say, however, that φ is an optimal small parameter in the problem under consideration. As it follows from Theorem 3.1, not only the parameter φ but the decay of the corresponding function (e.g., the function $q(x)$ in the special case) play an important role in the analysis and it is rather δ should be taken as the appropriate small parameter.

To clarify this, let us note that two terms N_1 and N_2 are of different orders with respect to the small parameter φ . Indeed: $N_1(\varphi) = O(\varphi^2)$ and $N_2(\varphi) = O(\varphi)$, as $\varphi \rightarrow 0$. As a result, one could construct another first order approximation of the factorization basing only on the term N_2 instead of $N_1 + N_2$. This gives the same first order estimate in terms of the small parameter φ . Simple calculations give the following (first-order) factorization terms in this case:

$$N_{1,\varphi}^{*-}(x) = \begin{pmatrix} 0 & -\frac{i(e^{-i\varphi x} - e^{-4\varphi})}{x-4i} + \frac{i(e^{-i\varphi x} - e^{-2\varphi})}{x+2i} \\ \frac{i(e^{-i\varphi x} - e^{-2\varphi})}{x-2i} - \frac{i(e^{-i\varphi x} - e^{-4\varphi})}{x+4i} & 0 \end{pmatrix}, \quad (5.1)$$

$$N_{1,\varphi}^{*+}(x) = \begin{pmatrix} 0 & -\frac{i(e^{i\varphi x} - e^{-2\varphi})}{x+2i} + \frac{i(e^{i\varphi x} - e^{-4\varphi})}{x-4i} \\ \frac{i(e^{i\varphi x} - e^{-4\varphi})}{x+4i} - \frac{i(e^{i\varphi x} - e^{-2\varphi})}{x-2i} & 0 \end{pmatrix}. \quad (5.2)$$

We estimate then a quality of the approximation of the factorization of the matrix-function $G_{1,\varphi}(x)$ basing now solely on the first order term with respect to the parameter φ . The new reminder is

$$\Delta K^*(x) = G_{1,\varphi}(x) - (I + N_{1,\varphi}^{*-}) (I + N_{1,\varphi}^{*+}) = N_1(x) - N_{1,\varphi}^{*-} N_{1,\varphi}^{*+}.$$

Note that the off-diagonal terms gives exact (identity) result for the terms (5.1), (5.2) in opposite to the factorization provided by the terms (4.12), (4.13). Thus for the first glance, the latter approach looks less beneficia than the former.

In Fig. 3 we compare the errors related to the diagonal elements of the reminders for the value of the parameter $\varphi = 10^{-3}$. The errors corresponding to the matrix-functions (5.1), (5.2) are given by doted line while those related to (4.12), (4.13) are depicted by the solid line. One can observe a striking difference in the accuracy of those two approximations. While for a small values of the variable $|x| < 1$ they are identical, for larger value of the argument the approximation given by the general procedure provide much better accuracy than that based on the only first order term, N_2 , with respect to the small parameter φ . Moreover, if one decide to continue asymptotic expansion, it may become a real issue in numerical computations of the next asymptotic terms as the decay is very slow. Thus, the procedure suggested here is in a sense optimal.

Note, that proposed procedure is working in a general setting discussed in Sec. 3.1 not only in the case of the Subsec. 3.2. In particular, the procedure is definitely valid when all entries of the matrices are quasi-polynomials, i.e. sum of different exponentials with meromorphic coefficients, providing that all conditions of the class $\mathcal{G}K_n$ are satisfied.

In the case of non-canonical factorization the algorithm becomes more cumbersome. This case will be considered elsewhere.

Acknowledgement. The work is supported by the FP7 PEOPLE IAPP project N 251475 "HYDROFRAC".

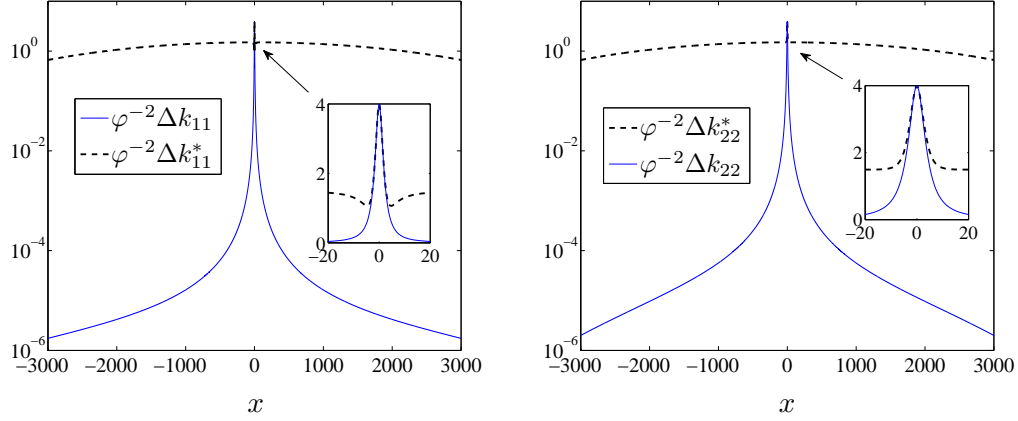


Figure 3: Comparison of the errors in the diagonal elements and different approximations for the value of the parameter $\varphi = 10^{-3}$. The errors corresponding to the factorization consisted of the matrix-functions (5.1), (5.2) are given by dotted line while those related to (4.12), (4.13) are depicted by the solid line.

References

- [1] Adukov, V.M.: Wiener-Hopf factorization of meromorphic matrix-functions. *Algebra Anal.* **4**, No. 1, 54–74 (1992) (Russian)
- [2] Aktosun, T., Klaus, M, van der Mee, C.: Explicit Wiener-Hopf factorization for certain non-rational matrix functions, *Integral Equations and Operator Theory*, **15**, 879–900 (1992)
- [3] Aleksandrov, A.B.: Norm of the Hilbert transformation in a space of Hölder functions. *Functional Analysis and Its Applications.* **9**, No. 2, 94–96 (1975)
- [4] Amirjanyan, H.A., Kamalyan, A.G.: Factorization of meromorphic matrix functions. *J. Contemp. Math. Anal., Armen. Acad. Sci.* **42**, No. 6, 303-319 (2007); translation from *Izv. Nats. Akad. Nauk Armen., Mat.* No. 6, 1–24 (2007)
- [5] Böttcher, A., Silberman, B.: *Analysis of Toeplitz Operators.* Springer Verlag, Heidelberg (1990)

- [6] Böttcher, A., Spitkovsky, I.M.: The factorization problem: some known results and open questions. In: *Advances in Harmonic Analysis and Operator Theory* (A.Almeida, L.Castro, F.-O.Speck eds.). Operator Theory: Advances and Applications, **229**. Birkhäuser, Basel, 101-122 (2013)
- [7] Camara, M.C., dos Santos, A.F., Manojlovic, N.: Generalized factorization for $n \times n$ Daniele-Khrapkov matrix functions. *Math. Mech. Appl. Sci.* **393**, 993–1020 (2001)
- [8] Chebotarev, G. N.: Partial indices of the Riemann boundary value problem with a triangular matrix of the second order, *Uspekhi mat. nauk*, **II**, No. 3, 192-202 (1956) (in Russian)
- [9] Chebotarev, G. N.: To the solution in an explicit form of the Riemann boundary value problem for a system of n pairs of functions. *Uch. zapiski Kazan State University. Matematika.* **116**, No. 4, 31–58 (1956) (in Russian)
- [10] Daniele, V.G.: On the factorization of Wiener-Hopf matrices in problems solvable with Hurd’s method. *IEEE Trans. Antennas Propagat.* **26**, 614–616 (1978)
- [11] Ehrhardt, T., Speck, F.-O.: Transformation techniques towards the factorization of non-rational 2×2 matrix functions. *Linear Algebra Appl.* **353**, No. 1-3, 53–90 (2002)
- [12] Ehrhardt, T., Spitkovsky, I.M.: Factorization of piece-wise constant matrix-functions and systems of differential equations. ; transl. from *Algebra Anal.* **13**, No. 6, 56-123 (2001); Engl. transl. in *St. Petersburg. Math. J.* **13**, No. 6, 939–991 (2001)
- [13] Feldman, I., Gohberg, I., Krupnik, N.: A method of explicit factorization of matrix functions and applications. *Integral Equations Oper. Theory.* **18**, No. 3, 277–302 (1994)
- [14] Feldman, I., Gohberg, I., Krupnik, N.: Convolution equations on finite intervals and factorization of matrix functions. *Integral Equations Oper. Theory.* **36**, No. 2, 201–211 (2000)
- [15] Feldman, I., Gohberg, I., Krupnik, N.: An explicit factorization algorithm. *Integral Equations Oper. Theory.* **49**, 149–164 (2004)

- [16] Gakhov, F.D.: Riemann boundary value problem for a system of n pairs of functions. *Uspekhi mat. nauk.* **VII**, No. 4 (50), 3–54 (1952) (in Russian)
- [17] Gakhov, F.D.: *Boundary Value Problems*, Nauka, Moscow (3rd ed.) (1977) (in Russian)
- [18] Gohberg, I.C., Feldman, I.A.: *Convolution equations and projection methods for their solution*. Transl. Math. Monographs, **41**. AMS, Providence, R.I. (1974)
- [19] Gohberg, I. Ts., Krein, M. G.: Systems of integral equations on a half-line with kernels depending on the difference of arguments, *Uspekhi mat. nauk*, **XIII**, No. 2 (80), 1958, 3–72 (in Russian)
- [20] Janashia, G.A., Lagvilava, E.T.; Ephremidze, L.N.: The approximate factorization of positive-definite matrix functions. *Russ. Math. Surv.* **54**, No. 6, 1246–1247 (1999); translation from *Usp. Mat. Nauk* **54**, No. 6, 161–162 (1999)
- [21] Jones, D. S.: Commutative WienerHopf factorization of a matrix. *Proc. R. Soc. A* **393**, 185–192 (1984)
- [22] Khrapkov, A.A.: Certain cases of the elastic equilibrium of an infinite wedge with a non-symmetric notch at the vertex, subjected to concentrated forces. *Appl. Math. Mech. (PMM)* **35**, 625–637 (1971)
- [23] Khrapkov, A.A.: Closed form solutions of problems on the elastic equilibrium of an infinite wedge with nonsymmetric notch at the apex. *Appl. Math. Mech. (PMM)* **35**, 1009–1016 (1971)
- [24] Kiyasov, S.N.: Linear fractional Riemann boundary value problem and its application to factorization of certain classes of Hölder matrix functions of the second order. *Russ. Math.* **39**, No. 9, 21–27 (1995); translation from *Izv. Vyssh. Uchebn. Zaved., Mat.* No. 9 (400), 23–29 (1995)
- [25] Kiyasov, S.N.: Effective factorization of some classes of third-order matrix-functions. *Uch. Zap. Kazan. Gos. Univ., Ser. Fiz.-Mat. Nauki.* **150**, No. 1, 65–70 (2008) (Russian)

- [26] Kiyasov, S.N.: Some cases of efficient factorization of second-order matrix functions. *Russian Mathematics (Izv. VUZ)*. **56**, No. 6, 30–36 (2012); transl. from *Izv. vyssh. uch. zav. Matematika*. No. 6, 36–43 (2012)
- [27] Lawrie, J.B., Abrahams, I.D.: A brief historical perspective of the Wiener-Hopf technique. *J. Eng. Math.* **59**, No. 4, 351–358 (2007)
- [28] Litvinchuk, G.S., Spitkovsky, I.M.: *Factorization of measurable matrix functions*. Birkhäuser, Basel-Boston (1987)
- [29] Markus, A., Matsaev, V.: The failure of factorization of positive matrix functions on noncircular contours. *Linear Algebra Appl.* 208–209, 231–237 (1994)
- [30] Mishuris, G.S., Movchan, N.V. and Movchan, A.B.: Dynamic mode-III interface crack in a bi-material strip. *International Journal of Fracture*, **166**, 121-133 (2010)
- [31] Muskhelishvili, N.I.: *Singular Integral Equations*. Nauka, Moscow (3rd ed.) (1968) (in Russian)
- [32] Noble, B.: *Methods based on the Wiener-Hopf Technique*. Chelsea Publishing Company, New York (2nd ed.) (1988)
- [33] Piccolroaz, A., Mishuris, G. and Movchan, A.: Evaluation of the Lazarus-Leblond constants in the asymptotic model of the interfacial wavy crack. *Journal of the Mechanics and Physics of Solids*, **55**, 1575-1600, (2007)
- [34] Piccolroaz, A., Mishuris, G. and Movchan, A.: Symmetric and skew-symmetric weight functions in 2D perturbation models for semi-infinite interfacial cracks. *Journal of the Mechanics and Physics of Solids*, **57**, 1657-1682, (2009)
- [35] Piccolroaz, A., Mishuris, G., Movchan, A., and Movchan, N.: Perturbation analysis of Mode III interfacial cracks advancing in a dilute heterogeneous material. *International Journal of Solids and Structures*, **49**, 244-255 (2012)

- [36] Prössdorf, S., Speck, F.-O.: A factorisation procedure for two by two matrix functions on the circle with two rationally independent entries. *Proc. Roy. Soc. Edinburgh. Sec. A* **115**, No. 1-2, 119–138 (1990)
- [37] Rawlins, A.D.: A note on Wiener-Hopf matrix factorization, *Quart. J. Mech. Appl. Math.* **38**, No. 3, 433–437 (1985)
- [38] Rawlins, A.D., Williams, W.E.: Matrix Wiener-Hopf factorisation, *Quart. J. Mech. Appl. Math.* **34**, No. 1, 1–8 (1981)
- [39] Speck, F.-O.: General Wiener-Hopf factorization methods. Pitman, Boston (1985)
- [40] Simonenko, I.B.: Some general questions of the theory of Riemann boundary value problem. *Mat. USSR Izv.* **2**, No. 5, 1091–1099 (1968); transl. from *Izv. AN SSSR, ser. mat.* **32**, No. 5, 1138–1146 (1968)
- [41] Spitkovskij, I.M., Tashbaev, A.M.: Factorization of certain piecewise constant matrix functions and its applications. *Math. Nachr.* **151**, 241–261 (1991)
- [42] Veitch, B.H., Abrahams, I.D.: On the commutative factorization of $n \times n$ matrix Wiener-Hopf kernels with distinct eigenvalues. *Proc. R. Soc. A* **463**, 613–639 (2007)
- [43] Vekua, N.P.: *Systems of singular integral equations*, P. Noordhoff, Groningen (1967)
- [44] Voronin, A.F.: A method of determination of partial indices of symmetric matrix-functions. *Siberian Math. J.* **52**, No. 1, 41–53 (2011); transl. from *Sib. mat. zh.* **52**, No. 1, 54–68 (2011)