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MODELS OF NONLINEAR ACOUSTICS VIEWED AS APPROXIMATIONS OF THE KUZNETSOV EQUATION

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ABSTRACT. We relate together different models of non linear acoustic in thermo-elastic media as the Kuznetsov equation, the Westervelt equation, the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation and the Nonlinear Progressive wave Equation (NPE) and estimate the time during which the solutions of these models keep closed in the L^2 norm. The KZK and NPE equations are considered as paraxial approximations of the Kuznetsov equation. The Westervelt equation is obtained as a nonlinear approximation of the Kuznetsov equation. Aiming to compare the solutions of the exact and approximated systems in found approximation domains the well-posedness results (for the Kuznetsov equation in a half-space with periodic in time initial and boundary data) are obtained.

1. Introduction. One of the most general model to describe an acoustic wave propagation in an homogeneous thermo-elastic medium is the compressible Navier-Stokes system in \mathbb{R}^n

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0, \quad (1)$$

$$\rho[\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla p + \eta \Delta \mathbf{v} + \left(\zeta + \frac{\eta}{3} \right) \nabla \cdot \operatorname{div}(\mathbf{v}), \quad (2)$$

$$\begin{aligned} \rho T[\partial_t S + (\mathbf{v} \cdot \nabla) S] &= \kappa \Delta T + \zeta (\operatorname{div} \mathbf{v})^2 \\ &+ \frac{\eta}{2} \left(\partial_{x_k} v_i + \partial_{x_i} v_k - \frac{2}{3} \delta_{ik} \partial_{x_i} v_i \right)^2, \end{aligned} \quad (3)$$

$$p = p(\rho, S), \quad (4)$$

where the pressure p is given by the state law $p = p(\rho, S)$. The density ρ , the velocity \mathbf{v} , the temperature T and the entropy S are unknown functions in system (1)–(4). The coefficients ζ , κ and η are constant viscosity coefficients. For the acoustical

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framework the wave motion is supposed to be potential and the viscosity coefficients are supposed to be small in terms of a dimensionless small parameter $\varepsilon > 0$, which also characterizes the size of the perturbations near the constant state $(\rho_0, 0, S_0, T_0)$. Here the velocity \mathbf{v}_0 is taken equal to 0 just using a Galilean transformation.

Actually, ε is the Mach number, which is supposed to be small [5] ($\varepsilon = 10^{-5}$ for the propagation in water with an initial power of the order of 0.3 W/cm^2):

$$\frac{\rho - \rho_0}{\rho_0} \sim \frac{T - T_0}{T_0} \sim \frac{|\mathbf{v}|}{c} \sim \varepsilon,$$

where $c = \sqrt{p'(\rho_0)}$ is the speed of sound in the unperturbed media.

Hence as in [12, 43], system (1)–(4) becomes an isentropic Navier-Stokes system

$$\partial_t \rho_\varepsilon + \operatorname{div}(\rho_\varepsilon \mathbf{v}_\varepsilon) = 0, \tag{5}$$

$$\rho_\varepsilon [\partial_t \mathbf{v}_\varepsilon + (\mathbf{v}_\varepsilon \cdot \nabla) \mathbf{v}_\varepsilon] = -\nabla p(\rho_\varepsilon) + \varepsilon \nu \Delta \mathbf{v}_\varepsilon, \tag{6}$$

with the approximate state equation $p(\rho, S) = p(\rho_\varepsilon) + O(\varepsilon^3)$:

$$p(\rho_\varepsilon) = p_0 + c^2(\rho_\varepsilon - \rho_0) + \frac{(\gamma - 1)c^2}{2\rho_0}(\rho_\varepsilon - \rho_0)^2, \tag{7}$$

where $\gamma = C_p/C_V$ denotes the ratio of the heat capacities at constant pressure and at constant volume respectively and with a small enough and positive viscosity coefficient:

$$\varepsilon \nu = \beta + \kappa \left(\frac{1}{C_V} - \frac{1}{C_p} \right).$$

If we go on physical assumptions of the wave motion [5, 18, 31, 50] for the perturbations of the density or of the velocity or of the pressure, the isentropic system (5)–(6) gives

1. the Westervelt equation for the potential of the velocity, derived initially by Westervelt [50] and later by other authors [1, 49]:

$$\partial_t^2 \Pi - c^2 \Delta \Pi = \varepsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta \Pi + \frac{\gamma + 1}{2c^2} (\partial_t \Pi)^2 \right) \tag{8}$$

with the same constants introduced for the Navier-Stokes system.

2. the Kuznetsov equation also for the potential of the velocity, firstly introduced by Kuznetsov [31] for the velocity potential, see also Refs. [18, 23, 28, 33] for other different methods of its derivation:

$$\partial_t^2 u - c^2 \Delta u = \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right). \tag{9}$$

3. the Khokhlov-Zabolotskaya-Kuznetsov (KZK) [5, 42] for the density:

$$c \partial_{\tau z}^2 I - \frac{(\gamma + 1)}{4\rho_0} \partial_\tau^2 I^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^3 I - \frac{c^2}{2} \Delta_y I = 0. \tag{10}$$

4. the Nonlinear Progressive wave Equation (NPE) derived in Ref. [39] also for the density:

$$\partial_{\tau z}^2 \xi + \frac{(\gamma + 1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0. \tag{11}$$

For higher order models as the nonlinear Jordan-Moore-Gibson-Thompson (JMGT) equation, containing the Kuznetsov equation as a particular or a limit case, see [24, 26, 27] and their references. In this article we don't consider such higher order

models and focus our attention on the Kuznetsov equation considered here as the most complete equation.

In [12] it is shown that the Kuznetsov equation comes from the Navier-Stokes or Euler system only by small perturbations, but to obtain the KZK and the NPE equations we also need to perform in addition to the small perturbations a paraxial change of variables. In this article we derive the KZK and the NPE equations from the Kuznetsov equation just performing the corresponding paraxial change of variables and show that the Westervelt equation can be also viewed as an approximation of the Kuznetsov equation by a nonlinear perturbation.

The physical context and the physical usage of the KZK and the NPE equations are different: the NPE equation is helpful to describe short-time pulses and a long-range propagation, for instance, in an ocean wave-guide, where the refraction phenomena are important [7, 38], while the KZK equation typically models the ultrasonic propagation with strong diffraction phenomena, combining with finite amplitude effects (see [42] and the references therein). But in the same time [12], there is a bijection between the variables of these two models and they can be presented by the same type differential operator with constant positive coefficients:

$$Lu = 0, \quad L = \partial_{tx}^2 - c_1 \partial_x (\partial_x \cdot)^2 - c_2 \partial_x^3 \pm c_3 \Delta_y \quad \text{for } t \in \mathbb{R}^+, \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^{n-1}.$$

Therefore, the results on the solutions of the KZK equation from [21, 41] are valid for the NPE equation.

The interest to study how closed are the solutions of the general model of the non-linear wave motion, described by the Kuznetsov equation, and of simplified models with more particular area of application (such as the KZK equation and the NPE equation which are valid only with additional assumptions on the wave propagation describing by the paraxial changes of variables) is naturally motivated by the questions about the accuracy of the approximations and of a comparative analysis of the solutions of these models.

If we formally consider the differential operators in the Kuznetsov equation (9) and the Westervelt equation (8), we notice that the Westervelt equation keeps only one of two non-linear terms of the Kuznetsov equation, producing cumulative effects in a progressive wave propagation [1]. The question how closed are the solutions of these two models, which differ on presence of local nonlinear term, was also open so far.

The mentioned approximation questions are treated theoretically in this article.

Let us also notice that the Kuznetsov equation (9) (and also the Westervelt equation (8)) is a non-linear wave equation with terms of different order (the wave operator is of order ε^0 and the nonlinear and viscosity terms are of order ε). But the KZK- and NPE-paraxial approximations allow to have the approximate equations with all terms of the same order, *i.e.* the KZK and NPE equations. For the well posedness of the Cauchy problem for the Kuznetsov equation we cite [11] and for boundary value problems in regular bounded domains see [25, 29, 40].

We present the structure of the paper and its mains results in the next subsection.

1.1. Main results. To keep a physical sense of the approximation problems, we consider especially the two or three dimensional cases, *i.e.* \mathbb{R}^n with $n = 2$ or 3 , and in the following we use the notation $x = (x_1, x') \in \mathbb{R}^n$ with one propagative axis $x_1 \in \mathbb{R}$ and the traversal variable $x' \in \mathbb{R}^{n-1}$.

To be able to consider the approximation of the Kuznetsov equation by the KZK equation (see Section 2), we establish (see Theorems A.3 and A.6 in Appendix A)

TABLE 1. Approximation results for models derived from the Kuznetsov equation

	KZK		NPE	Westervelt	
	periodic boundary condition problem	initial boundary value problem		viscous and inviscid case	viscous case inviscid case
Theorem	Theorem 2.1	Theorem 2.2	Theorem 3.1	Theorem 4.2	
Derivation	paraxial approximation $u = \Phi(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} \mathbf{x}')$		paraxial approximation $u = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon} \mathbf{x}')$	$\Pi = u + \frac{1}{2} \varepsilon u \partial_t u$	
Approximation domain	the half space $\{x_1 > 0, x' \in \mathbb{R}^{n-1}\}$		$\mathbb{T}_{x_1} \times \mathbb{R}^2$	\mathbb{R}^n	
Approximation order	$O(\varepsilon)$		$O(\varepsilon)$	$O(\varepsilon^2)$	
Estimation	$\ I - I_{approx}\ _{L^2(\mathbb{T}_t \times \mathbb{R}^{n-1})} \leq \varepsilon$ $z \leq K$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+ \ \nabla(u - \bar{u})(t)\ _{L^2} \leq K\varepsilon.$ $t < \frac{T}{\varepsilon}$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+ \ \nabla(u - \bar{u})(t)\ _{L^2} \leq K\varepsilon$ $t < \frac{T}{\varepsilon}$	$\ (u - \bar{u})_t(t)\ _{L^2}$ $+ \ \nabla(u - \bar{u})(t)\ _{L^2} \leq K\varepsilon$ $t < \frac{T}{\varepsilon}$	
Initial data regularity	$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}^{n-1})$ for $s > \max(\frac{n}{2}, 2)$	$I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ for $[\frac{s}{2}] > \frac{n}{2} + 2$	$\xi_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$ for $s > \frac{n}{2} + 1$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+3}(\mathbb{R}^3)$ for $s > \frac{n}{2}$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+2}(\mathbb{R}^3)$ for $s > \frac{n}{2}$
Data regularity for remainder boundness	$I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}^{n-1})$ for $s > \max(\frac{n}{2}, 2)$	$I_0 \in H^6(\mathbb{T}_t \times \mathbb{R}^{n-1})$ for $n = 2, 3,$ $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ for $[\frac{s}{2}] > \frac{n}{2} + 1, n \geq 4$	$\xi_0 \in H^4(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$ for $n = 2, 3,$ $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$ for $s > \frac{n}{2} + 2, n \geq 4.$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+3}(\mathbb{R}^n)$ for $s > \frac{n}{2}$	$u_0 \in H^{s+3}(\mathbb{R}^n)$ $u_1 \in H^{s+2}(\mathbb{R}^n)$ for $s > \frac{n}{2}$

global well posedness results for the Kuznetsov equation in the half space similar to the previous framework for the KZK and the Navier-Stokes system considered in [12]. Theorem A.3 corresponds to the well posedness of the periodic in time Dirichlet boundary valued problem for the Kuznetsov equation in the half space $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ (see Eq. (21)) for small enough boundary data. In this case the boundary condition is considered as the initial condition of the corresponding Cauchy problem in \mathbb{R}^n . The proof is based on the maximal regularity result for the corresponding linear problem given in Theorem A.1 and on the application of a result of the nonlinear functional analysis from [48, 1.5. Cor., p. 368] (see also [11, Thm. 4.2]). We also applied it to prove (see Theorem A.6) the well posedness for the initial boundary valued problem for the Kuznetsov equation in the half space (28), once again combining with the maximal regularity result for the linear problem (see the proof of Lemma A.4).

In Subsection 2.1 we derive the KZK equation from the Kuznetsov equation by introducing the paraxial change of variables (34).

For the approximation framework for their solutions we study two cases. The first case is treated in Sub-subsection 2.2.1 and it considers the purely time periodic boundary problem in the *ansatz* variables (z, τ, y) moving with the wave, where we use the well-posedness result of Theorem A.3. In this case the only viscous medium can be considered as the condition to be periodic in time is not compatible with shock formations providing the loss of the regularity which may occur in the inviscid medium (see [41, Thm. 1.3]). The approximation results are formulated in Theorem 2.1.

The second case (see Sub-subsection 2.2.2) studies the initial boundary-value problem for the Kuznetsov equation in the initial variables (t, x_1, x') with data coming from the solution of the KZK equation, using this time the well posedness results of Theorem A.6. This time we have the approximation results for the viscous and inviscid cases (see Theorem 2.2 and Remark 2).

In Section 3 we establish the approximation result between the Kuznetsov equation and NPE equation in the viscous and inviscid cases (see Theorem 3.1).

Finally in Section 4 we compare the solutions of the Westervelt and the Kuznetsov equations. We derive the Westervelt equation from the Kuznetsov equation by a nonlinear change of variables in Subsection 4.1 and we validate the approximation in Subsection 4.2 (see Theorem 4.2 for viscous and inviscid cases).

We denote by u a solution of the “exact” problem for the Kuznetsov equation $Exact(u) = 0$ and by \bar{u} an approximate solution, constructed by the derivation *ansatz* from a regular solution of one of the approximate models (for instance of the KZK or of the NPE equations), *i.e.* \bar{u} is a function which solves the Kuznetsov equation up to εR :

$$Approx(\bar{u}) = Exact(\bar{u}) - \varepsilon R = 0.$$

In the approximation between the solutions of the Kuznetsov equation and of the Westervelt equation the remainder term appears with the size ε^2 (it is natural since both models contain terms of order ε^0 and ε).

We can summarize the obtained approximation results of the Kuznetsov equation in the following way: if, once again, u is a solution of the Kuznetsov equation and \bar{u} is a solution of the NPE or of the KZK (for the initial boundary value problem) or of the Westervelt equations found for rather closed initial data

$$\|\nabla_{t,\mathbf{x}}(u(0) - \bar{u}(0))\|_{L^2(\Omega)} \leq \delta \leq \varepsilon,$$

then there exist constants $K, C_1, C_2, C > 0$ independent of ε, δ and on time, such that for all $t \leq \frac{C}{\varepsilon}$ it holds

$$\|\nabla_{t,\mathbf{x}}(u - \bar{u})\|_{L^2(\Omega)} \leq C_1(\varepsilon^2 t + \delta)e^{C_2 \varepsilon t} \leq K\varepsilon.$$

For a more detailed comparison between different models we include the main points of our results to the comparative Table 1.

In Table 1 the line named “Initial data regularity” gives the information about the regularity of the initial data for the approximate model, which ensure the same regularity of the solutions of an approximate model and of the solution of the Kuznetsov equation, taken with the same initial data $u(0) = \bar{u}(0)$, coming from the corresponding *ansatz*.

To have the remainder term $R \in C([0, T], L^2(\Omega))$ we ensure that $Exact(\bar{u}) \in C([0, T], L^2(\Omega))$, i.e. we need a sufficiently regular solution \bar{u} . The minimal regularity of the initial data to have a such \bar{u} is given in Table 1 in the last line named “Data regularity for remainder boundness”.

To summarize, the rest of the paper is organized as follows. Section 2 considers the derivation (Subsection 2.1) of the KZK equation from the Kuznetsov equation and two types of approximation results for the solutions of the Kuznetsov equation approximated by the solutions of the KZK equation in Subsection 2.2. The approximation by the solutions of the NPE equation is considered in Section 3. Section 4 contains the derivation of the Westervelt equation and the approximation result for the solutions of the Kuznetsov and the Westervelt equations. The well posedness results for the Kuznetsov equation needed for the approximation results of Section 2 are detailed in Appendix A.

2. The Kuznetsov equation and the KZK equation.

2.1. **Derivation of the KZK equation from the Kuznetsov equation.** If the velocity potential is given [31] by

$$u(x, t) = \Phi(t - x_1/c, \varepsilon x_1, \sqrt{\varepsilon} x') = \Phi(\tau, z, y), \tag{12}$$

we directly obtain from the Kuznetsov equation (9) via the paraxial change of variables

$$\tau = t - \frac{x_1}{c}, \quad z = \varepsilon x_1, \quad y = \sqrt{\varepsilon} x', \tag{13}$$

that

$$\begin{aligned} & \partial_t^2 u - c^2 \Delta u - \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \\ &= \varepsilon \left[2c \partial_{\tau z}^2 \Phi - \frac{\gamma + 1}{2c^2} \partial_\tau (\partial_\tau \Phi)^2 - \frac{\nu}{\rho_0 c^2} \partial_\tau^3 \Phi - c^2 \Delta_y \Phi \right] + \varepsilon^2 R_{Kuz-KZK} \end{aligned} \tag{14}$$

with

$$\begin{aligned} \varepsilon^2 R_{Kuz-KZK} = & \varepsilon^2 \left(-c^2 \partial_z^2 \Phi + \frac{2}{c} \partial_\tau (\partial_\tau \Phi \partial_z \Phi) - \partial_\tau (\nabla_y \Phi)^2 + \frac{2\nu}{c\rho_0} \partial_\tau^2 \partial_z \Phi - \frac{\nu}{\rho_0} \partial_\tau \Delta_y \Phi \right) \\ & + \varepsilon^3 \left(-\partial_\tau (\partial_z \Phi)^2 - \frac{\nu}{\rho_0} \partial_\tau \partial_z^2 \Phi \right). \end{aligned} \tag{15}$$

Let us notice that the paraxial change of variables (13) defines the axis of the propagation x_1 along which the wave changes its profile much slower than along the transversal axis x' . This is typical for the propagation of ultrasound waves.

Therefore, we find that the right-hand side ε -order terms in Eq. (14) is exactly the KZK equation (10). Thanks to [41, Thms. 1.1–1.3] we have the well posedness

result for the KZK equation in the half space with periodic boundary conditions of a period L on τ and of mean value zero.

Due to the well posedness domain $(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ of the KZK equation, to validate the approximation between the solutions of the KZK and the Kuznetsov equations, we need to have the well posedness of the Kuznetsov equation on the half space with boundary conditions coming from the initial condition for the KZK equation. For these well posedness results see Appendix A.

2.2. Approximation of the solutions of the Kuznetsov equation by the solutions of the KZK equation. Let us consider the Cauchy problem associated with the KZK equation

$$\begin{cases} c\partial_z I - \frac{(\gamma+1)}{4\rho_0} \partial_\tau I^2 - \frac{\nu}{2c^2\rho_0} \partial_\tau^2 I - \frac{c^2}{2} \partial_\tau^{-1} \Delta_y I = 0 \text{ on } \mathbb{T}_\tau \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ I(\tau, 0, y) = I_0(\tau, y) \text{ on } \mathbb{T}_\tau \times \mathbb{R}^{n-1}, \end{cases} \tag{16}$$

for small enough initial data in order to have by [41, Thm. 1.2] a time periodic solution I defined on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$. As it was mentioned in Introduction 1.1, if $\nu > 0$, to compare the solutions of the Kuznetsov and the KZK equations we consider two cases. The first case (see Sub-subsection 2.2.1) consists in studies of the time periodic boundary problem for the Kuznetsov equation (21) with the boundary condition imposed by the initial condition I_0 of the KZK equation. In Sub-subsection 2.2.2 we study the second case, when the solution of the KZK equation, taken for $\tau = 0$, gives $I(0, z, y)$ defined on $\mathbb{R}_+ \times \mathbb{R}^{n-1}$, from which we deduce, according to the derivation *ansatz*, both an initial condition for the Kuznetsov equation at $t = 0$ and a corresponding boundary condition. In this second situation, it also makes sense to consider the inviscid case, briefly commented in the end of Sub-subsection 2.2.2.

2.2.1. *Approximation problem for the Kuznetsov with periodic boundary conditions.* By [41, Thm. 1.2] there is a unique solution $I(\tau, z, y)$ of the Cauchy problem for the KZK equation (16) such that

$$z \mapsto I(\tau, z, y) \in C([0, \infty[, H^s(\Omega_1)) \tag{17}$$

with $\int_{\mathbb{T}_\tau} I(l, z, y) dl = 0$ and $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$, where \mathbb{T}_τ represents the periodicity in τ of period L . The operator ∂_τ^{-1} is defined by

$$\partial_\tau^{-1} I(\tau, z, y) := \int_0^\tau I(l, z, y) dl + \int_0^L \frac{\ell}{L} I(\ell, z, y) d\ell. \tag{18}$$

Formula (18), which implies that $\partial_\tau^{-1} I$ is L -periodic in τ and of mean value zero, gives us the estimate

$$\|\partial_\tau^{-1} I\|_{H^s(\Omega_1)} \leq C \|\partial_\tau \partial_\tau^{-1} I\|_{H^s(\Omega_1)} = C \|I\|_{H^s(\Omega_1)}. \tag{19}$$

So $\partial_\tau^{-1} I|_{z=0} \in H^s(\Omega_1)$, and hence by (17)

$$z \mapsto \partial_\tau^{-1} I(\tau, z, y) \in C([0, \infty[, H^s(\Omega_1)),$$

with $\int_{\mathbb{T}_\tau} \partial_\tau^{-1} I(s, z, y) ds = 0$.

We define on $\mathbb{T}_t \times \mathbb{R}_+ \times \mathbb{R}^{n-1}$

$$\bar{u}(t, x_1, x') := \frac{c^2}{\rho_0} \partial_\tau^{-1} I(\tau, z, y) = \frac{c^2}{\rho_0} \partial_\tau^{-1} I\left(t - \frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} x'\right) \tag{20}$$

with the paraxial change of variable (13) associated with the KZK equation. Thus \bar{u} is L -periodic in time and of mean value zero. Now we consider the following Kuznetsov problem

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \cdot \nabla u_t & \text{on } \mathbb{T}_t \times \mathbb{R}_+ \times \mathbb{R}^{n-1}, \\ u|_{x_1=0} = g & \text{on } \mathbb{T}_t \times \mathbb{R}^{n-1}, \end{cases} \tag{21}$$

in which the boundary condition is imposed by the initial condition for the KZK equation:

$$g(t, x') := \bar{u}(t, 0, x') = \frac{c^2}{\rho_0} \partial_\tau^{-1} I_0(\tau, y). \tag{22}$$

Let us define (see Eq. (12), and subsection 4.1 in [12] for more details)

$$\tilde{I} := \frac{\rho_0}{c^2} \partial_\tau \Phi. \tag{23}$$

Then \tilde{I} is the solution of the Kuznetsov equation written in the following form with the remainder $R_{Kuz-KZK}$ defined in Eq. (15):

$$\begin{cases} c \partial_z \tilde{I} - \frac{(\gamma+1)}{4\rho_0} \partial_\tau \tilde{I}^2 - \frac{\nu}{2c^2 \rho_0} \partial_\tau^2 \tilde{I} - \frac{c^2}{2} \Delta_y \partial_\tau^{-1} \tilde{I} + \varepsilon \frac{\rho_0}{2c^2} R_{Kuz-KZK} = 0, \\ \tilde{I}|_{z=0} = I_0. \end{cases} \tag{24}$$

In Eq. (24) we can recognize the system associated with the KZK equation (10).

Now we can formulate the following approximation result between the solutions of the KZK and Kuznetsov equations.

Theorem 2.1. *Let $\nu > 0$. For $s > \max(\frac{n}{2}, 2)$ and $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ small enough in $H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$, there exists a unique global solution I of the Cauchy problem for the KZK equation (16) such that*

$$z \mapsto I(\tau, z, y) \in C([0, \infty[, H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

In addition, there exists a unique global solution \tilde{I} of the Kuznetsov problem (24), in the sense $\tilde{I} := \frac{\rho_0}{c^2} \partial_\tau \Phi$, with $\Phi(\tau, z, y) := u(t, x_1, x')$ with the paraxial change of variable (13) and

$$u \in H^2(\mathbb{T}_t; H^s(\mathbb{R}^+ \times \mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2}(\mathbb{R}^+ \times \mathbb{R}^{n-1})),$$

is the global solution of the periodic problem (21) for the Kuznetsov equation with g defined by I_0 as in Eq. (22). Moreover there exist $C_1, C_2 > 0$ such that

$$\frac{1}{2} \frac{d}{dz} \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}^2 \leq C_1 \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}^2 + C_2 \varepsilon \|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})},$$

which implies

$$\|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}(z) \leq \frac{C_2}{2} \varepsilon z e^{\frac{C_1}{2} z} \leq \frac{C_2}{C_1} \varepsilon (e^{\frac{C_1}{2} z} - 1)$$

and

$$\|I - \tilde{I}\|_{L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})}(z) \leq K \varepsilon \text{ while } z \leq C$$

with $K > 0$ and $C > 0$ independent of ε .

Proof. For $s > \max(\frac{n}{2}, 2)$, the global well-posedness of I comes from [41, Thm. 1.2] if $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ is small enough. Moreover, since g is given by Eq. (22), thanks to the definition of ∂_τ^{-1} in (18) and the fact that $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$, we have

$$g \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}^{n-1}) \text{ and } \partial_t g \in H^{s+\frac{3}{2}}(\mathbb{T}_t \times \mathbb{R}^{n-1}).$$

And thus

$$g \in H^{\frac{7}{4}}(\mathbb{T}_t; H^s(\mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2-\frac{1}{2}}(\mathbb{R}^{n-1})).$$

Therefore we can use Theorem A.3, which implies the global existence of the periodic in time solution

$$u \in H^2(\mathbb{T}_t; H^s(\mathbb{R}^+ \times \mathbb{R}^{n-1})) \cap H^1(\mathbb{T}_t; H^{s+2}(\mathbb{R}^+ \times \mathbb{R}^{n-1})), \tag{25}$$

of the Kuznetsov periodic boundary value problem (21) as I_0 is small enough in $H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$. Therefore, it also implies the global existence of \tilde{I} , defined in (23), which is the solution of the exact Kuznetsov system (24).

Now we subtract the equations in systems (16) and (24) to obtain

$$\begin{aligned} c\partial_z(I - \tilde{I}) - \frac{\gamma + 1}{2\rho_0}(I - \tilde{I})\partial_\tau I - \frac{\gamma + 1}{2\rho_0}\tilde{I}\partial_\tau(I - \tilde{I}) - \frac{\nu}{2c^2\rho_0}\partial_\tau^2(I - \tilde{I}) \\ - \frac{c^2}{2}\partial_\tau^{-1}\Delta_y(I - \tilde{I}) = \varepsilon\frac{\rho_0}{2c^2}R_{Kuz-KZK}. \end{aligned}$$

Denoting $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$, we multiply this equation by $(I - \tilde{I})$, integrate over $\mathbb{T}_\tau \times \mathbb{R}^{n-1}$ and perform a standard integration by parts, which gives

$$\begin{aligned} \frac{c}{2}\frac{d}{dz}\|I - \tilde{I}\|_{L^2(\Omega_1)}^2 - \frac{\gamma + 1}{2\rho_0}\int_{\Omega_1}\partial_\tau I(I - \tilde{I})^2 d\tau dy \\ - \frac{\gamma + 1}{2\rho_0}\int_{\Omega_1}\tilde{I}(I - \tilde{I})\partial_\tau(I - \tilde{I}) d\tau dy \\ + \frac{\nu}{2c^2\rho_0}\int_{\Omega_1}(\partial_\tau(I - \tilde{I}))^2 d\tau dy = \varepsilon\frac{\rho_0}{2c^2}\int_{\Omega_1}R_{Kuz-KZK}(I - \tilde{I}) d\tau dy. \end{aligned}$$

Let us notice that

$$\begin{aligned} \int_{\Omega_1}\tilde{I}(I - \tilde{I})\partial_\tau(I - \tilde{I}) d\tau dy &= \int_{\Omega_1}[(\tilde{I} - I) + I]\frac{1}{2}\partial_\tau(I - \tilde{I})^2 d\tau dy = \\ &= -\frac{1}{2}\int_{\Omega_1}\partial_\tau I(I - \tilde{I})^2 d\tau dy. \end{aligned}$$

By (25) with $s > \max(\frac{n}{2}, 2)$, u is sufficiently regular to ensure

$$R_{Kuz-KZK} \in C(\mathbb{R}_+; L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1})). \tag{26}$$

This comes from the fact that in system (24) the “worst” term, asking the most regularity of Φ , inside the remainder $R_{Kuz-KZK}$ (see Eq. (15)) is $\partial_\tau\partial_z^2\Phi$ with \tilde{I} given by Eq. (23). As $\partial_t^3 u \in L^2(\mathbb{T}_t; H^{s-2}(\Omega))$, we need to take $s > \max(\frac{n}{2}, 2)$ to have $\partial_\tau\partial_z^2\Phi$ in $L^\infty(\mathbb{R}_+; L^2(\mathbb{T}_\tau \times \mathbb{R}^{n-1}))$. Therefore, it holds

$$\left| \int_{\Omega_1} R_{Kuz-KZK}(I - \tilde{I}) d\tau dy \right| \leq \|R_{Kuz-KZK}\|_{L^2(\Omega_1)} \|I - \tilde{I}\|_{L^2(\Omega_1)} \leq C \|I - \tilde{I}\|_{L^2(\Omega_1)}$$

with a constant $C > 0$ independent of z thanks to (26). It leads to the estimate

$$\frac{1}{2}\frac{d}{dz}\|I - \tilde{I}\|_{L^2(\Omega_1)}^2 \leq K \sup_{(\tau,y) \in \Omega_1} |\partial_\tau I(\tau, z, y)| \|I - \tilde{I}\|_{L^2(\Omega_1)}^2 + C\varepsilon \|I - \tilde{I}\|_{L^2(\Omega_1)},$$

in which, due to the regularity of I for s and I_0 (see [41]) the term

$$\sup_{(\tau,y) \in \Omega_1} |\partial_\tau I(\tau, z, y)|$$

is bounded by a constant $C > 0$ independent of z . Consequently, we have the desired estimate and the other results follow from Gronwall’s Lemma. \square

Remark 1. The regularity $I_0 \in H^{s+\frac{3}{2}}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ for $s > \max(\frac{n}{2}, 2)$, imposed in Theorem 2.1, is the minimal regularity to ensure (26).

2.2.2. *Approximation problem for the Kuznetsov equation with initial-boundary conditions.* Let the function $I_0(t, y) = I_0(t, \sqrt{\varepsilon}x')$ be L -periodic on t and such that

$$I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$$

for $s \geq [\frac{n+1}{2}]$ and $\int_0^L I_0(s, y) ds = 0$. Hence [41], there is a unique solution $I(\tau, z, y)$ of the Cauchy problem (16) for the KZK equation satisfying (17). We define \bar{u} and g as in Eqs. (20) and (22) respectively. Thus, for $R_{Kuz-KZK}$ defined in Eq. (15), \bar{u} is the solution of the following system

$$\begin{cases} \partial_t^2 \bar{u} - c^2 \Delta \bar{u} - \varepsilon \partial_t \left((\nabla \bar{u})^2 + \frac{\gamma-1}{2c^2} (\partial_t \bar{u})^2 + \frac{\nu}{\rho_0} \Delta \bar{u} \right) = \varepsilon^2 R_{Kuz-KZK} & \text{in } \mathbb{T}_t \times \Omega, \\ \bar{u} = g & \text{on } \mathbb{T}_t \times \partial\Omega. \end{cases} \tag{27}$$

In the same time let us consider for a sufficiently large $T > 0$ the solution u (see Theorem A.6 for its global existence and uniqueness) of the Dirichlet boundary-value problem for the Kuznetsov equation

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = \alpha \varepsilon u_t u_{tt} + \beta \varepsilon \nabla u \nabla u_t & \text{in } [0, +\infty[\times \Omega, \\ u = g & \text{on } [0, \infty[\times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega, \end{cases} \tag{28}$$

taking $u_0 := \bar{u}(0)$ and $u_1 := \bar{u}_t(0)$ and considering the time periodic function g defined by Eq. (22) as a function on $[0, T]$.

To compare u and \bar{u} , we obtain the following stability result:

Theorem 2.2. *Let $T, \nu > 0, n \geq 2, s \in \mathbb{R}^+, \Omega = \mathbb{R}^+ \times \mathbb{R}^{n-1}$ and $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$.*

Then, the following statements are valid.

1. *If $s \geq 6$ for $n = 2$ and 3, or else $[\frac{s}{2}] > \frac{n}{2} + 1$, there exists a constant $C_0 > 0$ such that $\|I_0\|_{H^s} < C_0$ implies the global well-posedness of the Cauchy problem for the KZK equation with the following regularity:*

$$\text{for } 0 \leq k \leq [\frac{s}{2}] \quad I \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

Moreover it implies the well-posedness of (27) with

$$\bar{u} \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})), \quad \partial_t \bar{u} \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})),$$

or again

$$\bar{u} \in H^2(\mathbb{T}_t, H^{[\frac{s}{2}]-1}(\Omega)) \cap H^1(\mathbb{T}_t, H^{[\frac{s}{2}]}(\Omega)). \tag{29}$$

The imposed regularity of I_0 (see Table 1) is minimal to ensure that $R_{Kuz-KZK}$ (see Eq. (15) for the definition) is in $C([0, +\infty[; L^2(\mathbb{R}_+ \times \mathbb{R}^{n-1}))$.

2. If $[\frac{s}{2}] > \frac{n}{2} + 2$, taking the same initial data for the exact boundary-value problem for the Kuznetsov equation (28) as for \bar{u} , i.e.

$$u(0) = \bar{u}(0) = \frac{c^2}{\rho_0} \partial_\tau^{-1} I(-\frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} x') \in H^{[\frac{s}{2}]}(\Omega),$$

$$u_t(0) = \bar{u}_t(0) = \frac{c^2}{\rho_0} \partial_\tau I(-\frac{x_1}{c}, \varepsilon x_1, \sqrt{\varepsilon} x') \in H^{[\frac{s}{2}]-1}(\Omega),$$

there exists $C_0 > 0$ such that $\|I_0\|_{H^s} < C_0$ implies the well-posedness of the exact Kuznetsov equation (28) supplemented with the Dirichlet boundary condition

$$g = \frac{c^2}{\rho_0} \partial_\tau^{-1} I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}) \subset H^{7/4}(\]0, T[; H^{[\frac{s}{2}]-2}(\partial\Omega))$$

$$\cap H^1(\]0, T[; H^{[\frac{s}{2}]-2+3/2}(\partial\Omega))$$

ensuring the regularity

$$u \in H^2(\]0, T[, H^{[\frac{s}{2}]-1}(\Omega)) \cap H^1(\]0, T[, H^{[\frac{s}{2}]}(\Omega)). \tag{30}$$

Moreover, there exist constants $K, C, C_1, C_2 > 0$, all independent of ε , such that for all $t \leq \frac{C}{\varepsilon}$

$$\sqrt{\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2} \leq C_1 \varepsilon^2 t e^{C_2 \varepsilon t} \leq K \varepsilon. \tag{31}$$

3. In addition, let u be a solution of the Dirichlet boundary-value problem (28) for the Kuznetsov equation, with g defined by Eq. (22) and with initial data $u_0 \in H^{m+2}(\Omega)$, $u_1 \in H^{m+1}(\Omega)$ for $m > \frac{n}{2}$ such that

$$\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2 \leq \delta^2 \leq \varepsilon^2. \tag{32}$$

Then there exist constants $K, C, C_1, C_2 > 0$, all independent of ε , such that for all $t \leq \frac{C}{\varepsilon}$

$$\sqrt{\|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2} \leq C_1(\varepsilon^2 t + \delta^2) e^{C_2 \varepsilon t} \leq K \varepsilon. \tag{33}$$

Proof. Let \bar{u} and g be defined by Eqs. (20) and (22) respectively by the solution I of the Cauchy problem (16) for the KZK equation with $I|_{z=0} = I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ and $s \geq 6$ for $n = 2$ and 3 , or else $[\frac{s}{2}] > \frac{n}{2} + 1$. In this case, \bar{u} is the global solution of the approximated Kuznetsov system (27), what is a direct consequence of Theorem 1.2 in Ref. [41]. If $I_0 \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1})$ with the chosen s , then $I \in C(\{z > 0\}; H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1}))$. But knowing, thanks to estimate (19), that $\Delta_y^k I_0 \in H^{s-2k}(\mathbb{T}_t \times \mathbb{R}^{n-1})$ implies also $\partial_t^{-k} \Delta_y^k I_0 \in H^{s-2k}(\mathbb{T}_t \times \mathbb{R}^{n-1})$ for $1 \leq k \leq [\frac{s}{2}]$, the condition in [41, Thm. 1.2, Point 4] is verified and thus we have the following regularity of I on z : for $0 \leq k \leq [\frac{s}{2}]$

$$I(\tau, z, y) \in C^k(\{z > 0\}; H^{s-2k}(\mathbb{T}_\tau \times \mathbb{R}^{n-1})).$$

As \bar{u} is defined by (20), we deduce (using as previously the notation $\Omega_1 = \mathbb{T}_\tau \times \mathbb{R}^{n-1}$)

$$\bar{u}(\tau, z, y) \text{ and } \partial_\tau \bar{u}(\tau, z, y) \in C^k(\{z > 0\}; H^{s-2k}(\Omega_1)), \text{ if } 0 \leq k \leq \left[\frac{s}{2}\right],$$

$$\partial_\tau^2 \bar{u}(\tau, z, y) \in C^k(\{z > 0\}; H^{s-1-2k}(\Omega_1)), \text{ if } 0 \leq k \leq \left[\frac{s}{2}\right] - 1,$$

but we can also say [21, 41], thanks to the exponential decay of the solution of the KZK equation on z , that

$$\begin{aligned} \bar{u}(\tau, z, y) \text{ and } \partial_\tau \bar{u}(\tau, z, y) &\in H^k(\{z > 0\}; H^{s-2k}(\Omega_1)), \\ \partial_\tau^2 \bar{u}(\tau, z, y) &\in H^k(\{z > 0\}; H^{s-1-2k}(\Omega_1)). \end{aligned}$$

This implies for the chosen s that

$$\begin{aligned} \bar{u}(t, x_1, x') \text{ and } \partial_t \bar{u}(t, x_1, x') &\in L^2(\mathbb{T}_t; H^{\lfloor \frac{s}{2} \rfloor}(\Omega)) \cap H^2(\mathbb{T}_t; H^{\lfloor \frac{s}{2} \rfloor - 1}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in L^2(\mathbb{T}_t; H^{\lfloor \frac{s}{2} \rfloor - 1}(\Omega)) \cap H^2(\mathbb{T}_t; H^{\lfloor \frac{s}{2} \rfloor - 2}(\Omega)). \end{aligned}$$

Therefore

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C^1([0, +\infty[; H^{\lfloor \frac{s}{2} \rfloor - 1}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^{\lfloor \frac{s}{2} \rfloor - 2}(\Omega)). \end{aligned}$$

For the chosen s these regularities of $\bar{u}(t, x_1, x')$ give us regularity (29) and allow to have all left-hand terms in the approximated Kuznetsov system (27) of the desired regularity, *i.e.* $C([0, +\infty[; L^2(\Omega))$. In addition for $\lfloor \frac{s}{2} \rfloor > \frac{n}{2} + 2$ with the chosen g , $u_0 = \bar{u}(0)$ and $u_1 = \bar{u}_t(0)$ in the conditions of the theorem we have

$$u_0 \in H^{\lfloor \frac{s}{2} \rfloor}(\Omega), \quad u_1 \in H^{\lfloor \frac{s}{2} \rfloor - 1}(\Omega)$$

with

$$g \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}) \text{ and } \partial_t g \in H^s(\mathbb{T}_t \times \mathbb{R}^{n-1}).$$

This implies

$$g \in H^{7/4}([0, T]; H^{\lfloor \frac{s}{2} \rfloor - 2}(\partial\Omega)) \cap H^1([0, T]; H^{\lfloor \frac{s}{2} \rfloor - 2 + 3/2}(\partial\Omega))$$

with $\lfloor \frac{s}{2} \rfloor - 2 > \frac{n}{2}$, as required by Theorem A.6 to have the well-posedness of the solution of the Kuznetsov equation u during the time $t \in [0, T]$ associated with system (28). This completes the well-posedness results and we deduce that u have the desired regularity (30), announced in the theorem. Moreover, we have $R_{Kuz-KZK}$ in $C([0, +\infty[, L^2(\Omega))$.

Let us now prove (33) from point 3 as it directly implies estimate (31) from point 2. We subtract the Kuznetsov equation from the approximated Kuznetsov equation (see system (27)), multiply by $(u - \bar{u})_t$ and integrate over Ω to obtain, as in Ref. [11], the following stability estimate:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\int_{\Omega} A(t, x) (u - \bar{u})_t^2 + c^2 (\nabla(u - \bar{u}))^2 dx \right) \\ \leq C\varepsilon \sup(\|u_{tt}\|_{L^\infty(\Omega)}; \|\Delta u\|_{L^\infty(\Omega)}; \|\nabla \bar{u}_t\|_{L^\infty(\Omega)}) \\ \cdot \left(\|(u - \bar{u})_t\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})\|_{L^2(\Omega)}^2 \right) \\ + \varepsilon^2 \int_{\Omega} R_{Kuz-KZK}(u - \bar{u})_t dx, \end{aligned}$$

where $\frac{1}{2} \leq A(t, x) \leq \frac{3}{2}$ for $0 \leq t \leq T$ and $x \in \Omega$. By regularity of the solutions, $\sup(\|u_{tt}\|_{L^\infty(\Omega)}; \|\Delta u\|_{L^\infty(\Omega)}; \|\nabla \bar{u}_t\|_{L^\infty(\Omega)})$ is bounded in time on $[0, T]$. Moreover,

we have $\|R_{Kuz-KZK}(t)\|_{L^2(\Omega)}$ bounded for $t \in [0, T]$ by the regularity of \bar{u} , where $R_{Kuz-KZK}$ is defined in Eq. (15). Then after integration on $[0, t]$, we can write

$$\begin{aligned} & \|(u - \bar{u})_t(t)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(t)\|_{L^2(\Omega)}^2 \\ & \leq 3(\|(u - \bar{u})_t(0)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(0)\|_{L^2(\Omega)}^2) \\ & \quad C_1 \varepsilon \int_0^t \|(u - \bar{u})_t(s)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(s)\|_{L^2(\Omega)}^2 ds \\ & \quad + C_2 \varepsilon^2 \int_0^t \sqrt{\|(u - \bar{u})_t(s)\|_{L^2(\Omega)}^2 + \|\nabla(u - \bar{u})(s)\|_{L^2(\Omega)}^2} ds. \end{aligned}$$

Thanks to (32) we finally find by the Gronwall Lemma that for $t \leq \frac{C}{\varepsilon}$ estimate (33) holds true, thereby concluding the proof. \square

Remark 2. Let us discuss the corresponding approximation results in the inviscid case. We have two approximation results:

1. between the solutions $\bar{\mathbf{U}}_{KZK}$ of the KZK equation and \mathbf{U}_{Euler} of the Euler system [12, 42] (see [12, Thm 6.8] for the definitions of \mathbf{U}_{Euler} and $\bar{\mathbf{U}}_{KZK}$) in a cone

$$C(T) = \{0 < t < T \mid T < \frac{T_0}{\varepsilon}\} \times Q_\varepsilon(t)$$

with

$$Q_\varepsilon(s) = \{x = (x_1, x') : |x_1| \leq \frac{R}{\varepsilon} - Ms, M \geq c, x' \in \mathbb{R}^{n-1}\}$$

and with

$$\|\nabla \mathbf{U}_{Euler}\|_{L^\infty([0, \frac{T_0}{\varepsilon}]; H^{s-1}(Q_\varepsilon))} < \varepsilon C \text{ for } s > \left[\frac{n}{2}\right] + 1;$$

2. between the solutions \mathbf{U}_{Euler} of the Euler system and $\bar{\mathbf{U}}_{Kuzn}$ of the Kuznetsov equation [12] (see [12, Thm. 6.6] for the definitions of \mathbf{U}_{Euler} and $\bar{\mathbf{U}}_{Kuzn}$) in $[0, \frac{T_0}{\varepsilon}] \times \mathbb{R}^n$ containing $C(T)$.

Consequently, we obtain the approximation result between the solutions $\bar{\mathbf{U}}_{KZK}$ of the KZK equation and the solutions $\bar{\mathbf{U}}_{Kuzn}$ of the Kuznetsov equation in $C(T)$ by the triangular inequality:

$$\|\bar{\mathbf{U}}_{Kuzn} - \bar{\mathbf{U}}_{KZK}\|_{L^2(Q_\varepsilon(t))}^2 \leq K(\varepsilon^3 t + \delta^2)e^{K\varepsilon t} \leq 9\varepsilon^2,$$

as soon as $\|(\bar{\mathbf{U}}_{Kuzn} - \bar{\mathbf{U}}_{KZK})(0)\|_{L^2(Q_\varepsilon(0))} \leq \delta < \varepsilon$. The initial data are constructed on the initial data I_0 for the KZK equation. More precisely we take $I_0 \in H^s(\mathbb{T}_\tau \times \mathbb{R}^{n-1})$ for $s > \max\{10, [\frac{n}{2}] + 1\}$, which ensures in the case of the same initial data

$$\bar{\mathbf{U}}_{Kuzn}(0) = \bar{\mathbf{U}}_{KZK}(0) = \mathbf{U}_{Euler}(0)$$

the existence with necessary regularity of all solutions: of the KZK equation, of the Euler system and of the Kuznetsov equation. Otherwise, to ensure the boundness and the minimal regularity $C([0, \frac{T_0}{\varepsilon}]; L^2(Q_\varepsilon))$ of the remainder terms it sufficient to impose $s \geq 6$.

3. Approximation of the solutions of the Kuznetsov equation with the solutions of the NPE equation. Now let us go back to the NPE equation (11) and consider its *ansatz* (see [12] for the derivation of the NPE equation from the isentropic Navier-Stokes system or the Euler system). In contrast with Eq. (12) for the KZK equation, this time the velocity potential is given [43] by

$$u(x, t) = \Psi(\varepsilon t, x_1 - ct, \sqrt{\varepsilon}x') = \Psi(\tau, z, y). \tag{34}$$

Thus we directly obtain from the Kuznetsov equation (9) with the paraxial change of variable

$$\tau = \varepsilon t, \quad z = x_1 - ct, \quad y = \sqrt{\varepsilon}x', \tag{35}$$

that

$$\begin{aligned} & \partial_t^2 u - c^2 \Delta u - \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma - 1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \\ &= \varepsilon \left(-2c \partial_{\tau z}^2 \Psi - c^2 \Delta_y \Psi + \frac{\nu}{\rho_0} c \partial_z^3 \Psi + \frac{\gamma + 1}{2} c \partial_z (\partial_z \Psi)^2 \right) + \varepsilon^2 R_{Kuz-NPE} \end{aligned}$$

with

$$\begin{aligned} \varepsilon^2 R_{Kuz-NPE} &= \varepsilon^2 \left(\partial_\tau^2 \Psi - \frac{\nu}{\rho_0} \partial_z^2 \partial_\tau \Psi + \frac{\nu}{\rho_0} c \Delta_y \partial_z \Psi - (\gamma - 1) \partial_\tau \Psi \partial_z^2 \Psi \right. \\ &\quad \left. - 2(\gamma - 1) \partial_z \Psi \partial_{\tau z}^2 \Psi - 2 \partial_z \Psi \partial_{\tau z}^2 \Psi + 2c \nabla_y \Psi \nabla_y \partial_z \Psi \right) \\ &\quad + \varepsilon^3 \left(-\frac{\nu}{\rho_0} \Delta_y \partial_\tau \Psi + 2 \frac{\gamma - 1}{c} \partial_\tau \Psi \partial_{\tau z}^2 \Psi + \frac{\gamma - 1}{c} \partial_z \Psi \partial_\tau^2 \Psi \right. \\ &\quad \left. - 2 \nabla_y \Psi \nabla_y \partial_\tau \Psi \right) + \varepsilon^4 \left(-\frac{\gamma - 1}{c^2} \partial_\tau \Psi \partial_\tau^2 \Psi \right). \end{aligned} \tag{36}$$

We obtain the NPE equation satisfying by $\partial_z \Psi$ modulo a multiplicative constant:

$$\partial_{\tau z}^2 \Psi - \frac{\gamma + 1}{4} \partial_z (\partial_z \Psi)^2 - \frac{\nu}{2\rho_0} \partial_z^3 \Psi + \frac{c}{2} \Delta_y \Psi = 0.$$

In the sequel we work with ξ defined by

$$\xi(\tau, z, y) = -\frac{\rho_0}{c} \partial_z \Psi, \tag{37}$$

which solves the Cauchy problem for the NPE equation

$$\begin{cases} \partial_{\tau z}^2 \xi + \frac{(\gamma + 1)c}{4\rho_0} \partial_z^2 [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^3 \xi + \frac{c}{2} \Delta_y \xi = 0 \text{ on } \mathbb{R}_+ \times \mathbb{T}_z \times \mathbb{R}^{n-1}, \\ \xi(0, z, y) = \xi_0(z, y) \text{ on } \mathbb{T}_z \times \mathbb{R}^{n-1}, \end{cases} \tag{38}$$

in the class of L -periodic functions with respect to the variable z and with mean value zero along z . The introduction of the operator ∂_z^{-1} defined similarly to ∂_τ^{-1} in Eq. (18) allows us to consider instead of Eq. (11) the following equivalent equation

$$\partial_\tau \xi + \frac{(\gamma + 1)c}{4\rho_0} \partial_z [(\xi)^2] - \frac{\nu}{2\rho_0} \partial_z^2 \xi + \frac{c}{2} \partial_z^{-1} \Delta_y \xi = 0 \text{ on } \mathbb{R}_+ \times \mathbb{T}_z \times \mathbb{R}^{n-1}.$$

This time, in comparison with the KZK equation, we use the bijection between this two models (see [12]). We also update our notation for $\Omega_1 = \mathbb{T}_z \times \mathbb{R}_y^{n-1}$ and take $s > \frac{n}{2} + 1$. Suppose that

$$\xi_0 \in H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1}) \quad \text{and} \quad \int_{\mathbb{T}_z} \xi_0(z, y) dz = 0.$$

Consequently there exists a constant $r > 0$ such that if $\|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})} < r$, then, by [41, Thms. 1.1, 1.2], there exists a unique solution

$$\xi \in C([0, \infty[; H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1}))$$

of the NPE Cauchy problem (38) satisfying

$$\int_{\mathbb{T}_z} \xi(\tau, z, y) dz = 0 \quad \text{for any } \tau \geq 0, y \in \mathbb{R}^{n-1}.$$

We define $\partial_{x_1} \bar{u}(t, x_1, x') := -\frac{c}{\rho_0} \xi(\tau, z, y)$ with the change of variable (35) and

$$\bar{u}(t, x_1, x') = -\frac{c}{\rho_0} \partial_z^{-1} \xi(\tau, z, y) = \left(-\frac{c}{\rho_0}\right) \left(\int_0^z \xi(\tau, s, y) ds + \int_0^L \frac{s}{L} \xi(\tau, s, y) ds\right).$$

We take $u_1(x_1, x') := \partial_t \bar{u}(0, x_1, x')$ and $u_0(x_1, x') := -\frac{c}{\rho_0} \partial_z^{-1} \xi_0(z, y)$, which implies

$$u_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}) \text{ and } u_1 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}).$$

Thus for these initial data there exists

$$\bar{u} \in C([0, \infty[; H^{s+1}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})) \cap C^1([0, \infty[; H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})),$$

the unique solution on $\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$ of the approximated Kuznetsov system

$$\begin{cases} \bar{u}_{tt} - c^2 \Delta \bar{u} - \nu \varepsilon \Delta \bar{u}_t - \alpha \varepsilon \bar{u}_t \bar{u}_{tt} - \beta \varepsilon \nabla \bar{u} \nabla \bar{u}_t = \varepsilon^2 R_{Kuz-NPE}, \\ \bar{u}(0) = u_0 \in H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}), \quad \bar{u}_t(0) = u_1 \in H^{s+1}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}) \end{cases} \quad (39)$$

with $R_{Kuz-NPE}$ defined in Eq. (36). If we consider the Cauchy problem

$$\begin{cases} \partial_t^2 u - c^2 \Delta u = \varepsilon \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right), \\ u(0) = u_0, \quad u_t(0) = u_1, \end{cases} \quad (40)$$

for the Kuznetsov equation on $\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$ with u_0 and u_1 derived from ξ_0 , we have

$$\|u_0\|_{H^{s+2}(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})} + \|u_1\|_{H^s(\mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1})} \leq C \|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})}.$$

Hence, if $\|\xi_0\|_{H^{s+2}(\mathbb{T}_z \times \mathbb{R}_y^{n-1})}$ is small enough [11], we have a unique bounded in time solution

$$u \in C([0, \infty[; H^{s+1}(\Omega)) \cap C^1([0, \infty[; H^s(\Omega))$$

of the Cauchy problem for the Kuznetsov equation (40).

Theorem 3.1. *For $\nu \geq 0$ let u and \bar{u} be the defined above solutions of the exact Cauchy problem (40) and of the approximated Cauchy problem (39) for the Kuznetsov equation on $\Omega = \mathbb{T}_{x_1} \times \mathbb{R}_{x'}^{n-1}$ respectively. Then for $\nu > 0$ there exist $K, C, C_1, C_2 > 0$ such that for all $t < \frac{C}{\varepsilon}$ estimate (31) is valid and in addition it holds Point 3 of Theorem 2.2.*

Moreover, if for $n \leq 3$, and $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$ with $s \geq 4$, then the approximated solution satisfies

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^4(\Omega)), \quad \partial_t \bar{u}(t, x_1, x') \in C([0, +\infty[; H^2(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; L^2(\Omega)). \end{aligned}$$

If for $n \geq 4$ $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$ with $s \geq \frac{n}{2} + 2$, then the approximated solution satisfies

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^s(\Omega)), \partial_t \bar{u}(t, x_1, x') \in C([0, +\infty[; H^{s-2}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^{s-4}(\Omega)). \end{aligned}$$

Under these conditions for $n \geq 1$

$$R_{Kuz-NPE} \in C([0, +\infty[; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})).$$

For $\nu = 0$ all previous results stay true on a finite time interval $[0, T]$.

Proof. For $\nu > 0$ the global existence of u and of \bar{u} has already been shown. The proof of the approximation estimate follows exactly the proof given for Theorem 2.2 and thus is omitted. The case $\nu = 0$ implies the same approximation result except that u and \bar{u} are only locally well posed on an interval $[0, T]$.

We can see for $n = 2$ or 3 , using the previous arguments that the minimum regularity of the initial data (see Table 1) to have the remainder terms

$$R_{Kuz-NPE} \in C([0, +\infty[; L^2(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1}))$$

corresponds to $\xi_0 \in H^s(\mathbb{T}_{x_1} \times \mathbb{R}^{n-1})$ with $s \geq 4$, since then for $0 \leq k \leq 2$

$$\xi(\tau, z, y) \in C^k([0, +\infty[; H^{s-2k}(\mathbb{T}_z \times \mathbb{R}^{n-2})),$$

which finally implies with formula $\bar{u} = -\frac{c}{\rho_0} \partial_z^{-1} \xi$ that

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^4(\Omega)), \partial_t \bar{u}(t, x_1, x') \in C([0, +\infty[; H^2(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; L^2(\Omega)). \end{aligned}$$

In the same way for $n \geq 4$ we find the minimal regularity for $\xi_0 \in H^s(\Omega)$ with $s > \frac{n}{2} + 2$ as it implies

$$\begin{aligned} \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^s(\Omega)), \partial_t \bar{u}(t, x_1, x') \in C([0, +\infty[; H^{s-2}(\Omega)), \\ \partial_t^2 \bar{u}(t, x_1, x') &\in C([0, +\infty[; H^{s-4}(\Omega)). \end{aligned}$$

The optimality of the previously chosen s also comes from the fact that in Eq. (36) the least regular term in $R_{Kuz-NPE}$ is $\partial_\tau \Psi \partial_\tau^2 \Psi$ presenting for both viscous and inviscid cases. □

4. The Kuznetsov equation and the Westervelt equation.

4.1. Derivation of the Westervelt equation from the Kuznetsov equation.

Let u be a solution of the Kuznetsov equation (9). Similarly as in Ref. [1] we set

$$\bar{\Pi} = u + \frac{1}{2c^2} \varepsilon \partial_t [u^2] \tag{41}$$

and obtain

$$\partial_t^2 \bar{\Pi} - c^2 \Delta \bar{\Pi} = \varepsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta u + \frac{\gamma + 1}{2c^2} (\partial_t u)^2 + \frac{1}{c^2} u (\partial_t^2 - c^2 \Delta u) \right).$$

By definition (41) of $\bar{\Pi}$ we have

$$\partial_t^2 \bar{\Pi} - c^2 \Delta \bar{\Pi} = \varepsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta \bar{\Pi} + \frac{\gamma + 1}{2c^2} (\partial_t \bar{\Pi})^2 \right) + \varepsilon^2 R_{Kuz-Wes}, \tag{42}$$

where

$$\begin{aligned} \varepsilon^2 R_{Kuz-Wes} = & \varepsilon^2 \partial_t \left[-\frac{1}{2c^2} \frac{\nu}{\rho_0} \Delta(u \partial_t u) - \frac{\gamma+1}{2c^4} \partial_t u \partial_t^2(u^2) \right. \\ & \left. + \frac{1}{c^2} u \partial_t \left((\nabla u)^2 + \frac{\gamma-1}{2c^2} (\partial_t u)^2 + \frac{\nu}{\rho_0} \Delta u \right) \right] \\ & + \varepsilon^3 \partial_t \left[-\frac{\gamma+1}{8c^6} [\partial_t^2(u^2)]^2 \right]. \end{aligned} \tag{43}$$

We recognize the Westervelt equation (8) obtained up to remainder terms of order ε^2 .

4.2. Approximation of the solutions of the Kuznetsov equation by the solutions of the Westervelt equation. The well-posedness of the Westervelt equation follows directly from [11]. For a solution of the Cauchy problem (40) for the Kuznetsov equation u we define as in Subsection 4.1 $\bar{\Pi}$ by Eq. (41). Hence $\bar{\Pi}$ is the solution of the approximated Cauchy problem for the Westervelt equation (42) with the initial data

$$\bar{\Pi}(0) = \Pi_0, \quad \partial_t \bar{\Pi}(0) = \Pi_1, \tag{44}$$

defined by

$$\Pi_0 = u_0 + \frac{1}{c^2} \varepsilon u_0 u_1, \tag{45}$$

$$\begin{aligned} \Pi_1 &= u_1 + \frac{1}{c^2} \varepsilon u_1^2 + \frac{1}{c^2} \varepsilon u_0 \partial_t^2 u(0) \\ &= u_1 + \frac{1}{c^2} \varepsilon u_1^2 + \frac{1}{c^2} \varepsilon u_0 \frac{1}{1 - \frac{\gamma-1}{c^2} \varepsilon u_1} \left(c^2 \Delta u_0 + \frac{\nu}{\rho_0} \varepsilon \Delta u_1 + 2\varepsilon \nabla u_0 \nabla u_1 \right) \end{aligned} \tag{46}$$

with u_0 and u_1 initial data of the Cauchy problem (40) for the Kuznetsov equation.

For $s > \frac{n}{2}$ and $\nu > 0$, if we take $u_0 \in H^{s+3}(\mathbb{R}^n)$ and $u_1 \in H^{s+3}(\mathbb{R}^3)$, we have $\Pi_0 \in H^{s+3}(\mathbb{R}^n) \subset H^{s+2}(\mathbb{R}^n)$ and $\Pi_1 \in H^{s+1}(\mathbb{R}^n)$ with

$$\|\Pi_0\|_{H^{s+2}(\mathbb{R}^n)} + \|\Pi_1\|_{H^{s+1}(\mathbb{R}^n)} \leq C(\|u_0\|_{H^{s+3}(\mathbb{R}^n)} + \|u_1\|_{H^{s+3}(\mathbb{R}^n)}).$$

In the inviscid case when $\nu = 0$, for $s > \frac{n}{2}$ if we still take $u_0 \in H^{s+3}(\mathbb{R}^n)$, but $u_1 \in H^{s+2}(\mathbb{R}^3)$, we have $\Pi_0 \in H^{s+2}(\mathbb{R}^n)$ and $\Pi_1 \in H^{s+1}(\mathbb{R}^n)$ with the estimate

$$\|\Pi_0\|_{H^{s+2}(\mathbb{R}^n)} + \|\Pi_1\|_{H^{s+1}(\mathbb{R}^n)} \leq C(\|u_0\|_{H^{s+3}(\mathbb{R}^n)} + \|u_1\|_{H^{s+2}(\mathbb{R}^n)}).$$

Then, similarly to our previous work [11], we obtain the following result.

Theorem 4.1. *Let $n \geq 1$ and $s > \frac{n}{2}$.*

1. *If $\nu > 0$, $u_0 \in H^{s+3}(\mathbb{R}^n)$ and $u_1 \in H^{s+3}(\mathbb{R}^n)$, then there exists a constant $k_2 > 0$ such that for*

$$\|u_0\|_{H^{s+4}(\mathbb{R}^n)} + \|u_1\|_{H^{s+3}(\mathbb{R}^n)} < k_2, \tag{47}$$

the exact Cauchy problem for the Westervelt equation

$$\begin{cases} \partial_t^2 \Pi - c^2 \Delta \Pi = \varepsilon \partial_t \left(\frac{\nu}{\rho_0} \Delta \Pi + \frac{\gamma+1}{2c^2} (\partial_t \Pi)^2 \right), \\ \Pi(0) = \Pi_0, \partial_t \Pi(0) = \Pi_1 \end{cases} \tag{48}$$

with Π_0 and Π_1 defined by Eqs. (45) and (46), has a unique global in time solution

$$\Pi \in H^2(]0, +\infty[, H^s(\mathbb{R}^n)) \cap H^1(]0, +\infty[, H^{s+2}(\mathbb{R}^n)) \tag{49}$$

and if $s \geq 1$

$$\Pi \in C([0, +\infty[, H^{s+2}(\mathbb{R}^n)) \cap C^1([0, +\infty[, H^{s+1}(\mathbb{R}^n)) \cap C^2([0, +\infty[, H^{s-1}(\mathbb{R}^n)) \tag{50}$$

Moreover, $\bar{\Pi}$, obtained from the solution u of the Kuznetsov equation with Eq. (41), is the unique global in time solution of the approximated Cauchy problem (42), (44) with the same regularity as Π .

2. Let $\nu = 0$, $u_0 \in H^{s+3}(\mathbb{R}^n)$ and $u_1 \in H^{s+2}(\mathbb{R}^n)$. Then there exists a constant $k_2 > 0$ such that if

$$\|u_0\|_{H^{s+3}(\mathbb{R}^n)} + \|u_1\|_{H^{s+2}(\mathbb{R}^n)} < k_2, \tag{51}$$

then the Cauchy problem (48) for the Westervelt equation with Π_0 and Π_1 , defined by Eqs. (45) and (46), has a unique solution on all finite time interval $[0, T]$

$$\Pi \in C([0, T], H^{s+2}(\mathbb{R}^n)) \cap C^1([0, T], H^{s+1}(\mathbb{R}^n)) \cap C^2([0, T], H^s(\mathbb{R}^n)). \tag{52}$$

Moreover, $\bar{\Pi}$, defined by Eq. (41), is the unique local in time solution of the approximated Cauchy problem (42), (44) with the same regularity as Π .

For Π , solution of the Cauchy problem for the Westervelt equation (48), we set \bar{u} such that

$$\Pi = \bar{u} + \frac{\varepsilon}{c^2} \bar{u} \partial_t \bar{u} \tag{53}$$

and we obtain

$$\begin{aligned} \partial_t^2 \bar{u} - c^2 \Delta \bar{u} - \varepsilon \frac{\nu}{\rho_0} \Delta \partial_t \bar{u} - \varepsilon \frac{\gamma - 1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - 2\varepsilon \nabla \bar{u} \cdot \nabla \partial_t \bar{u} \\ + \varepsilon \left(\frac{1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - \partial_t \bar{u} \Delta \bar{u} + \frac{1}{c^2} \bar{u} \partial_t^3 \bar{u} - \bar{u} \Delta \partial_t \bar{u} \right) = \varepsilon^2 R_{1, Wes-Kuz} \end{aligned}$$

with

$$\begin{aligned} R_{1, Wes-Kuz} = & \left[\frac{\nu}{\rho_0 c^2} (2\partial_t \bar{u} \Delta \partial_t \bar{u} + 2(\nabla \partial_t \bar{u})^2 + \partial_t^2 \bar{u} \Delta \bar{u} + \bar{u} \Delta \partial_t^2 + 2\nabla \bar{u} \cdot \nabla \partial_t^2 \bar{u}) \right. \\ & \left. + \frac{\gamma + 1}{c^4} ((\partial_t \bar{u})^2 + \bar{u} \partial_t^2 \bar{u}) \partial_t^2 \bar{u} + \frac{\gamma + 1}{c^4} (3\partial_t \bar{u} \partial_t^2 \bar{u} + \bar{u} \partial_t^3 \bar{u}) \partial_t \bar{u} \right] \\ & + \varepsilon \frac{\gamma + 1}{c^6} ((\partial_t \bar{u})^2 + \bar{u} \partial_t^2 \bar{u}) (3\partial_t \bar{u} \partial_t^2 \bar{u} + \bar{u} \partial_t^3 \bar{u}). \end{aligned}$$

And as

$$\partial_t^2 \bar{u} - c^2 \Delta \bar{u} = O(\varepsilon),$$

by inserting this in the term $(\frac{1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - \partial_t \bar{u} \Delta \bar{u} + \frac{1}{c^2} \bar{u} \partial_t^3 \bar{u} - \varepsilon \bar{u} \Delta \partial_t \bar{u})$ we obtain

$$\partial_t^2 \bar{u} - c^2 \Delta \bar{u} - \varepsilon \frac{\nu}{\rho_0} \Delta \partial_t \bar{u} - \varepsilon \frac{\gamma - 1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - 2\varepsilon \nabla \bar{u} \cdot \nabla \partial_t \bar{u} = \varepsilon^2 R_{Wes-Kuz} \tag{54}$$

with

$$\varepsilon^2 R_{Wes-Kuz} = \varepsilon^2 R_{1, Wes-Kuz} - \varepsilon \left(\frac{1}{c^2} \partial_t \bar{u} \partial_t^2 \bar{u} - \partial_t \bar{u} \Delta \bar{u} + \frac{1}{c^2} \bar{u} \partial_t^3 \bar{u} - \bar{u} \Delta \partial_t \bar{u} \right).$$

Now we can write the following result for the approximation of the Kuznetsov equation by the Westervelt equation.

Theorem 4.2. *Let $n \geq 2$, $s > \frac{n}{2}$ with $s \geq 1$ and $\nu \geq 0$.*

Let $\bar{u}_0 \in H^{s+3}(\mathbb{R}^n)$ and $\bar{u}_1 \in H^{s+3}(\mathbb{R}^n)$ if $\nu > 0$ and let $\bar{u}_1 \in H^{s+2}(\mathbb{R}^n)$ if $\nu = 0$ be small enough in the sense of the existence of Π the solution of the Cauchy problem for the Westervelt equation (48) with Π_0 and Π_1 defined by Eqs. (45) and (46). Let \bar{u} be defined by (53).

Consequently \bar{u} is a solution of the approximated Kuznetsov equation (54) with $\bar{u}(0) = \bar{u}_0$, $\partial_t \bar{u}(0) = \bar{u}_1$. If the initial data for u , the solution of the Cauchy problem (40) for the Kuznetsov equation, and for \bar{u} satisfy (32), there exist $K, C, C_1, C_2 > 0$, all independent of ε , such that for all $t \leq \frac{C}{\varepsilon}$ it holds estimate (33).

Proof. The existence of u and \bar{u} has already been shown in [11] and given in Theorem 4.1. The proof of the approximation estimate follows exactly the proof of Theorem 2.2 and hence it is omitted. The regularity on \bar{u}_0 and \bar{u}_1 (see expressions of u_0 and u_1 in Table 1) is minimal to ensure that $R_{Wes-Kuz}$ (see Eq. (54)) is in $C([0, +\infty[; L^2(\mathbb{R}^n))$. Indeed, with Π_0 and Π_1 defined by Eqs. (45) and (46) it is necessary to impose these regularities in order to have the well-posedness of Π with the same regularity as in Theorem 4.1. □

Appendix A. Well posedness of the Kuznetsov equation in the half space.

We establish here the well posedness results for the Kuznetsov equation in the framework of the KZK-approximation considered in Subsection 2.2. For two type of approximations we need two different well posedness results.

A.1. Periodic boundary problem. Let us consider the following periodic in time problem for the Kuznetsov equation in the half space $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ with periodic in time Dirichlet boundary conditions given by (21), where g is an L -periodic in time and of mean value zero function. To show the well-posedness of problem (21) we study the maximal regularity of the associated linear operator and then use an equivalent to the fixed point theorem. Using [9, Lem. 3.5 p. 13], we directly obtain the following result of maximal regularity:

Theorem A.1. *Let $n = 3$ and $p \in]1, +\infty[$. Then there exists a unique solution $u \in W_p^2(\mathbb{T}_t; L^p(\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^2(\Omega))$ with the mean value zero*

$$\int_{\mathbb{T}_t} u(s, x) ds = 0 \quad \forall x \in \Omega \tag{55}$$

of the following periodic boundary value problem

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = f & \text{on } \mathbb{T}_t \times \Omega, \\ u = g & \text{on } \mathbb{T}_t \times \partial\Omega \end{cases} \tag{56}$$

if and only if the functions f and g satisfy

$$f \in L^p(\mathbb{T}_t; L^p(\Omega)) \text{ and } g \in W_p^{2-\frac{1}{2p}}(\mathbb{T}_t; L^p(\partial\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^{2-\frac{1}{p}}(\partial\Omega)) \tag{57}$$

and are of mean value zero:

$$\int_{\mathbb{T}_t} f(l, x) dl = 0 \quad \forall x \in \Omega \text{ and } \int_{\mathbb{T}_t} g(l, x') dl = 0 \quad \forall x' \in \partial\Omega. \tag{58}$$

Moreover, the following stability estimate holds

$$\|u\|_{W_p^2(\mathbb{T}_t; L^p(\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^2(\Omega))} \leq C \left(\|f\|_{L^p(\mathbb{T}_t; L^p(\Omega))} + \|g\|_{W_p^{2-\frac{1}{2p}}(\mathbb{T}_t; L^p(\partial\Omega)) \cap W_p^1(\mathbb{T}_t; W_p^{2-\frac{1}{p}}(\partial\Omega))} \right).$$

Proof. On one hand, if f and g satisfy (57)–(58), the necessity of the conditions is shown in Ref. [9]. On the other hand, the conditions (57)–(58) are sufficient by a direct application of the trace theorems recalled in Ref. [9] pp. 6–7 and proved in Ref. [13] Section 3 for example. \square

The results of Ref. [9] allow to see that Theorem A.1 does not depend on n , moreover if we look at the case $p = 2$ the linearity of the operator $\partial_t^2 - c^2\Delta - \nu\Delta\partial_t$ from Eq. (56) implies that we can work with $H^s(\Omega)$ instead of $L^2(\Omega)$:

Lemma A.2. *Let $n \in \mathbb{N}^*$ and $s \geq 0$. There exists a unique solution of the periodic in time boundary value problem for the linear strongly damped wave equation (56)*

$$u \in X = \left\{ u \in H^2(\mathbb{T}_t; H^s(\Omega)) \cap H^1(\mathbb{T}_t; H^{s+2}(\Omega)) \mid \int_{\mathbb{T}_t} u(s, x) ds = 0 \ \forall x \in \Omega \right\} \tag{59}$$

if and only if f and g satisfy

$$f \in L^2(\mathbb{T}_t; H^s(\Omega)) \text{ and } g \in \mathbb{F}_{\mathbb{T}} = H^{\frac{7}{4}}(\mathbb{T}_t; H^s(\partial\Omega)) \cap H^1(\mathbb{T}_t; H^{s+\frac{3}{2}}(\partial\Omega)) \tag{60}$$

along with (58).

Moreover the following stability estimate holds

$$\|u\|_X \leq C(\|f\|_{L^2(\mathbb{T}_t; H^s(\Omega))} + \|g\|_{\mathbb{F}_{\mathbb{T}}}).$$

Here $H^2(\mathbb{T}_t; H^s(\Omega)) \cap H^1(\mathbb{T}_t; H^{s+2}(\Omega))$ is endowed with its usual norm denoted here and in the sequel by $\|\cdot\|_X$.

To prove the global well-posedness of the periodic in time boundary value problem (21) for the Kuznetsov equation we use its boundary condition as the initial condition of the corresponding Cauchy problem in \mathbb{R}^n and we combine the maximal regularity result for system (56) with [48, 1.5 Cor., p. 368] (see also [11, Thm. 4.2]) applying the same method as previously done for the Cauchy problem associated with the Kuznetsov equation [11].

Theorem A.3. *Let $\nu > 0$, $n \in \mathbb{N}^*$ and $s > \frac{n}{2}$. Let X be defined by (59) and the boundary condition $g \in \mathbb{F}_{\mathbb{T}}$ be defined by (60) and in addition, let g be of mean value zero (see Eq. (58)).*

Then there exist $r^ = O(1)$ and $C_1 = O(1)$ such that for all $r \in [0, r^*]$, if $\|g\|_{\mathbb{F}_{\mathbb{T}}} \leq \frac{\sqrt{\nu\varepsilon}}{C_1}r$, there exists a unique solution $u \in X$ of the periodic problem (21) for the Kuznetsov equation such that $\|u\|_X \leq 2r$.*

Proof. For $g \in \mathbb{F}_{\mathbb{T}}$ defined in (60) and satisfying (58), let us denote by $u^* \in X$ the unique solution of the linear problem (56) with $f = 0$ and $g \in \mathbb{F}_{\mathbb{T}}$.

In addition, according to Theorem A.2, we take X defined in (59), this time for $s > \frac{n}{2}$ (we need this regularity to control the non-linear terms), and introduce the Banach spaces

$$X_0 := \{u \in X \mid u|_{\partial\Omega} = 0 \text{ on } \mathbb{T}_t \times \partial\Omega\} \tag{61}$$

and

$$Y = \left\{ f \in L^2(\mathbb{T}_t; H^s(\Omega)) \mid \int_{\mathbb{T}_t} f(s, x) ds = 0 \ \forall x \in \Omega \right\}.$$

Then by Lemma A.2, the linear operator

$$L : X_0 \rightarrow Y, \quad u \in X_0 \mapsto L(u) := u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t \in Y,$$

is a bi-continuous isomorphism.

Let us now notice that if v is the unique solution of the non-linear Dirichlet problem

$$\begin{cases} v_{tt} - c^2 \Delta v - \nu \varepsilon \Delta v_t = & \alpha \varepsilon (v + u^*)_t (v + u^*)_{tt} & \text{on } \mathbb{T}_t \times \Omega, \\ & + \beta \varepsilon \nabla (v + u^*) \cdot \nabla (v + u^*)_t \\ v = 0 \text{ on } \mathbb{T}_t \times \partial \Omega, \end{cases} \quad (62)$$

then $u = v + u^*$ is the unique solution of the periodic problem (21). Let us prove the existence of a such v , using [48, 1.5 Cor., p. 368].

We suppose that $\|u^*\|_X \leq r$ and define for $v \in X_0$

$$\Phi(v) := \alpha \varepsilon (v + u^*)_t (v + u^*)_{tt} + \beta \varepsilon \nabla (v + u^*) \cdot \nabla (v + u^*)_t.$$

For w and z in X_0 such that $\|w\|_X \leq r$ and $\|z\|_X \leq r$, we estimate the norm $\|\Phi(w) - \Phi(z)\|_Y$. By applying the triangular inequality we have

$$\begin{aligned} \|\Phi(w) - \Phi(z)\|_Y &\leq \alpha \varepsilon \left(\|u_t^* (w - z)_{tt}\|_Y + \|(w - z)_t u_{tt}^*\|_Y \right. \\ &\quad \left. + \|w_t (w - z)_{tt}\|_Y + \|(w - z)_t z_{tt}\|_Y \right) \\ &\quad + \beta \varepsilon \left(\|\nabla u^* \nabla (w - z)_t\|_Y + \|\nabla (w - z) \nabla u_t^*\|_Y \right. \\ &\quad \left. + \|\nabla w \nabla (w - z)_t\|_Y + \|\nabla (w - z) \nabla z_t\|_Y \right). \end{aligned}$$

Now, for all a and b in X with $s \geq s_0 > \frac{n}{2}$ it holds

$$\begin{aligned} \|a_t b_{tt}\|_Y &\leq \|a_t\|_{L^\infty(\mathbb{T}_t \times \Omega)} \|b_{tt}\|_Y \\ &\leq C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \|a_t\|_{H^1(\mathbb{T}_t; H^{s_0}(\Omega))} \|b\|_X \\ &\leq C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \|a\|_X \|b\|_X, \end{aligned}$$

where $C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}$ is the embedding constant of $H^1(\mathbb{T}_t; H^{s_0}(\Omega))$ in $L^\infty(\mathbb{T}_t \times \Omega)$, independent of s , but depending only on the dimension n . In the same way, for all a and b in X it holds

$$\|\nabla a \nabla b_t\|_Y \leq C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \|a\|_X \|b\|_X.$$

Taking a and b equal to u^* , w , z or $w - z$, as $\|u^*\|_X \leq r$, $\|w\|_X \leq r$ and $\|z\|_X \leq r$, we obtain

$$\|\Phi(w) - \Phi(z)\|_Y \leq 4(\alpha + \beta) C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)} \varepsilon r \|w - z\|_X.$$

By the fact that L is a bi-continuous isomorphism, there exists a minimal constant $C_\varepsilon = O\left(\frac{1}{\varepsilon \nu}\right) > 0$, coming from the inequality

$$C_0 \varepsilon \nu \|u\|_X^2 \leq \|f\|_Y \|u\|_X$$

for u , a solution of the linear problem (56) with homogeneous boundary data (for a maximal constant $C_0 = O(1) > 0$) such that

$$\|u\|_X \leq C_\varepsilon \|Lu\|_Y \quad \forall u \in X_0.$$

Hence, for all $f \in Y$

$$P_{LU_{X_0}}(f) \leq C_\varepsilon P_{U_Y}(f) = C_\varepsilon \|f\|_Y.$$

Then we find for w and z in X_0 , such that $\|w\|_X \leq r$, $\|z\|_X \leq r$, and also for $\|u^*\|_X \leq r$, that with the notation

$$\Theta(r) := 4C_\varepsilon(\alpha + \beta)C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}\varepsilon r$$

it holds

$$P_{LU_{X_0}}(\Phi(w) - \Phi(z)) \leq \Theta(r)\|w - z\|_X.$$

Thus we apply [48, 1.5 Cor., p. 368] with $f(x) = L(x) - \Phi(x)$ and $x_0 = 0$. Therefore, knowing that $C_\varepsilon = \frac{C_0}{\varepsilon\nu}$, we have, that for all $r \in [0, r_*[$ with

$$r_* = \frac{\nu}{4C_0(\alpha + \beta)C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}} = O(1), \tag{63}$$

for all $y \in \Phi(0) + w(r)LU_{X_0} \subset Y$ with

$$w(r) = r - 2\frac{C_0}{\nu}C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)r^2,$$

there exists a unique $v \in 0 + rU_{X_0}$ such that $L(v) - \Phi(v) = y$. Since we are seeking v , which solves the non-linear problem (62), we need to impose $y = 0$, *i.e.* that v be the solution of the non-linear problem (62), then we need to impose $y = 0$ and thus, to ensure that

$$0 \in \Phi(0) + w(r)LU_{X_0}.$$

Since $-\frac{1}{w(r)}\Phi(0)$ is an element of Y and $LX_0 = Y$, there exists a unique $z \in X_0$ such that

$$Lz = -\frac{1}{w(r)}\Phi(0). \tag{64}$$

Let us show that $\|z\|_X \leq 1$, what will implies that $0 \in \Phi(0) + w(r)LU_{X_0}$. Noticing that

$$\begin{aligned} \|\Phi(0)\|_Y &\leq \alpha\varepsilon\|v_t v_{tt}\|_Y + \beta\varepsilon\|\nabla v \nabla v_t\|_Y \\ &\leq (\alpha + \beta)\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}\|v\|_X^2 \\ &\leq (\alpha + \beta)\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}r^2 \end{aligned}$$

and using (64) we find

$$\begin{aligned} \|z\|_X &\leq C_\varepsilon \|Lz\|_Y = C_\varepsilon \frac{\|\Phi(0)\|_Y}{w(r)} \\ &\leq \frac{C_\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)\varepsilon r}{(1 - 2C_\varepsilon C_{H^1(\mathbb{T}_t; H^{s_0}(\Omega)) \rightarrow L^\infty(\mathbb{T}_t \times \Omega)}(\alpha + \beta)\varepsilon r)} < \frac{1}{2}, \end{aligned}$$

as soon as $r < r^*$.

Consequently, $z \in U_{X_0}$ and $\Phi(0) + w(r)Lz = 0$. Then we conclude that for all $r \in [0, r_*[$, if $\|u^*\|_X \leq r$, there exists a unique $v \in rU_{X_0}$ such that $L(v) - \Phi(v) = 0$, *i.e.* v is the solution of the non-linear problem (62). Thanks to the maximal regularity and a priori estimate following from Theorem A.2 with $f = 0$, there exists a constant $C_1 = O(\varepsilon^0) > 0$, such that

$$\|u^*\|_X \leq \frac{C_1}{\sqrt{\nu\varepsilon}}\|g\|_{\mathbb{F}_T}.$$

Thus, for all $r \in [0, r_*[$ and $\|g\|_{\mathbb{F}_T} \leq \frac{\sqrt{\nu\varepsilon}}{C_1}r$, the function $u = u^* + v \in X$ is the unique solution of the time periodic problem for the Kuznetsov equation and $\|u\|_X \leq 2r$. \square

A.2. Initial boundary value problem. We still work on $\Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ and we study the initial boundary value problem for the Kuznetsov equation on this space, *i.e.* the perturbation of an imposed initial condition by a source on the boundary, which in Subsection 2.2.2 was determined by the solution of the KZK equation.

Lemma A.4. *Let $s \geq 0, n \in \mathbb{N}$. There exists a unique solution*

$$u \in \mathbb{E} := H^2(\mathbb{R}_+; H^s(\Omega)) \cap H^1(\mathbb{R}_+; H^{s+2}(\Omega)) \tag{65}$$

of the linear problem

$$\begin{cases} u_{tt} - c^2 \Delta u - \nu \varepsilon \Delta u_t = f & \text{in } \mathbb{R}_+ \times \Omega, \\ u = g & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega \end{cases} \tag{66}$$

if and only if the data satisfy the following conditions

- $f \in L^2(\mathbb{R}_+; H^s(\Omega))$,
- for the boundary condition

$$g \in \mathbb{F}_{\mathbb{R}_+} = H^{7/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap H^1(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)), \tag{67}$$

- $u_0 \in H^{s+2}(\Omega)$ and $u_1 \in H^{s+1}(\Omega)$,
- $g(0) = u_0$ and $g_t(0) = u_1$ on $\partial\Omega$ in the trace sense.

In addition, the solution satisfies the stability estimate

$$\|u\|_{\mathbb{E}} \leq C(\|f\|_{L^2(\mathbb{R}_+; H^s(\Omega))} + \|g\|_{\mathbb{F}_{\mathbb{R}_+}} + \|u_0\|_{H^{s+2}} + \|u_1\|_{H^{s+1}}).$$

In order to prove this result we will use the following lemma to remove the inhomogeneity g .

Lemma A.5. *Let $s \geq 0, n \in \mathbb{N}$ and \mathbb{E} defined in (65). There exists a unique solution $w \in \mathbb{E}$ of the following linear problem*

$$\begin{cases} w_{tt} - \nu \varepsilon \Delta w_t = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ w = g & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ w(0) = 0, \quad w_t(0) = 0 & \text{in } \Omega \end{cases} \tag{68}$$

if and only if

$g \in \mathbb{F}_{\mathbb{R}_+}$ (the space $\mathbb{F}_{\mathbb{R}_+}$ is defined in (67)) and it holds the following compatibility conditions:

for all $x \in \partial\Omega, g(0) = 0$ and $g_t(0) = 0$.

Moreover, the solution w satisfies the stability estimate

$$\|w\|_{\mathbb{E}} \leq C\|g\|_{\mathbb{F}_{\mathbb{R}_+}}.$$

Proof. First we prove the sufficiency. By assumption (67), we have

$$\partial_t g \in H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)).$$

Thanks to § 3 in Ref. [32, p. 288], we obtain a unique solution

$$v \in H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega))$$

of the parabolic problem

$$v_t - \nu\varepsilon\Delta v = 0 \text{ in } \mathbb{R}_+ \times \Omega, \quad v = \partial_t g \text{ on } \mathbb{R}_+ \times \partial\Omega, \quad v(0) = 0 \text{ in } \Omega.$$

Next we define for $t \in \mathbb{R}_+$ and $x \in \Omega$ the function

$$w(t, x) := \int_0^t v(l, x) dl.$$

We have $w(0) = 0$ and $w_t(0) = 0$. Moreover, it satisfies

$$w_{tt} - \nu\varepsilon\Delta w_t = 0, \quad w(t)|_{\partial\Omega} = \int_0^t g_t(l) dl = g(t),$$

as $g(0) = 0$. Therefore, w is a solution of problem (68). The necessity follows from the spatial trace theorem ensuring that the trace operator $Tr_{\partial\Omega} : u \mapsto u|_{\partial\Omega}$, considering as a map

$$H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \rightarrow H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)), \tag{69}$$

is bounded and surjective by [13, Lem. 3.5]. For the compatibility condition, thanks to [14, Lem. 11], we also know that the temporal trace $Tr_{t=0} : g \mapsto g|_{t=0}$, considered as a map

$$H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)) \rightarrow H^{s+1/2}(\partial\Omega), \tag{70}$$

is well defined and bounded. Moreover, the spatial trace

$$H^{s+1/2}(\Omega) \rightarrow H^s(\partial\Omega) \tag{71}$$

is bounded by [16, Thm. 1.5.1.1].

To obtain uniqueness, let w be a solution to (68) with $g = 0$. Since w_t solves the heat problem with homogeneous data, we obtain $w_t = 0$ and therefore also $w = 0$ by the initial condition $w(0) = 0$. The stability estimate follows from the closed graph theorem. \square

Let us prove Lemma A.4:

Proof. We obtain the uniqueness of the solution of the boundary value problem for the linear strongly damped equation (66) from the fact that in the case $g = 0$ we can consider $-\Delta$ as a self-adjoint and non negative operator with homogeneous Dirichlet boundary conditions and we can use [15].

To verify the necessity of the conditions on the data, we suppose that $u \in \mathbb{E}$ (see Eq. (65) for the definition of \mathbb{E}) is a solution of (66). Then

$$u, u_t \in H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \text{ and thus } f \in L^2(\mathbb{R}_+; H^s(\Omega)).$$

Taking as in the previous proof the spatial trace $Tr_{\partial\Omega}$ as in Eq. (69) we have

$$g, g_t \in H^{3/4}(\mathbb{R}_+; H^s(\partial\Omega)) \cap L^2(\mathbb{R}_+; H^{s+3/2}(\partial\Omega)), \text{ which implies } g \in \mathbb{F}_{\mathbb{R}_+}.$$

By the Sobolev embedding $H^1(\mathbb{R}_+; H^{s+2}(\Omega)) \hookrightarrow C(\mathbb{R}_+; H^{s+2}(\Omega))$, it follows that $u_0 \in H^{s+2}(\Omega)$ and we also have the temporal trace

$$u \mapsto u|_{t=0} : H^1(\mathbb{R}_+; H^s(\Omega)) \cap L^2(\mathbb{R}_+; H^{s+2}(\Omega)) \rightarrow H^{s+1}(\Omega)$$

by [13, Lem. 3.7]. Following the proof of Lemma A.5, we use Eqs. (70) and (71) to obtain the compatibility conditions.

It remains to prove the sufficiency of the conditions. We extend u_0, u_1 and f in odd functions among x_1 on \mathbb{R}^n so that we have

$$\tilde{u}_0 \in H^{s+2}(\mathbb{R}^n), \tilde{u}_1 \in H^{s+1}(\mathbb{R}^n) \text{ and } \tilde{f} \in L^2(\mathbb{R}_+; H^s(\mathbb{R}^n)).$$

Considering the non homogeneous linear Cauchy problem

$$\begin{cases} \tilde{u}_{tt} - c^2 \Delta \tilde{u} - \nu \varepsilon \Delta \tilde{u}_t = \tilde{f} & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ \tilde{u}(0) = \tilde{u}_0, \tilde{u}_t(0) = \tilde{u}_1 & \text{in } \mathbb{R}^n, \end{cases}$$

by [11, Thm. 4.1] we obtain the existence of its unique solution

$$\tilde{u} \in H^2(\mathbb{R}_+; H^s(\mathbb{R}^n)) \cap H^1(\mathbb{R}_+; H^{s+2}(\mathbb{R}^n)).$$

Let $\bar{u} \in \mathbb{E}$, denote the restriction of \tilde{u} to Ω and let $\bar{g} := g - \bar{u}|_{\partial\Omega}$. By the spatial trace theorem $\bar{u}|_{\partial\Omega} \in \mathbb{F}_{\mathbb{R}_+}$, and hence $\bar{g} \in \mathbb{F}_{\mathbb{R}_+}$. Then the solution u of the non homogeneous linear problem (66) is given by $u = v + \bar{u}$, where v solves problem (66) with $f = u_0 = u_1 = 0$ and $g = \bar{g}$. From Lemma A.5 we have a unique solution $\bar{v} \in \mathbb{E}_u$ of problem (68) with $g = \bar{g}$. Then the function $w := v - \bar{v}$ solves the following system

$$\begin{cases} w_{tt} - \Delta w - \nu \varepsilon \Delta w_t = c^2 \Delta \bar{v} & \text{in } \mathbb{R}_+ \times \Omega, \\ w = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ w(0) = 0, \quad w_t(0) = 0 & \text{in } \Omega, \end{cases}$$

which thanks to [15, Thm. 2.6] has a unique solution $w \in \mathbb{E}$ defined in (65). The function $u := w + \bar{v} + \bar{u}$ is the desired solution of system (66) and the stability estimate follows from the closed graph theorem. This concludes the proof of Lemma A.4. \square

The next theorem follows from the maximal regularity result of Lemma A.4 and of [48, 1.5. Cor., p. 368]. Its proof is similar to the proof of Theorem A.3 and hence is omitted.

Theorem A.6. *Let $\nu > 0, n \in \mathbb{N}^*, \Omega = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ and $s > \frac{n}{2}$. Considering the initial boundary value problem for the Kuznetsov equation in the half space with the Dirichlet boundary condition (28) the following results hold: there exist constants $r^* = O(1)$ and $C_1 = O(1)$, such that for all initial data satisfying*

- $g \in \mathbb{F}_{\mathbb{R}_+} := H^{7/4}([0, \infty[; H^s(\partial\Omega)) \cap H^1([0, \infty[; H^{s+3/2}(\partial\Omega))$,
- $u_0 \in H^{s+2}(\Omega), u_1 \in H^{s+1}(\Omega)$,
- $g(0) = u_0|_{\partial\Omega}$ and $g_t(0) = u_1|_{\partial\Omega}$,

and such that for $r \in [0, r^*[$

$$\|u_0\|_{H^{s+2}(\Omega)} + \|u_1\|_{H^{s+1}(\Omega)} + \|g\|_{\mathbb{F}_{[0,r]}} \leq \frac{\nu \varepsilon}{C_1} r,$$

there exists a unique solution of problem (28) for the Kuznetsov equation

$$u \in H^2([0, \infty[; H^s(\Omega)) \cap H^1([0, \infty[; H^{s+2}(\Omega)),$$

such that

$$\|u\|_{H^2([0, \infty[; H^s(\Omega)) \cap H^1([0, \infty[; H^{s+2}(\Omega))} \leq 2r.$$

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