Adaptive Fuzzy Interpolation and Extrapolation with Multiple-antecedent Rules
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Adaptive Fuzzy Interpolation and Extrapolation
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Abstract—Adaptive fuzzy interpolation strengthens the potential of fuzzy interpolative reasoning owning to its efficient identification and correction of defective interpolated rules during the interpolation process [11]. This approach assumes that: i) two closest adjacent rules which flank the observation or a previously inferred result are always available; ii) only single-antecedent rules are involved. In practice, however, variable values of these rules may lie just on one side of the observation or inferred result. Also, there may be certain rules with multiple antecedents in the rule base. This paper extends the adaptive approach, in order to cover fuzzy extrapolation and to support rule base with multiple-antecedent rules. Adaptive fuzzy interpolation and extrapolation complement each other, which jointly improve the applicability of fuzzy interpolative reasoning, as it significantly reduces the restriction over the given rule base.

I. INTRODUCTION

Fuzzy rule interpolation enhances the robustness of fuzzy reasoning. When given observations have no overlap with any antecedent values, no rule can be fired in classical inference. However, interpolative reasoning through a sparse rule base may still obtain certain conclusions and thus improve the applicability of fuzzy models. Also, with the help of fuzzy interpolation, the complexity of a rule base can be reduced by omitting those fuzzy rules which may be approximated from their neighboring ones. A number of important interpolating approaches have been presented in the literature, including [1], [2], [3], [4], [7], [8], [9], [10]. In particular, the scale and move transformation-based approach can handle both interpolation and extrapolation which involve multiple fuzzy rules, with each rule consisting of multiple antecedents. This approach also guarantees the uniqueness as well as normality and convexity of the resulting interpolated fuzzy sets. Yet, it is possible that more than one object value of a single variable may be derived or observed in fuzzy interpolation. This implies that certain inconsistencies may result.

To address this problem, recently, adaptive interpolative reasoning has been proposed [11]. This approach is capable of efficiently detecting inconsistencies, locating possible fault candidates and modifying the candidates in an effort to remove all the inconsistencies. It works by artificially viewing the interpolative inference procedures as system components, and then utilizing an assumption-based truth maintenance system (ATMS) [5] to record the dependencies between an interpolated value (including any contradiction) and its proceeding interpolation components. From this, the classical algorithm of general diagnostic engine (GDE) [6] is employed to manipulate the sets of dependent components of contradictions to hypothesize all possible candidates of defective rules.

However, the adaptive approach of [11] is limited in its implementation in that fuzzy models are assumed to involve only single-antecedent rules and to reason only based on neighboring rules which strictly flank the given observation or a previously inferred result. Nevertheless, fundamentally, this is not restricted by the underlying approach. This work extends that of [11], in order to allow for the use of rules with multiple antecedents and to reason based on two rules both of which lie on one side of the observation or the inferred result. This will considerably widen the scope of the existing approach for adaptive fuzzy interpolation. This is because in many practical applications of fuzzy systems, multiple-antecedent rules are common and distributions of rules in a rule base can be very irregular.

The rest of this paper is structured as follows. Sec. II reviews the background of adaptive fuzzy interpolation. Sec. III describes the generalization of the existing approach to allow multiple-antecedent rules and cover fuzzy extrapolative reasoning. Sec. IV gives an example to illustrate the utility of this work. Sec. V concludes the paper and points out important future research.

II. OVERVIEW OF ADAPTIVE INTERPOLATION

Adaptive interpolative reasoning [11] provides a way to ensure inference results being consistent during the fuzzy interpolative process. In implementing fuzzy interpolation, each pair of neighboring rules is defined as a fuzzy reasoning component which takes a fuzzy set (an observation or a previously inferred result, which is hereafter referred to as an observation for simplicity) as input and produces another (the consequent of the interpolated rule) as output. The process of adaptive interpolation can be summarized in Fig. 1. Firstly, the interpolator carries out interpolation and passes the interpolated results to the ATMS for dependency-recording. Then, the ATMS relays any \( \beta_0 \)-contradictions (i.e. inconsistency between two different values for a common variable at least to the degree of a given threshold \( \beta_0 \) \((0 \leq \beta_0 \leq 1))\) as well as their dependent fuzzy reasoning components to the GDE which diagnoses the problem and generates all possible component candidates. After that, a modification process takes place to correct a certain candidate to restore consistency. A brief description of each of these key methods is given below.
ATMS is utilized to record the dependency of the interpolated results, including any contradictions, upon those fuzzy reasoning components from which they are inferred. Any ATMS node with an inferred proposition is represented by an ATMS justification:

\[ O, R_i R_j \Rightarrow C, \]  

(1)

where \( R_i R_j \) stands for the fuzzy reasoning component containing the two neighboring rules \( R_i \) and \( R_j \) (\( i \neq j \)) that have been used to infer the outcome \( C \) from the observation \( O \). Accordingly, a \( \beta_0 \)-contradiction is represented as:

\[ P, P' \Rightarrow \beta_0 \perp. \]  

(2)

In ATMS terms, a label is a set of environments, each supporting the node that it is associated with. An environment contains a minimal set of fuzzy reasoning components that jointly entail the concerned node, thereby describing how the node ultimately depends on those fuzzy reasoning components. An environment is said to be \( \beta_0 \)-inconsistent if \( \beta_0 \)-contradiction is derivable propositionally by the environment and a given justification. An environment is said to be \( (1 - \beta_0) \)-consistent if it is not \( \beta_0 \)-inconsistent.

The label of each node is guaranteed to be \( (1 - \beta_0) \)-consistent, sound, minimal and complete by the algorithm that ATMS updates node labels, except that the label of the special “false” node is guaranteed to be \( \beta_0 \)-inconsistent rather than \( (1 - \beta_0) \)-consistent. In particular, the label of the special “false” node gathers all \( \beta_0 \)-inconsistent environments. Its corresponding label-updating process is given as follows. Whenever a \( \beta_0 \)-contradiction is detected, each environment in its label is added into the label of “false” node and all such environments and their supersets are removed from the label of every other node. Also, any such environment which is a superset of another is removed from the label of the node “false”.

B. Candidate generation

A candidate in GDE [6] is a set of assumptions which may be responsible for the whole set of current contradictions. GDE generates minimal candidates by manipulating the label of the specific “false” node. Because a \( \beta_0 \)-inconsistent environment indicates that at least one of its assumptions is faulty, a candidate must have a nonempty intersection with each \( \beta_0 \)-inconsistent environment. Thus, each candidate is constructed by taking one assumption from each environment in the label of “false” node. Supersets removal then ensures such generated candidates to be minimal. In light of this, a successful correction of any single candidate will remove all the contradictions (see later).

C. Candidate modification

Consistency can be restored by successfully correcting any single candidate because each such candidate explains the entire set of current contradictions. Suppose that MODIFY(\( f \)) is the modification procedure for a given fuzzy reasoning component (\( f \)), which returns true when the modification succeeds and false otherwise. Let \( Q \) be a priority queue whose elements are ordered such that those of the smallest cardinality have the highest priority. Given a set of candidates \( S \), each of which (\( C \)) is a set of fuzzy reasoning components, the modification procedure is shown in Fig. 2.

**Fig. 2. The CONSISTENCY RESTORING procedure**

For convenience, in the rest of this paper, let \( A_{ij}^+ \) denote the modified consequence of a culprit interpolated rule whose consequent value is \( A_{ij} \), and \( A_{ij}' \) and \( \lambda_{ij}^* \) denote the corresponding modified intermediate rule consequence and the relative placement factor of \( A_{ij}' \), respectively. Suppose that the neighboring rules \( (x_1 = A_{11}) \Rightarrow (x_2 = A_{21}) \) and \( (x_1 = A_{1n}) \Rightarrow (x_2 = A_{2n}) \) are the two rules used by a defective fuzzy reasoning component, that \( A_{12}, A_{13}, \ldots, A_{1(n-1)} \) are observations located between \( A_{11} \) and \( A_{1n} \), and that \( A_{1j} \) is the middle most one. In carrying out interpolation, the presumed linear relation between an antecedent variable and the corresponding consequent variable can be represented by a line segment which starts from \( (x_1, x_2) \) and ends by \( (A_{1n}, A_{2n}) \) in the \( x_1, x_2 \) plane. The modification breaks this straight line segment into two connected straight line segments: one from \( (A_{11}, A_{12}) \) to \( (A_{1j}, A_{2j}) \) and the other from \( (A_{1j}, A_{2j}) \) to \( (A_{1n}, A_{2n}) \). That is, it uses a first-order piecewise linear approximation to replace the original linear method. The modification procedure for a single fuzzy reasoning component is summarized as follows.

1. Find the rule \( (A_{ij} \Rightarrow A_{2j}) \) whose antecedent locates in the middle most of the neighborhood of the antecedents of any two rules that may be used for interpolation, with respect to their representative values. Assume that the relative placement factor of its consequence \( \lambda_{2j} \) is modified to \( \lambda_{2j}' \).

2. Calculate the correction rate pair according to the relative placement factor modification of rule \( A_{1j} \Rightarrow A_{2j} \):

\[
\begin{align*}
\epsilon^- &= \frac{\lambda_{2j}'}{\lambda_{2j}} \\
\epsilon^+ &= \frac{1 - \lambda_{2j}'}{1 - \lambda_{2j}}
\end{align*}
\]  

(3)
3. Calculate the modified relative placement factors of consequences of all other interpolated rules which are generated from the same defective fuzzy reasoning component as per the correction rate pair computed above, where $i \in \{2, 3, ..., j - 1\}$ and $k \in \{j + 1, j + 2, ..., n - 1\}$:

$$\lambda_{2i}^* = \lambda_{2i} \cdot c^-$$
$$1 - \lambda_{2k}^* = (1 - \lambda_{2k}) \cdot c^+.$$  

(4)

4. Calculate the modified consequences of all interpolated rules which are generated from the same defective fuzzy reasoning component in accordance with their modified relative placement factors:

$$A_{2x'}^* = (1 - \lambda_{2x})A_{21} + \lambda_{2x}A_{2n},$$
$$T(A_{1x'}, A_{1x}) = T(A_{2x'}, A_{2x}),$$  

(5)

where $x \in \{2, 3, ..., n - 1\}$, and $T(A', A)$ represents scale and move transformations from fuzzy set $A'$ to $A$.

5. Restrict the modified consequence to be consistent with the context. Suppose that $m$ object values $A_{11}, A_{12}, ..., A_{im}$ are obtained for variable $x_i$. If they are $(1 - \beta_0)$-consistent, they must satisfy:

$$\bigcap_{j=1}^{m} (A_{ij})_{\beta_0} \neq \emptyset,$$  

(6)

where $(A_{ij})_{\beta_0}$ denotes the $\beta_0$-cut of fuzzy set $A_{ij}$.

6. Restrict the propagations of all modified consequences to be consistent with the context. For simplicity, let function $I(A_{ij}, R_i R_r) = A_{kj}$ denote the standard interpolation from the antecedent fuzzy set $A_{ij}$ to the consequent value $A_{kj}$, based on fuzzy reasoning component $R_i R_r$. Suppose that $m$ object values $A_{11}, A_{12}, ..., A_{im}$ of variable $x_i$ are modified which are located between the antecedent values of rules $R_i$ and $R_r$, that the corresponding modified object values of variable $x_k$ are $A_{kj}', j \in \{1, 2, ..., m\}$, and that $n$ object values $A_{ki}, l \in \{1, 2, ..., n\}$ of variable $x_k$ are already obtained one way or another. If the modified consequences $A_{kj}'$ are all $(1 - \beta_0)$-consistent, then they must satisfy:

$$A_{kl}' = I(A_{kj}', R_i R_r)$$
$$\left(\bigcap_{j=1}^{m} (A_{kj}')_{\beta_0}\right) \cap \left(\bigcap_{l=1}^{n} (A_{ki})_{\beta_0}\right) \neq \emptyset.$$  

(7)

7. Solve all these simultaneous equations as generated above. The result is the modified solution which ensures inconsistency-free.

III. EXTENSIONS

The approach described above assumes that each rule in the rule base involves only one antecedent variable. Also, the two closest adjacent rules must flank the observation. These limitations inevitably restrict the potential application of the existing techniques. However, the present approach is readily extendable to deal with these situations. Thus, the work [11] is extended herein to allow both interpolation and extrapolation with rules that involve multiple-antecedent attributes.

A. Interpolation with multiple-antecedent rules

If only one antecedent is involved in each rule in the rule base, given an observation, it is straightforward to find the flank rules to fire in the rule base. However, when multiple conditional variables are involved, the situation is rather different. It is too restrict to find such a pair of rules that every pair of their counterpart antecedents flanks the corresponding term of the observation, also in the same order.

In order to remove such limitations, two closest rules rather than strictly two flanked rules are employed for multiple-antecedent rule interpolation. Once the two closest rules are chosen, the intermediate rule can then be constructed. From this, the resultant fuzzy set can be transformed from the consequent of the intermediate rule. The procedures of how to achieve these are briefly outlined as follows:

1. Choose the closest two rules: Without losing generality, suppose that a rule and an observation are represented by:

Rule $R_t$: If $x_1$ is $A_{111}, ..., x_m$ is $A_{11m}$, then $x_n$ is $A_{11n}$  

(8)

Observation: $x_1$ is $A_{1kx}$ and ... and $x_m$ is $A_{1nx}$.  

(9)

According to the work in [7], the distance $d(A_{ki}, A_{kx}) (k \in \{1, 2, ..., m\})$ between two fuzzy sets $A_{ki}$ and $A_{kx}$ can be calculated by:

$$d_k = d(A_{ki}, A_{kx}) = d(Rep(A_{ki}), Rep(A_{kx})),$$  

(10)

where $Rep(A_{ki})$ and $Rep(A_{kx})$ are the representative values of fuzzy sets $A_{ki}$ and $A_{kx}$, respectively. As attributes have different domains, the absolute distances may not be compatible with each other. Therefore, a normalized distance measure (range of 0 to 1) is defined by:

$$d'_k = \frac{d(A_{ki}, A_{kx})}{\max_k - \min_k} = \frac{d(Rep(A_{ki}), Rep(A_{kx}))}{\max_k - \min_k},$$  

(11)

where $\max_k$ and $\min_k$ are the maximal and minimal values in the domain of attribute $k$, respectively. The distance $d$ between the antecedents of a rule and an observation can be calculated in accordance with the weights of the antecedent attributes. If all attributes are of the same importance, the distance $d$ is defined as the average of its all normalized attributes’ distances:

$$d = \sqrt{d'_{1}^2 + d'_{2}^2 + \cdots + d'_{m}^2}.$$  

(12)

With the above definition, the distances between a given observation and the antecedent values of all those rules which involve the same antecedent attributes in the rule base can be calculated. The two rules which have minimal distances are chosen as the closest two rules from the observation. Note that each pair of antecedent values of the two closest rules does not necessarily flank its corresponding term in the observation. In the extreme case, all the conditional attribute values of the two closest rules may locate in one side of the given observation, resulting in extrapolation rather than interpolation (see Sec. III-B).

2. Construct the intermediate rule: Having chosen the two closest rules, the next step is to construct the intermediate
rule. Suppose that rules $R_i$ and $R_j$ are the two closest rules for a given observation:

If $x_1$ is $A_{1i}$ and ... and $x_m$ is $A_{mi}$, then $x_n$ is $A_{ni}$;  
If $x_1$ is $A_{1j}$ and ... and $x_m$ is $A_{mj}$, then $x_n$ is $A_{nj}$.

When an observation $(A_{1z}, A_{2z}, ..., A_{nz})$ is given, by analogy to the single antecedent case, the object values $A_{ki}$ and $A_{kj}$ ($k \in \{1, 2, ..., m\}$) of antecedent variable $x_k$ of those two rules are used to obtain the new intermediate fuzzy set $A'_{xz}$:

$$A'_{xz} = (1 - \lambda_{xz})A_{iz} + \lambda_{xz}A_{jz},$$

where $\lambda_{xz}$ is the relative placement factor associated with the value $A_{xz}$ of the $k$th antecedent variable, that is:

$$\lambda_{xz} = \frac{d(A_{zi}, A_{xz})}{d(A_{zi}, A_{zj})}.$$  

It can be shown that the representative value of $A'_{xz}$ remains the same as that of $A_{xz}$. From this, the relative placement factor $\lambda_{xz}$ of the consequent is computed by the average of $\lambda_{xz}$:

$$\lambda_{xz} = \frac{1}{m} \sum_{k=1}^{m} \lambda_{xz}.$$  

Then the consequent of the intermediate rule is calculated by:

$$A'_{nxz} = (1 - \lambda_{nxz})A_{ni} + \lambda_{nxz}A_{nj}.$$  

3. Scale and move transformations: The main issue that remains is how to calculate the transformation rates after the intermediate rule has been constructed. The scale rate $s_{xz}$ and move rate $m_{xz}$ of each term $A_{xz}$ of the observation and its corresponding fuzzy set $A'_{xz}$ in the intermediate rule can be calculated in a way which is exactly the same as that of single-antecedent rule interpolation. From this, the combined scale rate $s_{nxz}$ and move rate $m_{nxz}$ over the $m$ conditional attributes are calculated as the arithmetic averages of $s_{xz}$ and $m_{xz}$, $k \in \{1, 2, ..., m\}$:

$$s_{nxz} = \frac{1}{m} \sum_{k=1}^{m} s_{xz},$$  

$$m_{nxz} = \frac{1}{m} \sum_{k=1}^{m} m_{xz}.$$  

Note that, other than using arithmetic average, different methods such as the medium value operator or weighted average operator may be employed for this purpose. Once the scale rate and move rate of the consequent are worked out, the rest of the interpolation process remains the same as that of single-antecedent rule interpolation, which is omitted here due to space limit.

These transformations can be concisely represented by an integrated transformation function $T$ such that the transformation from $(A_{1z}', ..., A_{nz}')$ to $(A_{1z}, ..., A_{nz})$ is denoted by $T((A_{1z}', ..., A_{nz}'), (A_{1z}, ..., A_{nz}))$. Note that the combined scale rate $s_{nxz}$ and move rate $m_{nxz}$ reflect the similarity degree between the observation and the antecedent values of the intermediate interpolated rule. The fuzzy set $A_{nxz}$ of the conclusion can then be estimated by transforming the consequent $A_{nxz}$ of the intermediate interpolated rule via the application of the same $s_{nxz}$ and $m_{nxz}$. Thus, the resultant fuzzy set $A_{nxz}$ can be transformed from its intermediate rule consequent by the same transformation function:

$$T(A_{nxz}', A_{nxz}) = T((A_{1z}', ..., A_{nz}'), (A_{1z}, ..., A_{nz})).$$  

B. Fuzzy extrapolation

The extension of the above to perform extrapolation is readily attainable. Computationally, it can be treated as a special case of fuzzy interpolation. Indeed, when all the object values of the conditional variables of the two closest rules lie on just one side of the given observation, the interpolation problem becomes extrapolation. However, other than such a strict extrapolation case, the problem becomes somewhat more complex when certain antecedent values lie between the two closest rules while the others lie on one side or another. Nevertheless, both choosing the closest rules and constructing the intermediate rules for these situations are carried out in exactly the same way as it for interpolation as described in the above subsection.

C. Truth maintenance and candidate generation

In order for the adaptive approach to handle interpolation based on rules with multiple antecedents, the concept of fuzzy reasoning component therefore is generalized as shown in Fig. 3. Here, Rules $i$ and $j$ are the two closest ones to the observation $(A_{1x}, A_{2x}, ..., A_{nx})$ according to the distance measure given in Eq. 12, and $A_{nx}$ is the inferred result based on these two rules from the observation. The truth maintenance and minimal candidate generation procedures of adaptive fuzzy interpolation/extrapolation with multi-antecedent rules are basically the same as the one used for fuzzy interpolation with single-antecedent rules. The difference only exists in the representation of fuzzy reasoning component. Thus they are omitted here (refer to Sec. II or [11] for details).

D. Candidate modification

The consistency-restoring algorithm outlined in Fig. 2, which is used for single-antecedent rule interpolation can also be used in principle, for multiple-antecedent interpolation with the generalized fuzzy reasoning component. However, it is not straightforward when it comes to the correction procedure for individual defective fuzzy reasoning component in a multiple-antecedent rule environment. There are more sophisticated situations which complicate the
choosing procedure of the first rule to modify and thereby
the correction rate pair. In particular, three cases need to
be considered: i) strict interpolation, that is all observations
lie on between the two rules; ii) strict extrapolation, that is all
observations lie one side or another but not in between;
iii) mixed interpolation and extrapolation, that is observations
may lie anywhere, but not as cases i and ii.

Fig. 4. Defective fuzzy reasoning component modification for interpolation

The problem space of \( n \)-antecedent \((n \geq 1)\) rule interpolation
is \((n+1)\)-dimensional. Without losing generality, for
simplicity, two-antecedent rules are taken here as an example. Suppose that \((A_{11}, A_{22}), (A_{13}, A_{23}), \ldots, (A_{1(n-1)}, A_{2(n-1)})\)
are observations, and that the neighboring rules \(A_{11}, A_{21} \Rightarrow A_{3j}\) and
\(A_{1n}, A_{2n} \Rightarrow A_{3n}\) are the two closest rules to these
observations. It is interesting to observe that in computing
interpolation involving two antecedent variables, the
presumed linear relation between the antecedent variables
and the corresponding consequent variable can be represented
by a line in a 3-dimensional space (line \( P_0P_1 \) in Fig. 4) if fuzzy
sets are represented using their representative values. Line \( P_0P_3\), the projection of line \( P_0P_1 \) onto plane \(x_1x_2\),
provides a partial order of all possible antecedent value
pairs of variables \(x_1\) and \(x_2\) by mapping them onto line
\( P_0P_3 \). In particular, as shown in Fig. 4, it has mapped
observations \((A_{1i}, A_{2j}), (A_{1j}, A_{2j})\) and \((A_{1k}, A_{2k})\) to points
\(A_{c1}, A_{cj}\) and \(A_{ck}\), respectively, on the line \( P_0P_3 \). This is
done by the combined relative placement factor \( \lambda_{3x} \) \((x \in \{2, 3, \ldots, n-1\})\) calculated from \( \lambda_{1x} \) and \( \lambda_{2x} \) (Eq. 15).

Note that it is not necessary that \( A_{1i} \leq A_{1j} \leq A_{1k} \) and
\( A_{2i} \leq A_{2j} \leq A_{2k} \) though \( A_{1j} \leq A_{cj} \leq A_{ck}\).

Suppose that \( A_{cj} \) \((2 \leq j \leq n-1)\) sits in the middle
most within all the observations on the line \( P_0P_3 \). Then,
interpolated rule \( A_{1j}, A_{2j} \Rightarrow A_{3j} \) will be modified first.

The modification breaks the straight interpolation line \( P_0P_1 \)
to two connected straight line segments \( P_0P_3 \) and \( P_3P_1 \) as
illustrated in Fig. 4. The effect of this proposed modification
method is to refine the defective fuzzy reasoning component
by dividing it into two more accurate fuzzy reasoning
components. This corresponds to refining the fuzzy reasoning
component represented by \( P_0P_1 \) into two represented by
\( P_0P_3 \) and \( P_3P_1 \). All interpolated rules based on the original
defective fuzzy reasoning component need to be modified by
the two replacement fuzzy reasoning components.

In order to facilitate the modification from the result of
the original defective fuzzy reasoning component to the result
of either of the two new replacements, a pair of correction rates
are defined as follows:

\[
\begin{align*}
  c^- &= \frac{\lambda_{2j}}{\lambda_{1j}} \\
  c^+ &= -\frac{\lambda_{1j}}{1-\lambda_{2j}}.
\end{align*}
\]

where \( c^- \) represents the modification rate of those interpolated
rules whose antecedents are less than the antecedent of
the first modified rule (i.e. \((A_{1j}, A_{2j})\)) by the partial order,
and \( c^+ \) represents the same meaning for the greater ones.
In other words, \( c^- \) measures the difference of the interpolated
results by interpolation lines \( P_0P_1 \) and \( P_0P_3 \) from those
antecedent pairs which are greater than \((A_{11}, A_{21})\) and
less than \((A_{1j}, A_{2j})\) according to the partial order, while \( c^+ \) does
the same but by interpolation lines \( P_3P_1 \) and \( P_3P_3 \) from those
pairs which are between \((A_{1j}, A_{2j})\) and \((A_{1n}, A_{2n})\).

Fig. 5. Defective fuzzy reasoning component modification for extrapolation

The case discussed above covers the first kind of distribution
of observations. For strict extrapolation, where all
observations lie on just one side of the the two closest rules in
accordance with the partial order, the linear relation between
the antecedent variables and the corresponding consequent
variable can also be represented by a straight line. However,
all the extrapolated rules lie on the extension (i.e. line \( P_1P_2 \)
in Fig. 5) of the line segment which connects the two closest
rules in the problem space (i.e. line \( P_0P_1 \) in Fig. 5). Because
no interpolation is possible between the two closest rules, the
extrapolated rule whose antecedent is located farthest from
both antecedents of these two rules is deemed to be the most
dissimilar to them and hence, should be modified the most.

Continue the example, and suppose that all interpolated
rules lie on just one side of the two rules for interpolation
and that \( A_{cj} \) \((2 \leq j \leq n-1)\) sits in the farthest place
to these two rules on the extension of line \( P_0P_1 \) \((\forall x \in \{2, 3, \ldots, n-1\}\)
\{2, 3, ..., n - 1\}, \text{ and } A_{cj} \geq A_{cx} \text{ by the partial order). Therefore, the interpolated rule } A_{1j}, A_{2j} \Rightarrow A_{3j} \text{ will be modified first. The modification replaces the interpolation line } P_1P_2 \text{ with } P_1P_3. \text{ All other interpolated rules based on the same fuzzy reasoning component will be modified by the same correction rate. Particularly in this example, all the observations are greater than the corresponding antecedent values of these rules with respect to the partial order, the correction rate is the same as that with single-antecedent situation, which is \( \frac{\text{correction rate}}{\text{values}} \text{ with these rules while the others are less than such values.} \) 

Finally, when some of the observations are located between the corresponding antecedent values of the two rules for interpolation and all antecedents are located outside with respect to the partial order, the interpolated rule whose antecedent sits in the middle most of the neighborhood of the two rules will be modified first. Suppose that \( A_{cj} \) \((2 \leq j \leq n - 1)\) sits in the middle most on the line \( P_0P_1 \), then the original interpolation line \( P_0P_1 \) is replaced by two line segments \( P_1P_5 \) and \( P_0P_3 \) as illustrated in Fig. 6. In this case, the correction rate pair is still the same as that in the strict interpolation situation, that is Eq. 20.

Having chosen the first rule to modify and calculated the correction rate pair, the rest of the modification is exactly the same as that with single-antecedent situation, which is outlined in Sec. II-C and thus omitted here.

IV. AN ILLUSTRATE EXAMPLE

To illustrate the potential of this extended adaptive fuzzy interpolation and extrapolation method for multiple-antecedent rules, the example given in [11] is extended. The rule base is given as follows:

\begin{align*}
R_1: & \text{ If } x_1 \text{ is } A_{11} \text{ and } x_2 \text{ is } A_{21}, \text{ then } x_3 \text{ is } A_{31}; \\
R_2: & \text{ If } x_1 \text{ is } A_{12} \text{ and } x_2 \text{ is } A_{22}, \text{ then } x_3 \text{ is } A_{32}; \\
R_3: & \text{ If } x_3 \text{ is } A_{35}, \text{ then } x_5 \text{ is } A_{51}; \\
R_4: & \text{ If } x_3 \text{ is } A_{36}, \text{ then } x_5 \text{ is } A_{52}; \\
R_5: & \text{ If } x_3 \text{ is } A_{33}, \text{ then } x_4 \text{ is } A_{41}; \\
R_6: & \text{ If } x_3 \text{ is } A_{34}, \text{ then } x_4 \text{ is } A_{42}; \\
R_7: & \text{ If } x_5 \text{ is } A_{53} \text{ and } x_6 \text{ is } A_{61}, \text{ then } x_7 \text{ is } A_{73}; \\
R_8: & \text{ If } x_5 \text{ is } A_{54} \text{ and } x_6 \text{ is } A_{62}, \text{ then } x_7 \text{ is } A_{74}; \\
R_9: & \text{ If } x_4 \text{ is } A_{43}, \text{ then } x_7 \text{ is } A_{71}; \\
R_{10}: & \text{ If } x_4 \text{ is } A_{44}, \text{ then } x_7 \text{ is } A_{72};
\end{align*}

Given \( \beta_0 = 0.5 \) and six observations: \( x_1 = A_{11} = (2.0, 3.0, 4.0), x_1 = A_{14} = (2.6, 3.6, 4.6), x_2 = A_{23} = (18.0, 19.0, 20.0), x_4 = A_{45} = (9.5, 10.5, 11.5), x_5 = A_{55} = (8.0, 9.0, 10.0), \) and \( x_6 = A_{63} = (12.0, 13.0, 14.0), \) the original observations as well as interpolated results by scale and move transformation-based interpolation technique are presented in Fig. 7 and the interpolation procedures are illustrated in Fig. 8. Here, rules \( R_1, R_2, R_7 \) and \( R_8 \) are of two antecedents each. For observations \( (A_{13}, A_{23}) \) and \( (A_{14}, A_{23}) \), \( R_1 \) and \( R_2 \) are the two closest rules while for \( (A_{55}, A_{63}) \) and \( (A_{56}, A_{63}) \), \( R_7 \) and \( R_8 \) are the two closest. Once obtaining the two closest rules, the relative placement factor, scale rate and move rate of the consequent of each observation can be calculated by following Eqs. 15, 17 and 18, respectively. From this, the rest of the interpolation procedure is the same as that of the single-antecedent one.

A. Dependency recording by ATMS

In Fig. 8, an arrowed line flanked by two rules \( R_i \) and \( R_j \) represents a fuzzy reasoning component, which is denoted as \( R_i R_j \), where \( R_i \) and \( R_j \) are the neighboring rules used for interpolation. ATMS nodes and contradictions are represented by circles. Particularly, each of \( F_i, i \in \{1, 2, ..., 5\}, \)
is a node denoting a fuzzy reasoning component; each of
$P_j$, $j \in \{1, 2, \ldots, 16\}$, is a node denoting a proposition; and
each of $\bot_k$, $k \in \{1, 2, \ldots, 9\}$, denotes a $\beta_0$-contradiction.
These ATMS nodes and contradictions are listed as follows,
with all justifications omitted:

$F_1 : \{R_1, R_2\}, \{\{R_1, R_2\}\}$;
$F_2 : \{R_3, R_4\}, \{\{R_3, R_4\}\}$;
$F_3 : \{R_5, R_6\}, \{\{R_5, R_6\}\}$;
$F_4 : \{R_7, R_8\}, \{\{R_7, R_8\}\}$;
$F_5 : \{R_9, R_{10}\}, \{\{R_9, R_{10}\}\}$;
$F_6 : \{x_1 = A_{11}, \{\}\}$;
$F_7 : \{x_3 = A_{13}, \{\}\}$;
$F_8 : \{x_3 = A_{23}, \{\}\}$;
$F_9 : \{x_3 = A_{35}, \{\}\}$;
$F_{10} : \{x_3 = A_{55}, \{\}\}$.

There are just two minimal environments in the label of
the “false” node:

$P_{11} : \{R_1, R_2, R_3, R_4\}$.

The label of $P_{11}$ means that at least one element of the set
$\{R_1, R_2, R_3, R_4\}$ and one element of the set $\{R_7, R_8, R_9, R_{10}\}$
are faulty simultaneously.

**B. Candidate generation by GDE**

Four minimal candidates are generated, each of which is
composed by taking one element from each environment in
the label of the “false” node:

$C_1 : [R_1, R_2, R_7, R_8]$;
$C_2 : [R_1, R_2, R_9, R_{10}]$;
$C_3 : [R_3, R_4, R_7, R_8]$;
$C_4 : [R_3, R_4, R_9, R_{10}]$.

**C. Candidate modification**

Any one of these four minimal candidates can be taken
for modification first because they are of the same size
in cardinality. Particularly in this example, $C_3$ is taken
randomly to modify first. Four rules have been interpolated
through the two fuzzy reasoning components that comprise
the candidate:

$IR_1$: If $x_3$ is $A_{37}$, then $x_4$ is $A_{46}$;
$IR_2$: If $x_3$ is $A_{38}$, then $x_4$ is $A_{47}$;
$IR_3$: If $x_5$ is $A_{55}$ and $x_6$ is $A_{63}$, then $x_7$ is $A_{78}$;
$IR_4$: If $x_5$ is $A_{56}$ and $x_6$ is $A_{63}$, then $x_7$ is $A_{79}$.

For fuzzy reasoning component $R_3, R_6$, the culprit interpo-
lated rule $IR_1$ will be modified first because fuzzy set $A_{37}$ is
located nearer the middle than $A_{46}$. Suppose that the relative
placement factor of the modified consequence is $\lambda_5^{+}$. Then
the correction rate pair is:

\[
\begin{align*}
    c^{+}_{R_3, R_6} &= \frac{\lambda_3^{+}}{1 - \lambda_3^{+}} \\
    c^{-}_{R_3, R_6} &= \frac{1 - \lambda_3^{+}}{1 - \lambda_3^{+}}.
\end{align*}
\]

Accordingly, $IR_2$ will be modified with respect to the
generated correction rate pair $(c^{+}_{R_3, R_6}, c^{-}_{R_3, R_6})$. The relative
placement factor $\lambda_7^{+}$ of the modified consequence satisfies:

\[
1 - \lambda_7^{+} = (1 - \lambda_7^{+}) \cdot c^{+}_{R_3, R_6}.
\]
The interpolated rule consequences after modification, $A^*_{46}$ and $A^*_{47}$ can thus be expressed by:

\[
\begin{align*}
A^*_{46} &= (1 - \lambda^*_{46}) A_{41} + \lambda^*_{46} A_{42} \\
A^*_{47} &= (1 - \lambda^*_{47}) A_{41} + \lambda^*_{47} A_{42} \\
T(A^*_{46}, A^*_{47}) &= T(A^*_{46}, A^*_{46}) \\
T(A^*_{48}, A^*_{48}) &= T(A^*_{47}, A^*_{47}).
\end{align*}
\]

Fuzzy sets $A_{46}$ and $A_{47}$ must satisfy the following constraints if they are $(1 - \beta_0)$-consistent:

\[
(A^*_{46})_{\beta_0} \cap (A^*_{47})_{\beta_0} \cap (A_{45})_{\beta_0} \neq \emptyset.
\]

Similarly, the culprit interpolated rules $IR_3$ and $IR_4$ are also modified by following the modification procedure outlined in Sec. III-D. The following constraints are hence generated:

\[
\begin{align*}
\frac{c_{R_2R_8}}{1 - \lambda^{T^*}_{78}} &= \frac{\lambda^{T^*}_{78}}{\lambda^{T^*}_{78}} \\
\frac{c_{R_2R_8}}{1 - \lambda^{T^*}_{79}} &= (1 - \lambda_{78}) \cdot \frac{c_{R_2R_8}}{1 - \lambda^{T^*}_{79}} \\
A^*_{78} &= (1 - \lambda^{T^*}_{78}) A_{73} + \lambda^{T^*}_{78} A_{74} \\
A^*_{79} &= (1 - \lambda^{T^*}_{79}) A_{73} + \lambda^{T^*}_{79} A_{74} \\
T((A^*_{55}, A^*_{65}), (A_{55}, A_{65})) &= T(A^*_{78}, A^*_{78}) \\
T((A^*_{56}, A^*_{66}), (A_{56}, A_{66})) &= T(A^*_{79}, A^*_{79}) \\
(A^*_{78})_{\beta_0} \cap (A^*_{79})_{\beta_0} \cap (A_{77})_{\beta_0} &\neq \emptyset.
\end{align*}
\]

The propagations of all these modified rules need to be $(1 - \beta_0)$-consistent as well, which can be ensured by introducing the following simultaneous equations:

\[
\begin{align*}
A_{75} &= I(A^*_{46}, R_9 R_{10}) \\
A_{76} &= I(A^*_{47}, R_9 R_{10}) \\
(A^*_{75})_{\beta_0} \cap (A^*_{76})_{\beta_0} \cap (A_{77})_{\beta_0} &\neq \emptyset.
\end{align*}
\]

One of the solutions led by solving these simultaneous equations is illustrated in Fig. 9. It is clear from this result that there is no $\beta_0$-contradiction any more and thus consistency has been restored. This means that the original inconsistent interpolation process has been corrected with consistent interpolated results throughout.

V. CONCLUSIONS

This paper has generalized the recent work on adaptive fuzzy interpolation [11]. This is achieved by introducing fuzzy extrapolation to the adaptive approach and extending the approach to involving multiple-antecedent rules. The work first uses the classical ATMS-based GDE approach to detect and locate faults during the process of fuzzy interpolation/extrapolation. It then modifies the identified culprit interpolated or extrapolated rule consequents by replacing the original linear interpolation/extrapolation with first-order piecewise linear approximation, in an effort to restore consistency. The working of this method is illustrated with a practically significant example.

Whilst the proposed work is promising, it relies upon the assumption that all rules for interpolation/extrapolation which are provided in the initial rule base are totally true and fixed. This may not be the case in some real-world problems, despite the fact that it is a common assumption made in the literature of interpolative reasoning. Thus, further development on the work may be desirable in allowing such rules to become themselves diagnosable and modifiable. It is also very interesting to develop an unified inconsistency diagnosis and fault correction mechanism on a fuzzy reasoning platform that implements both standard fuzzy inference and fuzzy interpolation/extrapolation.

REFERENCES