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## Classical and Quantum Stochastic Models of Resistive and Memristive Circuits

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The purpose of this paper is to examine stochastic Markovian models for circuits in phase space for which the drift term is equivalent to the standard circuit equations. In particular we include dissipative components corresponding to both a resistor and a memristor in series. We obtain a dilation of the problem for which is canonical in the sense that the underlying Poisson Brackets structure is preserved under the stochastic flow. We do this first of all for standard Wiener noise, but also treat the problem using a new concept of symplectic noise where the Poisson structure is extended to the noise as well as the circuit variables, and in particular where we have canonically conjugate noises. Finally we construct a dilation which describes the quantum mechanical analogue.

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## I. INTRODUCTION

Dissipation has long been realized as a feature, though also a resource, in the design of engineered systems,<sup>1-11</sup>. The question of how exactly to model dissipation, given that the fundamental physical dynamical equations of motion are Hamiltonian, remains of fundamental importance; the issue applies to both classical and quantum systems,<sup>12-22</sup>. Hamiltonian systems with a finite number of degrees of freedom have dynamical evolutions that preserve the canonical structure - that is, the Poisson brackets in classical theory, and the commutation relations in quantum theory. In this setting, dissipation is *per se* impossible. There exist a number of ingenious approaches to tackling this problem such as approximations using lossless systems<sup>22</sup> and redefining the Poisson bracket to include dissipative effects<sup>5</sup>.

The approach to introducing dissipation adopted in this paper is to embed the system in an environment (leading to a joint coupled Hamiltonian model) and average out the environment. In other words, we consider stochastic models which are Hamiltonian in structure which dilate the dissipative dynamics. It is well-known that under various assumptions about the bulk limit of the environment, negligible autocorrelation of the environment processes (memoryless property), ignoring rapid oscillations (the rotating wave approximation in the quantum case), etc., one may obtain limiting dynamical models with an irreversible semi-group evolution. Moreover, this semi-group may often be dilated to a stochastic Markov system: in the classical case, this may be described by stochastic differential equations of motion, for instance for a diffusion process, where the generator is the second-order differential operator on phase space coinciding with the generator of the semi-group; in the quantum setting, this may be a unitary quantum stochastic process, leading to an evolution described by quantum stochastic calculus<sup>23</sup>, where the generator is Lindbladian<sup>24,25</sup>.

The purpose of this paper is to carry out this programme in the setting of electronic circuit models, where we allow for dissipation beyond the usual ohmic damping. It is well known that constant inductance-capacitance (*LC*) circuits are Hamiltonian, and are readily quantized,<sup>26</sup> (Section 3.4.3). Markovian models that include dissipation may then be formulated on purely phenomenological grounds, however, it is clear that it is desirable to have a theory that is capable of dealing with non-linear dissipation, in particular, we wish to include charge and current dependent resistances, which brings us to the formalism introduced by Chua<sup>27</sup> for a unified axiomatic description of passive circuit components. While originally a theoretical construct<sup>27-34</sup>, there is evidence to suggest that ideal memristors also occur in quantum models<sup>35</sup>.

### A. Organization of the paper

The layout and main contributions of this paper are summarized as follows. In section II we recall the framework of Chua for describing general classical circuit models and present the canonical formulation in phase space.

In section III we show how we may obtain a stochastic dilation of these models for an arbitrary resistor and memristor in series, and remarkably, we are able to give explicit constructions based on the principle that the dilation should in some sense be Hamiltonian: more exactly, this emerges from the requirement that the stochastic evolution be canonical. In the theory of quantum stochastic evolutions<sup>23</sup>, this turns out to arise automatically from the requirement of a unitarity of the evolution process. The classical theory is more flexible, but less transparent as one has to put in the requirement of preservation of the Poisson brackets as an additional constraint on the stochastic evolution explicitly. Here the algebraic features encoded in the quantum case are replaced by geometric features imposed on the diffusion process.

Our first construction (Theorem 5) deals with the classical problem and utilizes a result by one of the authors in<sup>37</sup>. This however involves standard Wiener noise which has no symplectic structure of its own leading to a rather involved form of the dilation. Motivated by the situation that occurs in quantum models, we consider *canonically conjugate pairs of Wiener noise* that satisfy their own non-trivial Poisson bracket relations: these symplectic noises<sup>38</sup> were only recently introduced and they parallel the corresponding situation in quantum theory where the noise is actually a quantum electromagnetic field. We give the corresponding result, Theorem 9, for symplectic noise and this constitutes a genuine extension of Theorem 5 as we obtain the later in the case where only one of each of the canonical pairs of symplectic noise is present. The construction in Theorem 9 to obtain the general canonical dilation of a resistor and memristor in series is markedly more simple and natural than that of Theorem 5 due to the role of the canonically conjugate noises.

For completeness, in section IV we show how these models are readily quantized and establish the analogous result, Theorem 10. In Section V we discuss the origin of the stochastic models proposed in Sections III and IV.

There has been much interest into how Hamiltonian systems can approximate dissipative systems, see<sup>16</sup> for classical and<sup>39, 40</sup> for quantum, and there have been a control theoretic models of lossless approximations to dissipative systems<sup>20, 22</sup>. In section V, we show how these models may arise from approximations by physically realistic microscopic systems. This involves a Wong-Zakai type limit procedure - more exactly, the quantum stochastic analogue of this, which contains the classical result as a special case. Finally we conclude the paper in Section VI.

## II. GENERAL CIRCUIT THEORY

The concept of a memristor was introduced by Leon Chua in 1971 as a missing fourth element in the theory of idealized passive components in electronics<sup>27,28</sup>. There are four variables to consider in a circuit: the charge  $q$ , the current  $I$ , the flux  $\varphi$ , and the voltage  $V$ . They are actually paired naturally as  $(q, I)$  and  $(\varphi, V)$ , and we always have the fundamental relations

$$I = \dot{q}, \quad V = \dot{\varphi}, \quad (1)$$

however, it is mathematically useful to treat these as four independent variables.

An ideal *resistor*  $\mathcal{R}$  is a component that leads to a fixed  $I$ - $V$  characteristic, that is, the voltage  $V$  across the resistor and the current  $I$  through the resistor are constrained by an equation of the form

$$f_{\mathcal{R}}(I, V) = 0,$$

and assuming that the relation is differentiable and bijective (that is, each voltage determines a unique current and vice versa) we may always write

$$dV = R(I) dI.$$

Note that we can always choose  $R$  to be a function of  $I$  only by assumption. The coefficient  $R$  is the resistance, and in the case where it is a constant  $R$  we obtain Ohm's law  $V = RI$ .

Likewise, an ideal *capacitor*  $\mathcal{C}$  is a component that determines a  $q$ - $V$  characteristic  $f_{\mathcal{C}}(q, V) = 0$  and under similar assumptions we may write

$$dV = \frac{1}{C(q)} dq;$$

while an *inductor*  $\mathcal{L}$  is a component that determines a  $\varphi$ - $I$  characteristic  $f_{\mathcal{L}}(\varphi, I) = 0$  and under similar assumptions we have

$$d\varphi = L(I) dI.$$

The physical quantities  $C$  and  $L$  arising are the capacitance and inductance respectively, and it is convenient to take them as functions of  $q$  and  $I$  respectively.

Chua's insight was that the mathematical theory missed out an idealized component  $\mathcal{M}$  that determined a  $q$ - $\varphi$  characteristic: see Figure 1. This fourth element he called a *memristor* and it fixed a relationship  $f_{\mathcal{M}}(q, \varphi) = 0$ , and again assuming differentiability and bijective we are led to the differential relation

$$d\varphi = M(q) dq.$$

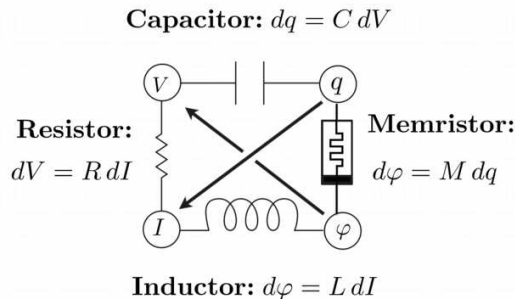


FIG. 1. Chua's fourfold way! The inclusion of the memristor conceptually completes the set of idealized passive circuit elements.

In fact, it immediately follows from (1) that for a memristor we have the relation

$$V = M(q) I,$$

and comparison with Ohm's law suggests that the physical quantity  $M(q)$  is a charge dependent resistance. Indeed as the memristor relates the integral of the voltage  $\varphi(t) = \int_{-\infty}^t V(t') dt'$  to the integral of the current  $q(t) =$

$\int_{-\infty}^t I(t') dt'$ , the component acts as a resistive element with memory of past voltages and currents - whence the name memristor.

The capacitor and inductor are both capable of storing energy and we have the associated energies

$$E_{\mathcal{C}} = \int qdV = \int \frac{qdq}{C(q)}, \quad E_{\mathcal{L}} = \int Id\varphi = \int L(I)IdI.$$

These four components will then form the building blocks for general passive electric circuits. While they are clearly idealizations, it is nevertheless the case that they are the fundamental elements that can be combined to produce physical systems.

We may summarize the main properties of the four components as follows:

<b>The Inductor</b> $\mathcal{L}$		<b>The Capacitor</b> $\mathcal{C}$	
Type	Energy Storing	Type	Energy Storing
Characteristic	$d\varphi = L(I)dI$	Characteristic	$dV = \frac{1}{C(q)}dq$
Voltage	$V_{\mathcal{L}} = \dot{\varphi}_{\mathcal{L}} = L(I_{\mathcal{L}})\dot{I}_{\mathcal{L}}$	Voltage	$V_{\mathcal{C}} = \int \frac{dq}{C(q)} \Big _{q_{\mathcal{C}}}$
<b>The Resistor</b> $\mathcal{R}$		<b>The Memristor</b> $\mathcal{M}$	
Type	Dissipative	Type	Dissipative
Characteristic	$dV = R(I)dI$	Characteristic	$d\varphi = M(q)dq$
Voltage	$V_{\mathcal{R}} = \int R(I)dI \Big _{I_{\mathcal{R}}}$	Voltage	$V_{\mathcal{M}} = \dot{\varphi}_{\mathcal{M}} = M(q_{\mathcal{M}})\dot{I}_{\mathcal{M}}$

The introduction of the memristor by Chua has the important theoretical implication that we can model a wider class of dissipative systems than just described by the resistor alone. We will explore this shortly, but the main feature is that we may have dissipative models where the damping is dependent on the charge in addition to the current.

## A. Lagrangian & Hamiltonian models for energy storing components

The situation where we have an inductor and capacitor in series is well-known to be a conservative system with a Lagrangian formulation, see also<sup>30,32,34</sup>. For a prescribed applied emf  $e(t)$ , we have  $V_{\mathcal{L}} + V_{\mathcal{C}} = e(t)$  and so the equation of motion is

$$L(\dot{q})\ddot{q} + \Phi'_{\mathcal{C}}(q) = e(t),$$

with  $\Phi'_{\mathcal{C}}(q) = \int \frac{dq}{C(q)}$ . We assume the existence of a twice-differentiable function  $K$  such that  $K'(I) > 0$  and  $K''(I) = L(I)$ . The equations of the  $\mathcal{L}\mathcal{C}$ -circuit then follow from the Euler-Lagrange equations  $\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}}) - \frac{\partial \mathcal{L}}{\partial q} = 0$  with Lagrangian

$$\mathcal{L}(q, \dot{q}, t) = K(\dot{q}) - \Phi_{\mathcal{C}}(q) + e(t)q.$$

The canonical momentum is then given by

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = K'(\dot{q}),$$

and by assumption on  $K$  this is a bijection with inverse  $\dot{q} = \mathbb{I}(p)$ , that is, the Lagrangian is hyper-regular. Note that

$$\dot{p} = L(\dot{q})\ddot{q} = L(I)\dot{I} \equiv V_{\mathcal{L}} = \dot{\varphi}_{\mathcal{L}},$$

so in a sense  $p$  may be identified with the inductor flux  $\varphi_{\mathcal{L}}$ . The corresponding Hamiltonian is then

$$H(q, p, t) = \mathbb{K}(p) + \Phi_{\mathcal{C}}(q) - e(t)q, \tag{2}$$

and we have the Legendre transform  $\mathbb{K}(p) = \sup_I \{pI - K(I)\} \equiv p\mathbb{I}(p) - K(\mathbb{I}(p))$ .

In the special case where the inductance is  $L(I) = L_0$  (constant) we take  $K(I) = \frac{1}{2}L_0I^2$ , and here

$$\mathbb{I}(p) = p/L_0, \quad \mathbb{K}(p) = \frac{1}{2L_0}p^2.$$

Likewise, in the special case where the capacitance is  $C(q) = C_0$ , we may take  $\Phi_{\mathcal{C}}(q) = \frac{1}{2C_0}q^2$ . Here, the Hamiltonian takes the explicit form

$$H_0(q, p, t) = \frac{p^2}{2L_0} + \frac{q^2}{2C_0} - q e(t), \quad (3)$$

with  $p = L_0\dot{q}$ , and that this is a driven harmonic oscillator with resonant frequency  $\omega_0 = (L_0C_0)^{-1/2}$ . Note that, with  $e(t) = 0$ ,  $H_0$  is equal to the physically stored energy  $E_{\mathcal{L}} + E_{\mathcal{C}}$  with the appropriate substitution  $\dot{q} = p/L_0$ .

## B. Dissipative Circuits

We begin by examining a circuit having all four components in series, see Figure 2.

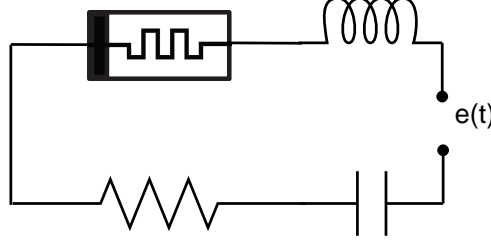


FIG. 2. All four idealized components in series.

The applied emf  $e(t)$  driving the circuit is related by Kirchhoff's voltage law to the total voltage drop over the circuit:

$$V_{\mathcal{L}} + V_{\mathcal{C}} + V_{\mathcal{R}} + V_{\mathcal{M}} = e(t).$$

Recalling that  $V_{\mathcal{L}} = \dot{p}$ , the dynamical equation may instead be expressed as

$$\dot{p} = - \int \frac{dq}{C(q)} - \int R(I)dI - M(q)\dot{q} + e(t).$$

Let us introduce the functions

$$\mathcal{D}'_{\mathcal{R}}(I) = \int R(I)dI, \quad \Psi'_{\mathcal{R}}(p) = \mathcal{D}'_{\mathcal{R}}(\mathbb{I}(p)), \quad (4)$$

then we obtain the system of equations in the  $(q, p)$  phase space of the circuit

$$\dot{q} = \mathbb{I}(p), \quad \dot{p} = -\Phi'_{\mathcal{C}}(q) - \Psi'_{\mathcal{R}}(p) - M(q)\mathbb{I}(p) + e(t). \quad (5)$$

The negative divergence of the velocity field of phase points is then

$$\gamma(q, p) = - \left( \frac{\partial}{\partial q}\dot{q} + \frac{\partial}{\partial p}\dot{p} \right) = \Psi''_{\mathcal{R}}(p) + M(q) \frac{\partial \mathbb{I}}{\partial p} = \frac{1}{\mathbb{L}(p)} (\mathbb{R}(p) + M(q)), \quad (6)$$

where we define the  $p$ -dependent resistance and inductance as

$$\mathbb{R}(p) = R(\mathbb{I}(p)), \quad \mathbb{L}(p) = L(\mathbb{I}(p)).$$

The equations of motion are now

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} - \mathbb{V}_{\mathcal{R}}(p) - \mathbb{V}_{\mathcal{M}}(q, p), \quad (7)$$

with  $H$  the Hamiltonian (2) for the energy storing components  $\mathcal{L}$  and  $\mathcal{C}$  and the voltages associated with the energy dissipating components  $\mathcal{R}$  and  $\mathcal{M}$  are

$$\mathbb{V}_{\mathcal{R}}(p) = \int R(I)dI \Big|_{I=\mathbb{I}(p)} \equiv \Psi'_{\mathcal{R}}(p), \quad (8)$$

$$\mathbb{V}_{\mathcal{M}}(q, p) = M(q)\mathbb{I}(p). \quad (9)$$

In the situation where the inductance is a constant  $L_0$  the system of equations (5) reduce to  $\dot{q} = v^q$ ,  $\dot{p} = v^p$  with

$$v^q = \frac{p}{L_0}, \quad v^p = -\Phi'_{\mathcal{E}}(q) - \Psi'_{\mathcal{R}}(p) - \frac{1}{L_0}M(q)p + e(t). \quad (10)$$

We shall generally be interested in the behaviour of the Poisson brackets  $\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} \frac{\partial f}{\partial p}$  under general dynamical flows on phase space.

In general, let  $w^q$  and  $w^p$  be twice-differentiable coefficients of a vector field in phase space along the coordinate axes, then the tangent vector field is the operator

$$w = w^q \frac{\partial}{\partial q} + w^p \frac{\partial}{\partial p}.$$

We say that the field is Hamiltonian if it can be written in terms of the Poisson brackets as  $w(\cdot) \equiv \{\cdot, H\}$ , for some smooth function  $H$ . In coordinate form this reads as  $w^q = \frac{\partial H}{\partial p}$ ,  $w^p = -\frac{\partial H}{\partial q}$ .

Given a general vector field  $w$ , the flow it generates is the family of diffeomorphisms  $\{\phi_t\}$ , for  $t$  in a neighbourhood of 0, such that  $t \mapsto (q_t, p_t) \equiv \phi_t(q_0, p_0)$  gives the integral curve of  $w$  passing through  $(q_0, p_0)$  at  $t = 0$ . The flow is canonical if it preserves the Poisson brackets. To obtain a necessary condition, we look at small times  $t$  where locally we have

$$q_t = q_0 + tv^q(q_0, p_0) + O(t^2), \quad p_t = p_0 + tv^p(q_0, p_0) + O(t^2)$$

and so

$$\begin{aligned} \frac{\partial q_t}{\partial q} &= 1 + tv^q_{,q} + O(t^2), & \frac{\partial q_t}{\partial p} &= tv^q_{,p} + O(t^2) \\ \frac{\partial p_t}{\partial q} &= tv^p_{,q} + O(t^2), & \frac{\partial p_t}{\partial p} &= 1 + tv^p_{,p} + O(t^2) \end{aligned}$$

We therefore see that

$$\{q_t, p_t\} = 1 + t(v^q_{,q} + v^p_{,p}) + O(t^2) = 1 - \gamma(q_0, p_0)t + O(t^2).$$

A necessary condition for the flow to be canonical is therefore that  $\gamma(q, p) \equiv 0$ , and, as is well-known, the possible solutions take the form  $v^q = \frac{\partial H}{\partial p}$ ,  $v^p = -\frac{\partial H}{\partial q}$  for some function  $H$ , in other words  $w$  must be a Hamiltonian vector field. More generally, the infinitesimal expression describing the preservation of the Poisson brackets under the flow integral to  $w$  is

$$w(\{f, g\}) = \{w(f), g\} + \{f, w(g)\} \quad (11)$$

for every pair of smooth functions  $f, g$  on phase space. Again it can be shown that this holds if and only if  $w$  is divergence free, that is,  $\frac{\partial}{\partial q} w^q + \frac{\partial}{\partial p} w^p = 0$ , and this of course is equivalent to  $w$  being a Hamiltonian vector field.

In general, the dissipation  $\gamma(q, p)$  gives the exponential rate at which phase area in the  $qp$  phase space is decreasing. As physically  $R \geq 0$  and  $M \geq 0$  for passive systems, we must have  $\gamma \geq 0$ . Geometrically,  $\gamma$  characterizes exactly the non-Hamiltonian nature of the dynamical flow on phase space, and we conclude that a dynamical flow on phase space will preserve the Poisson brackets if and only if it corresponds to the evolution governed by some Hamiltonian function  $H$ .

For the systems in series as above, we may decompose

$$\gamma(q, p) = \gamma_{\mathcal{R}}(p) + \gamma_{\mathcal{M}}(q, p)$$

with  $\gamma_{\mathcal{R}}(p) = \frac{\partial}{\partial p} \mathbb{V}_{\mathcal{R}}$  and  $\gamma_{\mathcal{M}}(q, p) = \frac{\partial}{\partial p} \mathbb{V}_{\mathcal{M}}$ . In the case where the inductances are constant we find

$$\gamma_{\mathcal{R}}(p) = \frac{1}{L_0} \mathbb{R}(p), \quad \gamma_{\mathcal{M}}(q, p) = \frac{1}{L_0} M(q),$$

and in particular these are functions of  $p$  only and, respectively,  $q$  only. Therefore we have a natural decomposition (up to an additive constant) of the resistor and memristor elements as providing the  $p$  only and  $q$  only contributions for the dissipation  $\gamma$ , respectively, when the inductances are constant.

It is fairly easy to construct dissipative components that do not have the special decomposition  $\gamma(q, p) = \gamma_{\mathcal{R}}(p) + \gamma_{\mathcal{M}}(q)$  for constant inductance. A simple example is a resistor and memristor in parallel, where now

$$\mathbb{V}(q, p) = (\mathbb{V}_{\mathcal{R}}(p)^{-1} + \mathbb{V}_{\mathcal{M}}(q)^{-1})^{-1},$$

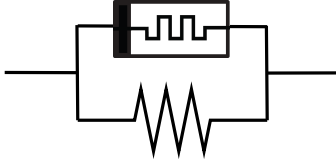


FIG. 3. A resistor and memristor in parallel.

see Figure 3.

Without going into explicit details, it is intuitively obvious that an arbitrary damping function  $\gamma(q, p) \geq 0$  may be approximated by a network of resistive and memristive components in series and parallel.

The dynamical equations may also be written in Lagrangian terms as

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = - \frac{\partial \mathcal{D}}{\partial \dot{q}},$$

where we must now introduce the so-called dissipation potential  $\mathcal{D}(q, \dot{q}) = \mathcal{D}_{\mathcal{R}}(\dot{q}) + \frac{1}{2}M(q)\dot{q}^2$  which is a sum of the resistance potential  $\mathcal{D}_{\mathcal{R}}$  introduced in (4) and a memristive Rayleigh type dissipation function. However, we shall work in the Hamiltonian formalism.

### C. Equivalent Circuit Theorem

Based on the above discussions, we have the following separation of arbitrary passive circuits into idealized energy storing and energy dissipating components.

**Theorem 1** *An arbitrary passive circuit may be decomposed in an energy storing components, the **Hamiltonor**, described by a Hamilton's function  $H$  and an energy dissipating component, the **Dissipator**, described by a voltage function  $\mathbb{V}_{\mathcal{D}}$ , see Figure 4, such that the circuit equations are*

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= - \frac{\partial H}{\partial q} - \mathbb{V}_{\mathcal{D}}(q, p). \end{aligned} \quad (12)$$

*The dissipation is then the negative divergence of the phase velocity, and this is given by*

$$\gamma(q, p) = \frac{\partial}{\partial p} \mathbb{V}_{\mathcal{D}}(q, p) \geq 0.$$

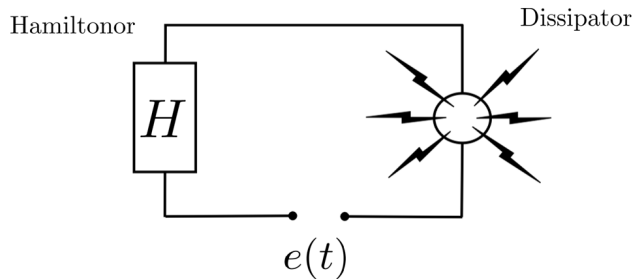


FIG. 4. A Hamiltonian component (Hamiltonor) and dissipative component (dissipator) in series.

From the mathematical perspective, the beauty of Chua's introduction of the memristor as a dissipative element is that it enlarges the class of possible dissipators in a maximal way.



### III. CLASSICAL STOCHASTIC MODELS

#### A. Diffusion Noise Models

We now consider a stochastic dynamics on phase space given by the Stratonovich stochastic differential equation (SDE)

$$dq = w^q dt + \sum_{\alpha} \sigma_{\alpha}^q \circ dB_t^{\alpha}, \quad dp = w^p dt + \sum_{\alpha} \sigma_{\alpha}^p \circ dB_t^{\alpha}, \quad (13)$$

where the coefficients  $w^q, w^p, \sigma_{\alpha}^q, \sigma_{\alpha}^p$  are assumed to be twice-differentiable functions of the coordinates  $(q, p)$  and satisfy Lipschitz and growth conditions to ensure existence and uniqueness of solution<sup>41</sup>. Here we take independent standard Wiener processes  $(B_t^{\alpha})_{t \geq 0}$ . The equation may be cast in Itô form as

$$\begin{aligned} dq &= \left( w^q + \frac{1}{2} \sum_{\alpha} \sigma_{\alpha}^q \frac{\partial \sigma_{\alpha}^q}{\partial q} + \frac{1}{2} \sum_{\alpha} \sigma_{\alpha}^p \frac{\partial \sigma_{\alpha}^q}{\partial p} \right) dt + \sum_{\alpha} \sigma_{\alpha}^q dB_t^{\alpha}, \\ dp &= \left( w^p + \frac{1}{2} \sum_{\alpha} \sigma_{\alpha}^q \frac{\partial \sigma_{\alpha}^p}{\partial q} + \frac{1}{2} \sum_{\alpha} \sigma_{\alpha}^p \frac{\partial \sigma_{\alpha}^p}{\partial p} \right) dt + \sum_{\alpha} \sigma_{\alpha}^p dB_t^{\alpha}. \end{aligned}$$

We now require that a stochastic differential flow preserves the Poisson structure in the sense that we now obtain a random family of canonical diffeomorphisms on phase space. In the Itô calculus, this implies that

$$d\{f_t, g_t\}_t = \{df_t, g_t\}_t + \{f_t, dg_t\}_t + \{df_t, dg_t\}_t.$$

The structure for symplectic manifolds is given in chapter V of Bismut's *Mechanique Aléatoire*<sup>36</sup> and essentially the vector fields  $w$  and  $\sigma_{\alpha}$  must be Hamiltonian vector fields, see also<sup>37</sup>, section III, for a discussion on general Poisson manifolds.

**Theorem 2** *A stochastic dynamics determined by the system of stochastic differential equations (13) on phase space will be canonical if and only if the vector fields  $w$  and  $\sigma_{\alpha}$  are Hamiltonian.*

We now suppose that this is the case and take  $w^q = \frac{\partial H}{\partial p}$ ,  $w^p = -\frac{\partial H}{\partial q}$  and  $\sigma_{\alpha}^q = \frac{\partial F_{\alpha}}{\partial p}$ ,  $\sigma_{\alpha}^p = -\frac{\partial F_{\alpha}}{\partial q}$ . In the Itô calculus we find the following Langevin equation for functions  $f_t = f(q_t, p_t)$

$$df_t = \left( \{f, H\}_t + \frac{1}{2} \sum_{\alpha} \{ \{f, F_{\alpha}\}_t, F_{\alpha}\}_t \right) dt + \sum_{\alpha} \{f, F_{\alpha}\}_t dB_t^{\alpha}.$$

The dissipative component of the evolution is described by the double Poisson bracket with respect to the  $F_{\alpha}$  in the drift ( $dt$ -term). This is the generator of the diffusion, which in fact is the second-order differential operator

$$\mathcal{L} = \{\cdot, H\} + \frac{1}{2} \sum_{\alpha} \{ \{ \cdot, F_{\alpha} \}, F_{\alpha} \}. \quad (14)$$

We have for instance,

$$\mathcal{L}(q) = v^q, \quad \mathcal{L}(p) = v^p,$$

where

$$v^q = \frac{\partial H}{\partial p} + \frac{1}{2} \sum_{\alpha} \left\{ \frac{\partial F_{\alpha}}{\partial p} \frac{\partial^2 F_{\alpha}}{\partial q \partial p} - \frac{1}{2} \frac{\partial F_{\alpha}}{\partial q} \frac{\partial^2 F_{\alpha}}{\partial p^2} \right\}, \quad v^p = -\frac{\partial H}{\partial q} - \frac{1}{2} \sum_{\alpha} \left\{ \frac{\partial F_{\alpha}}{\partial p} \frac{\partial^2 F_{\alpha}}{\partial q^2} + \frac{1}{2} \frac{\partial F_{\alpha}}{\partial q} \frac{\partial^2 F_{\alpha}}{\partial p \partial q} \right\}. \quad (15)$$

The Itô equations for the coordinates become

$$dq = v^q dt + \sum_{\alpha} \frac{\partial F_{\alpha}}{\partial p} dB_t^{\alpha}, \quad dp = v^p dt - \sum_{\alpha} \frac{\partial F_{\alpha}}{\partial q} dB_t^{\alpha}.$$

Note that (15) may be written more compactly as

$$v = J\nabla H + \frac{1}{2} \sum_{\alpha} JF_{\alpha}'' J\nabla F_{\alpha},$$

where the symplectic matrix  $J$  and the Hessian  $F''$  are defined respectively as

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad F'' = \begin{bmatrix} \frac{\partial^2 F}{\partial q^2} & \frac{\partial^2 F}{\partial q \partial p} \\ \frac{\partial^2 F}{\partial q \partial p} & \frac{\partial^2 F}{\partial p^2} \end{bmatrix}.$$

**Proposition 3** *The dissipation associated with the system of equations (15) is the sum of the Hessian determinants of the  $F_{\alpha}$*

$$\gamma(q, p) = \sum_{\alpha} \left( \frac{\partial^2 F_{\alpha}}{\partial q^2} \frac{\partial^2 F_{\alpha}}{\partial p^2} - \left( \frac{\partial^2 F_{\alpha}}{\partial q \partial p} \right)^2 \right). \quad (16)$$

**Proof.** This follows by substituting the expressions (15) for the Itô drift coefficients into  $\gamma(q, p) = - \left( \frac{\partial v^q}{\partial q} + \frac{\partial v^p}{\partial p} \right)$ . ■

The stochastic flow will, however, be canonical as the dissipation is balanced geometrically by the fluctuations in the noise term. That is,

$$dq_t dp_t = - \sum_{\alpha} \frac{\partial F_{\alpha}}{\partial p} \frac{\partial F_{\alpha}}{\partial q} dt. \quad (17)$$

The model is said to be *passive* if we have dissipation  $\gamma(q, p) \geq 0$  at all phase points  $(q, p)$ , otherwise it is *active*. The van der Pol oscillator has  $\gamma(q, p) = c(q^2 - a^2)$  for constants  $a, c > 0$  and is an example of an active model: this can be cast in the above form with a single Wiener process, see<sup>37</sup> (Section V.B) for detail.

We would now like to realize models of the form (12) as the stochastic canonical models. That is, to obtain the velocity fields  $v^q = \frac{\partial H}{\partial p}$  and  $v^p = -\frac{\partial H}{\partial q} - \mathbb{V}_{\mathcal{D}}$ , where  $\mathbb{V}_{\mathcal{D}}(q, p)$  is the voltage from the dissipative circuit elements, as the Itô drift of a given stochastic canonical model. This is possible if we can solve the following system of equations for  $\{F_{\alpha}\}$ :

$$\sum_{\alpha} \frac{\partial F_{\alpha}}{\partial p} \frac{\partial^2 F_{\alpha}}{\partial q \partial p} - \sum_{\alpha} \frac{\partial F_{\alpha}}{\partial q} \frac{\partial^2 F_{\alpha}}{\partial p^2} = 0, \quad (18)$$

$$\sum_{\alpha} \frac{\partial F_{\alpha}}{\partial p} \frac{\partial^2 F_{\alpha}}{\partial q^2} - \sum_{\alpha} \frac{\partial F_{\alpha}}{\partial q} \frac{\partial^2 F_{\alpha}}{\partial p \partial q} = 2\mathbb{V}_{\mathcal{D}}. \quad (19)$$

In the following, we shall restrict to the problem of a single Wiener process  $B_t$  and function  $F$ .

**Lemma 4** *The equation (18) requires that there exists a function  $\xi(q)$  such that  $\xi(q) \frac{\partial F}{\partial p} = \frac{\partial F}{\partial q}$ .*

**Proof.** We may rewrite the equation (18) as

$$\left( \frac{\partial F}{\partial p} \right)^{-1} \frac{\partial}{\partial p} \left( \frac{\partial F}{\partial p} \right) - \left( \frac{\partial F}{\partial q} \right)^{-1} \frac{\partial}{\partial p} \left( \frac{\partial F}{\partial q} \right) = 0,$$

and so

$$\frac{\partial}{\partial p} \log \left( \frac{\partial F}{\partial q} / \frac{\partial F}{\partial p} \right) = 0$$

and so  $\frac{\partial F}{\partial q} / \frac{\partial F}{\partial p} = \xi(q)$  for some function  $\xi$  of  $q$  only. ■

We see that we may now substitute the identity  $\xi(q) \frac{\partial F}{\partial p} = \frac{\partial F}{\partial q}$  into the second relation (19) to get  $\xi'(q) \left( \frac{\partial F}{\partial p} \right)^2 = 2\mathbb{V}_{\mathcal{D}}$ . However, we note that solving this partial differential equation for  $F$  is in general a difficult problem. We next show, remarkably, that there exist a broad class of solutions to physically important cases.

**Theorem 5** A stochastic evolution with Itô drift velocity  $(v^q, v^p)$  given by the phase velocity field (10) can be achieved by a stochastic canonical model driven by a pair of independent Wiener processes  $(B_t^1, B_t^2)$ , for the resistance and memristance components respectively, with the following choices:

$$\begin{aligned} H &= \frac{p^2}{2L_0} + \Phi_{\mathcal{E}}(q) + \frac{1}{2}W'(p)q + \frac{1}{2}G'(q)p - e(t)q, \\ F_1 &= \frac{q^2}{2c} + cW(p), \quad F_2 = \frac{p^2}{2\ell} + \ell G(q), \end{aligned} \quad (20)$$

where  $c$  and  $\ell$  are constants with units of capacitance and inductance, respectively, and we choose  $W(p)$  and  $G(q)$  such that

$$W'(p) = \Psi'_{\mathcal{R}}(p), \quad G''(q) = \frac{1}{L_0}M(q). \quad (21)$$

**Proof.** Substituting in the expressions for the Itô drift in (15) leads to

$$\begin{aligned} v^q &= \frac{p}{L_0}, \\ v^p &= -\Phi'_{\mathcal{E}}(q) - W'(p) - G''(q)p + e(t). \end{aligned}$$

The system of equations has the desired form, and it remains only to choose  $W$  and  $G$  as in Eq. (21) to specify the required resistance and memristance. ■

To see explicitly what the stochastic differential equations (SDEs) for  $q_t$  and  $p_t$  look like for this model we first observe that

$$\begin{aligned} dq_t &= v^q(q_t, p_t) dt + \{q, F_1\}_t dB_t^1 + \{q, F_2\}_t dB_t^2 \\ &= v^q(q_t, p_t) dt + cW'(q_t) dB_t^1 + \frac{p_t}{\ell} dB_t^2, \end{aligned}$$

with a similar expression for the momentum. According to Theorem 5, the phase coordinate SDEs are therefore

$$dq_t = \frac{p_t}{L_0} dt + cW'(p_t) dB_t^1 + \frac{p_t}{\ell} dB_t^2, \quad (22)$$

$$dp_t = \left( -\Phi'_{\mathcal{E}}(q_t) - \Psi'_{\mathcal{R}}(p_t) - \frac{1}{L_0}M(q_t)p_t + e(t) \right) dt - \frac{q_t}{c} dB_t^1 - \ell G'(q_t) dB_t^2. \quad (23)$$

We note that the dissipation here is

$$\gamma(q, p) = W''(p) + G''(q) = \frac{\mathbb{R}(p) + M(q)}{L_0}, \quad (24)$$

which is consistent to Eq. (6). Moreover, we also have non-trivial fluctuations, and for instance,

$$dq_t dp_t = -(q_t W'(p_t) + p_t G'(q_t)) dt. \quad (25)$$

The fluctuations and dissipation balance out to preserve the Poisson structure.

It is a feature of canonical diffusions that the stochastic process  $p_t$  is no longer simply  $\frac{1}{L_0}\dot{q}_t$ . We would need a noise that was more singular than Wiener processes if we wanted to have  $dq_t = \frac{1}{L_0}p_t dt$  and still preserve the Poisson structure. In fact, to ensure that the  $dB_t^\alpha$  terms did not arise in the  $dq_t$  SDE, we would need that  $\{q, F_\alpha\} = 0$  for each  $\alpha$ . But this would imply that each  $F_\alpha$  is a function of  $q$  only, in which case the drift terms  $v^q$  and  $v^p$  would be purely Hamiltonian, since we also have  $\{p, F_\alpha\}, F_\alpha \equiv 0$ .

## B. Symplectic Noise Models

In<sup>38</sup> the concept of canonically conjugate Wiener processes was introduced. Here one extends the Poisson bracket structure to include the noise so that in addition to each Wiener process  $B_k$ , which we now relabel as  $Q_k$ , we have a conjugate process  $P_k$  so the collection  $(Q_k, P_k)$  have the statistics of independent Wiener processes, so that

$$\begin{aligned} dQ_j(t) dQ_k(t) &= \delta_{jk} dt = dP_j(t) dP_k(t), \\ dQ_j(t) dP_k(t) &= 0 = dP_j(t) dQ_k(t), \end{aligned}$$

but additionally satisfy

$$\{Q_j(t), P_k(s)\} = \Gamma \delta_{jk} \min(t, s), \quad (26)$$

where  $\Gamma > 0$ . We shall restrict to just a single canonical pair  $(Q, P)$  in the following, although the generalization is straightforward. For stochastic processes  $X_t$  and  $Y_t$  adapted to the filtration generated by the Wiener processes  $(Q_s, P_s)_{s \leq t}$ , we have the infinitesimal relations

$$\begin{aligned} \{X_t dQ_j(t), Y_t\} &= \{X_t, Y_t\} dQ_j(t), \\ \{X_t dP_j(t), Y_t\} &= \{X_t, Y_t\} dP_j(t), \end{aligned}$$

and

$$\begin{aligned} \{X_t dQ_j(t), Y_t dP_k(t)\} &= \{X_t, Y_t\} dQ_j(t) dP_k(t) + X_t Y_t \{dQ_j(t), dP_k(t)\} \\ &= \delta_{jk} \Gamma X_t Y_t dt. \end{aligned}$$

Let us now consider a diffusion on phase space driven by a canonical pair of Wiener processes:

$$dq_t = w^q(q_t, p_t) dt + \sigma^q(q_t, p_t) dQ_t + \varsigma^q(q_t, p_t) dP_t, \quad (27)$$

$$dp_t = w^p(q_t, p_t) dt + \sigma^p(q_t, p_t) dQ_t + \varsigma^p(q_t, p_t) dP_t, \quad (28)$$

with the coefficients assumed to be Lipschitz, etc., so as to guarantee the existence and uniqueness of solution. The main result, Theorem 2, of<sup>38</sup> is stated below.

**Lemma 6** *The diffusion (27)-(28) on phase space driven by a canonically conjugate pair of Wiener processes is canonical for the full Poisson brackets if the vector fields  $w, \sigma$  and  $\varsigma$  are all Hamiltonian, say  $w(\cdot) = \{\cdot, H\}$ ,  $\sigma(\cdot) = \{\cdot, F\}$  and  $\varsigma(\cdot) = \{\cdot, G\}$ , in which case we have*

$$df_t = (\mathcal{L}f)_t dt + \{f, F\}_t dQ(t) + \{f, G\}_t dP(t)$$

with generator

$$\mathcal{L} = \{\cdot, H\} + u \cdot \nabla + \frac{1}{2} \{\{\cdot, F\}, F\} + \frac{1}{2} \{\{\cdot, G\}, G\}, \quad (29)$$

where  $u$  is a vector field with

$$\nabla \cdot u = -\Gamma \{F, G\}. \quad (30)$$

In this case, the dissipation function on phase space is given by

$$\gamma(q, p) = \Gamma \{F, G\} + \left\{ \frac{\partial F}{\partial q}, \frac{\partial F}{\partial p} \right\} + \left\{ \frac{\partial G}{\partial q}, \frac{\partial G}{\partial p} \right\}. \quad (31)$$

As an example, we consider the linear  $LC$  model with the Hamiltonian  $H_0$  defined in Eq. (3) and

$$F = p, \quad G = -q.$$

By Eq. (30) we have  $\nabla \cdot u = -\Gamma$ . A particular solution is given by  $u^q = 0, u^p = -\Gamma p$ . As a result, the equations of motion are

$$dq_t = \frac{p_t}{L_0} dt + dQ_t, \quad (32)$$

$$dp_t = - \left( \frac{1}{C_0} q_t + \Gamma p_t - e(t) \right) dt + dP_t. \quad (33)$$

Finally, it is easy to show that

$$\gamma(q, p) \equiv \Gamma.$$

It is easy to see that  $dq_t dp_t = 0$ . However, noises  $(Q_t, P_t)$  do leave their imprint by adding the term  $-\Gamma p_t$  to the momentum.

In the above example, the proposed particular solution to  $\nabla \cdot u = -\Gamma$  can be given by

$$u \cdot \nabla = -\Gamma p \frac{\partial}{\partial p}.$$

The next two results show how to find the vector field  $u$  in general.

**Lemma 7** Given the family of twice-differentiable functions  $(F_\alpha, G_\alpha)$ , a particular solution to the equation

$$\nabla \cdot u = -\Gamma \sum_{\alpha} \{F_\alpha, G_\alpha\} \quad (34)$$

is given by the vector field  $u_0$  with components

$$\begin{aligned} u_0^q &= \frac{1}{2} \Gamma \sum_{\alpha} \left( G_\alpha \frac{\partial F_\alpha}{\partial p} - F_\alpha \frac{\partial G_\alpha}{\partial p} \right), \\ u_0^p &= \frac{1}{2} \Gamma \sum_{\alpha} \left( F_\alpha \frac{\partial G_\alpha}{\partial q} - G_\alpha \frac{\partial F_\alpha}{\partial q} \right). \end{aligned}$$

The general solution is then the particular solution plus an arbitrary Hamiltonian vector field.

**Proof.** We verify directly that

$$\begin{aligned} \frac{\partial u_0^q}{\partial q} &= -\frac{1}{2} \Gamma \sum_{\alpha} \{F_\alpha, G_\alpha\} + \frac{1}{2} \Gamma \sum_{\alpha} \left( G_\alpha \frac{\partial^2 F_\alpha}{\partial q \partial p} - F_\alpha \frac{\partial^2 G_\alpha}{\partial q \partial p} \right), \\ \frac{\partial u_0^p}{\partial p} &= -\frac{1}{2} \Gamma \sum_{\alpha} \{F_\alpha, G_\alpha\} - \frac{1}{2} \Gamma \sum_{\alpha} \left( G_\alpha \frac{\partial^2 F_\alpha}{\partial p \partial q} - F_\alpha \frac{\partial^2 G_\alpha}{\partial p \partial q} \right), \end{aligned}$$

so adding the terms gives

$$\frac{\partial u_0^q}{\partial q} + \frac{\partial u_0^p}{\partial p} = -\Gamma \sum_{\alpha} \{F_\alpha, G_\alpha\}$$

as required. ■

**Corollary 8** Given the family of functions  $F_\alpha, G_\alpha$ , assumed to be twice-differentiable, a particular solution to the equation (34) is given by the vector field  $u$  with components

$$u^q = -\Gamma \sum_{\alpha} F_\alpha \frac{\partial G_\alpha}{\partial p}, \quad u^p = \Gamma \sum_{\alpha} F_\alpha \frac{\partial G_\alpha}{\partial q}. \quad (35)$$

**Proof.** This can be verified by direct substitution into the equation (34). Alternatively, note that

$$u^q = u_0^q + \frac{\partial K}{\partial p}, \quad u^p = u_0^p - \frac{\partial K}{\partial q},$$

with  $K = \sum_{\alpha} F_\alpha G_\alpha$ . ■

The following result gives the symplectic stochastic model which realizes the velocity fields (10).

**Theorem 9** A stochastic evolution with Itô drift velocity  $(v^q, v^p)$  given by the phase velocity field (10) can be achieved by the stochastic canonical model driven by a pair of independent symplectic Wiener processes  $(Q_1, P_1)$  and  $(Q_2, P_2)$ , for the resistance and memristance components respectively, with the following choices:

$$H = \frac{p^2}{2L_0} + \Phi_{\mathcal{E}}(q) - e(t)q, \quad (36)$$

$$F_1 = \varrho(p), \quad G_1 = -q, \quad (37)$$

$$F_2 = p, \quad G_2 = -\mu(q), \quad (38)$$

where  $\varrho(p) = \frac{1}{\Gamma} \Psi'_{\mathcal{R}}(p)$  and  $\mu'(q) = \frac{1}{\Gamma L_0} M(q)$ . Here we take  $u$  to be the vector field obtained from Corollary 8. The symplectic stochastic model of the system (10) is

$$dq_t = v^q dt + \frac{\mathbb{R}(p)}{\Gamma L_0} dQ_1(t) + dQ_2(t), \quad (39)$$

$$dp_t = v^p dt - dP_1(t) - \frac{M'(q)}{\Gamma L_0} dP_2(t), \quad (40)$$

with Itô drift

$$v^q = \frac{1}{L_0} p, \quad (41)$$

$$v^p = -\Phi'_{\mathcal{E}}(q) + e(t) - \Psi'_{\mathcal{Z}}(p) - \frac{1}{L_0} M(q) p. \quad (42)$$

The negative divergence of the velocity field is

$$\gamma(q, p) = \frac{1}{L_0} (\mathbb{R}(p) + M(q)). \quad (43)$$

**Proof.** For the choices  $(F_\alpha, G_\alpha)$  in the statement of the Theorem, we have that the vector field obtained in Corollary 8 will be

$$u^q = 0, \quad u^p = -\Gamma(\varrho(p) + p\mu'(q)) \equiv -\Psi'_{\mathcal{Z}}(p) - \frac{p}{L_0} M(q).$$

The Itô drift may be calculated from the generator in Lemma 6, for instance by  $v^q = \mathcal{L}q$  we have

$$v^q = \frac{\partial H}{\partial p} + u^q + \frac{1}{2} \sum_{\alpha} \left( \left\{ \frac{\partial F_{\alpha}}{\partial p}, F_{\alpha} \right\} + \left\{ \frac{\partial G_{\alpha}}{\partial p}, G_{\alpha} \right\} \right)$$

however  $\sum_{\alpha} \left\{ \frac{\partial F_{\alpha}}{\partial p}, F_{\alpha} \right\} = \{\varrho'(p), \varrho(p)\} + \{p, 1\} = 0$  and  $\sum_{\alpha} \left\{ \frac{\partial G_{\alpha}}{\partial p}, G_{\alpha} \right\} \equiv 0$ , since the  $G_{\alpha}$  do not depend on  $q$ .

So we have

$$v^q = \frac{\partial H}{\partial p} + u^q = \frac{1}{L_0} p. \quad (44)$$

Similarly

$$v^p = -\frac{\partial H}{\partial q} + u^p - \frac{1}{2} \sum_{\alpha} \left\{ \frac{\partial F_{\alpha}}{\partial q}, F_{\alpha} \right\} - \frac{1}{2} \sum_{\alpha} \left\{ \frac{\partial G_{\alpha}}{\partial q}, G_{\alpha} \right\}.$$

but again  $\sum_{\alpha} \left\{ \frac{\partial F_{\alpha}}{\partial q}, F_{\alpha} \right\} \equiv 0$ , as the functions  $F_{\alpha}$  do not depend on  $p$ , and  $\sum_{\alpha} \left\{ \frac{\partial G_{\alpha}}{\partial q}, G_{\alpha} \right\} = \{1, q\} + \{\mu'(q), \mu(q)\} = 0$ . Therefore, we have

$$v^p = -\frac{\partial H}{\partial q} + u^p = -\Phi'_{\mathcal{E}}(q) + e(t) - \Psi'_{\mathcal{Z}}(p) - \frac{1}{L_0} M(q) p. \quad (45)$$

Equations (44)-(45) are the desired form of the Itô drift (41)-(42). The stochastic evolution (39)-(40) follows immediately. Finally, We note that the dissipation will be

$$\gamma(q, p) = \Gamma \sum_{\alpha} \{F_{\alpha}, G_{\alpha}\} = \Gamma(\varrho'(p) + \mu'(q)) = \frac{1}{L_0} (\mathbb{R}(p) + M(q)) \quad (46)$$

which is the correct form, (recall that  $\Psi''_{\mathcal{Z}}(p) \equiv \mathbb{R}(p)/L_0$ , cf. Eq. (6)). Eq. (43) is established. ■

#### IV. QUANTUM STOCHASTIC MODELS

To quantize, we replace  $q$  and  $p$  by operators satisfying the commutation relations

$$[\hat{q}, \hat{p}] = i\hbar.$$

In the case of an  $LC$  circuit driven by a classical emf  $e$  we have the Hamiltonian

$$\hat{H}_0 = \frac{1}{2L_0} \hat{p}^2 + \frac{1}{2C_0} \hat{q}^2 - e(t)\hat{q}.$$

We may introduce annihilation operators, defined as

$$\hat{a} = (2\hbar\omega_0 L_0)^{-1/2} (\omega_0 L_0 \hat{q} + ip)$$

and so  $\hat{q} = \sqrt{\frac{\hbar\omega_0 C_0}{2}} (\hat{a} + \hat{a}^*)$ ,  $\hat{p} = i\sqrt{\frac{\hbar\omega_0 L_0}{2}} (\hat{a}^* - \hat{a})$ .

The Hamiltonian is then

$$\hat{H}_0 = \hbar\omega_0 \left( \hat{a}^* \hat{a} + \frac{1}{2} \right) - \sqrt{\frac{\hbar\omega_0 C_0}{2}} (\hat{a} + \hat{a}^*) e(t). \quad (47)$$

While the formalism is identical to the quantum mechanical oscillator, we may give an entirely different interpretation physically. The state  $|n\rangle$  for the quantized circuit describes the situation where there are  $n$  quanta (photons) in the circuit. In this case the number operator  $\hat{N} = \hat{a}^* \hat{a}$  is the observable corresponding to the number of photons in the circuit. Note that we may have zero photons, and this corresponds to a physical state  $|0\rangle$ . The source term in the Hamiltonian is proportional to  $e(t)$  and involves the creation  $\hat{a}^*$  and annihilation  $\hat{a}$  of photons.

We are in the situation that there exists a theory of quantum stochastic integration generalizing the Itô calculus. The mathematical theory was developed in 1984 by Hudson and Parthasarathy,<sup>23</sup> though was also derived on a physical basis for modelling physical noise in quantum photonic models by Gardiner and Collett in 1985,<sup>42</sup>. In essence, we have a quantum system with underlying Hilbert space  $\mathfrak{h}$  which is in interaction with an infinite environment modelled as a quantum field,<sup>23,43,26</sup>. Without going too far into the details we have quantum white noise fields  $\hat{a}_\alpha(t)$  which are operators satisfying commutation relations

$$[\hat{a}_\alpha(t), \hat{a}_\beta^*(s)] = \delta_{\alpha\beta} \delta(t-s).$$

We also fix the vacuum state which is the unique vector such that  $\hat{a}_\alpha(t)|\Omega\rangle = 0$ .

The integrated fields  $\hat{A}_\alpha(t) = \int_0^t \hat{a}_\alpha(s) ds$  and  $\hat{A}_\alpha^*(t) = \int_0^t \hat{a}_\alpha^*(s) ds$  are well defined operators on Bose Fock space  $\mathfrak{F}$  satisfying

$$[\hat{A}_\alpha(t), \hat{A}_\beta^*(s)] = \delta_{\alpha\beta} \min\{t, s\}.$$

The quadrature processes are defined by

$$\hat{Q}_\alpha(t) = \hat{A}_\alpha(t) + \hat{A}_\alpha^*(t), \quad \hat{P}_\alpha(t) = \frac{1}{i} (\hat{A}_\alpha(t) - \hat{A}_\alpha^*(t)),$$

both of which are self-commuting operator-valued processes, that is,

$$[\hat{Q}_\alpha(t), \hat{Q}_\beta(s)] = 0 = [\hat{P}_\alpha(t), \hat{P}_\beta(s)].$$

In the vacuum state of the noise, they both have the statistics of a standard Wiener process. However, the quadrature processes do not commute and we have instead

$$[\hat{Q}_\alpha(t), \hat{P}_\beta(s)] = 2i\delta_{\alpha\beta} \min\{t, s\}. \quad (48)$$

Hudson and Parthasarathy developed a theory of quantum stochastic integration wrt. these processes. This involves the following nontrivial product of Itô increments

$$d\hat{A}_\alpha(t) d\hat{A}_\beta^*(t) = \delta_{\alpha\beta} dt.$$

They show that a quantum stochastic process  $\hat{U}(t)$  can be defined on the joint system+noise space  $\mathfrak{h} \otimes \mathfrak{F}$  by

$$d\hat{U}(t) = \left( \sum_\alpha \hat{L}_\alpha d\hat{A}_\alpha^*(t) - \sum_\alpha \hat{L}_\alpha^* d\hat{A}_\alpha(t) - \frac{1}{2} \sum_\alpha \hat{L}_\alpha^* \hat{L}_\alpha dt - \frac{i}{\hbar} \hat{H}_0 dt \right) \hat{U}(t),$$

$$\hat{U}(0) = I_{\mathfrak{h}} \otimes I_{\mathfrak{F}},$$

and that the process is unitary. (Technically they require the system operators  $\hat{H}_0 = \hat{H}_0^*$  and  $\hat{L}_\alpha$  to be bounded, however, the theory extends to unbounded coefficients). The dynamical evolution of a system observable  $\hat{X}$  is given by

$$j_t(\hat{X}) = \hat{U}(t)^* (\hat{X} \otimes I_{\mathfrak{F}}) \hat{U}(t),$$

and one can deduce the following dynamical equations of motion:

$$dj_t(\hat{X}) = \sum_\alpha j_t([\hat{X}, \hat{L}_\alpha]) d\hat{A}_\alpha^*(t) + \sum_\alpha j_t([\hat{L}_\alpha^*, \hat{X}]) d\hat{A}_\alpha(t) + j_t(\mathcal{L}\hat{X}) dt \quad (49)$$

where the generator takes the form

$$\mathcal{L}\hat{X} = \frac{1}{2} \sum_{\alpha} [\hat{L}_{\alpha}^*, \hat{X}] \hat{L}_{\alpha} + \frac{1}{2} \sum_{\alpha} \hat{L}_{\alpha}^* [\hat{X}, \hat{L}_{\alpha}] + \frac{1}{i\hbar} [\hat{X}, \hat{H}_0].$$

This is the well-known GKS-Lindblad generator from the theory of quantum dynamical semi-groups,<sup>24, 25, 45, 46</sup>. The Hudson-Parthasarathy theory therefore offers a unitary dilation of such quantum Markov semi-groups.

We now show how to realize the analogue of the Itô drift corresponding to the phase velocity field (10). Note that in this case the terms  $pM(q)$  is ambiguous as  $\hat{p}$  and  $M(\hat{q})$  generally do not commute. We naturally interpret this as the symmetrically ordered term  $\frac{1}{2}\hat{p}M(\hat{q}) + \frac{1}{2}M(\hat{q})\hat{p}$ . Motivated by the previous Theorem 9 giving the construction of such a diffusion in the classical setting with two independent canonical pairs of symplectic Wiener noise, we now establish the corresponding quantum analogue.

**Theorem 10** *Given the function  $\Phi_{\mathcal{E}}$ ,  $\Psi_{\mathcal{R}}$  and  $M$  we obtain the Itô drift terms*

$$\begin{aligned} v^q &= \mathcal{L}\hat{q} = \frac{\hat{p}}{L_0}, \\ v^p &= \mathcal{L}\hat{p} = -\Phi'_{\mathcal{E}}(\hat{q}) - \Psi'_{\mathcal{R}}(\hat{p}) - \frac{1}{2L_0} (M(\hat{q})\hat{p} + \hat{p}M(\hat{q})) + e(t), \end{aligned}$$

for the choice  $\hat{H} = \hat{H}_0 + \hat{K}$  where  $\hat{H}_0$  is the Hamiltonian (47) and

$$\hat{K} = \frac{1}{2} [f(\hat{p})\hat{q} + \hat{q}f(\hat{p})] + \frac{1}{2} [\hat{p}g(\hat{q}) + g(\hat{q})\hat{p}]$$

and coupling terms

$$\hat{L}_1 = \hat{q} + i\frac{1}{\hbar}f(\hat{p}), \quad (50)$$

$$\hat{L}_2 = \frac{1}{\hbar}g(\hat{q}) + i\hat{p}, \quad (51)$$

with the functions  $f$  and  $g$  given by  $f(\hat{p}) = \frac{1}{2}\Psi'_{\mathcal{R}}(\hat{p})$ ,  $g'(\hat{q}) = \frac{1}{2L_0}M(\hat{q})$ . The quantum stochastic model of the system (10) is then

$$dj_t(\hat{q}) = j_t(v^q)dt - j_t(f'(\hat{p}))d\hat{Q}_1(t) - \hbar d\hat{Q}_2(t), \quad (52)$$

$$dj_t(\hat{p}) = j_t(v^p)dt - \hbar d\hat{P}_1(t) - j_t(g'(\hat{q}))d\hat{P}_2(t). \quad (53)$$

**Proof.** We have for instance

$$\frac{1}{i\hbar} [\hat{q}, \hat{H}] = \frac{\hat{p}}{L_0} + \frac{1}{2} [f'(\hat{p})\hat{q} + \hat{q}f'(\hat{p})] + g(\hat{q}),$$

and that

$$\begin{aligned} \frac{1}{2} [\hat{L}_1^*, \hat{q}] \hat{L}_1 + \frac{1}{2} \hat{L}_1^* [\hat{q}, \hat{L}_1] &= -\frac{f'(\hat{p})\hat{q} + \hat{q}f'(\hat{p})}{2}, \\ \frac{1}{2} [\hat{L}_2^*, \hat{q}] \hat{L}_2 + \frac{1}{2} \hat{L}_2^* [\hat{q}, \hat{L}_2] &= -g(\hat{q}) \end{aligned}$$

which combine to give  $\mathcal{L}\hat{q} = \frac{\hat{p}}{L_0}$ . Similarly, we have

$$\frac{1}{i\hbar} [\hat{p}, \hat{H}] = -\Phi'_{\mathcal{E}}(\hat{q}) + e(t) - f(\hat{p}) - \frac{1}{2} [g'(\hat{q})\hat{p} + \hat{p}g'(\hat{q})],$$

and

$$\begin{aligned} \frac{1}{2} [\hat{L}_1^*, \hat{p}] \hat{L}_1 + \frac{1}{2} \hat{L}_1^* [\hat{p}, \hat{L}_1] &= -f(\hat{p}) \\ \frac{1}{2} [\hat{L}_2^*, \hat{p}] \hat{L}_2 + \frac{1}{2} \hat{L}_2^* [\hat{p}, \hat{L}_2] &= -\frac{g'(\hat{q})\hat{p} + \hat{p}g'(\hat{q})}{2} \end{aligned}$$

which combine to give

$$\mathcal{L}\hat{p} = -\Phi'_{\mathcal{E}}(\hat{q}) + e(t) - 2f(\hat{p}) - g'(\hat{q})\hat{p} - \hat{p}g'(\hat{q})$$



so that the choices  $f(\hat{p}) = \frac{1}{2}\Psi'_{\mathcal{R}}(\hat{p})$  and  $g'(\hat{q}) = \frac{1}{2L_0}M(\hat{q})$  give the desired form. ■

If we choose  $\Gamma = 2$  in Eq. (26), (notice that there exists factor 2 for the canonical commutator in Eq. (48)) then Eqs. (37)-(38) become

$$F_1 = \frac{1}{2}\Psi'_R(p), \quad F_2 = p,$$

$$G_1 = q, \quad G_2 = \frac{1}{2L_0}M'(q).$$

As a result,

$$G_1 + \frac{i}{\hbar}F_1 = q + \frac{i}{\hbar}\frac{1}{2}\Psi'_R(p), \quad (54)$$

$$\frac{1}{\hbar}G_2 + iF_2 = \frac{1}{\hbar}\frac{1}{2L_0}M'(q) + ip. \quad (55)$$

Replacing  $q$  and  $p$  by  $\hat{q}$  and  $\hat{p}$  respectively in Eqs. (54)-(55) we get exactly operators  $\hat{L}_1$  and  $\hat{L}_2$  in Eqs. (50)-(51).

## V. APPROXIMATIONS

In Sections III and IV we have proposed stochastic models for dissipative electronic circuits. A natural question to ask is whether the noisy dynamics can be approximated by systems which are lossless, or more specifically, Hamiltonian. In this section we give an affirmative answer to this question.

We remark that we may readily find approximation schemes to Wiener noise. For instance, it is possible to construct processes  $B^{(n)}(t)$  that are continuously differentiable in the time  $t$  variable and which converge almost surely to a Wiener process  $B(t)$  uniformly in  $t$  in compacts. The random Hamiltonian

$$\Upsilon^{(n)}(q, p) = H(q, p) + F(q, p)B^{(n)}(t),$$

generates the following equations of motion for the state  $x^{(n)}(t) = (q^{(n)}(t), p^{(n)}(t))$

$$\frac{d}{dt}q^{(n)}(t) = \frac{\partial H}{\partial p}|_{x^{(n)}(t)} + \frac{\partial F}{\partial p}|_{x^{(n)}(t)}B^{(n)}(t),$$

$$\frac{d}{dt}p^{(n)}(t) = -\frac{\partial H}{\partial q}|_{x^{(n)}(t)} - \frac{\partial F}{\partial q}|_{x^{(n)}(t)}B^{(n)}(t),$$

By the Wong-Zakai theorem<sup>44</sup>, the process  $x^{(n)}(t)$  then converges uniformly in  $t$  on compact almost surely to the solution of the Stratonovich SDEs

$$dq(t) = \frac{\partial H}{\partial p}|_{x(t)}dt + \frac{\partial F}{\partial p}|_{x(t)} \circ dB(t),$$

$$dp(t) = -\frac{\partial H}{\partial q}|_{x(t)}dt - \frac{\partial F}{\partial q}|_{x(t)} \circ dB(t),$$

This is the single noise case of Theorem 2. The multiple noise case is the obvious generalization.

In order to describe the symplectic noise, we set about developing an approximation using lossless circuits. In particular, a continuous transmission line may be approximated by shunted LC circuits.

Let  $q_k, p_k$  be the canonical variables satisfying the relations  $\{q_j, q_k\} = 0 = \{p_j, p_k\}$  and  $\{q_j, p_k\} = \delta_{jk}$ . For  $t > 0$  and integer  $N > 0$ , let us set

$$Q^{(N)}(t) = \frac{1}{\sqrt{N}} \sum_k^{\lfloor Nt \rfloor} q_k, \quad P^{(N)}(t) = \frac{1}{\sqrt{N}} \sum_k^{\lfloor Nt \rfloor} p_k,$$

where  $\lfloor x \rfloor$  means rounding down to the nearest integer value. We evidently have  $\{Q^{(N)}(t), P^{(N)}(t)\} = \frac{1}{N} \lfloor Nt \rfloor$  which evidently converges to  $t$  as  $N \rightarrow \infty$ . Let us take the energy of the  $k$ th circuit to be  $\frac{1}{2L_0}p_k^2 + \frac{1}{2C_0}q_k^2$  and consider the canonical ensemble corresponding to circuits in thermal equilibrium at temperature  $T$ . (In practice, for  $t$  and  $N$  finite we only need a finite number in the assembly.) The pairs of variables  $(q_k, p_k)$  are then independent and identically distributed with mean zero and variance  $Var(q_k) = C_0k_B T$ ,  $Var(p_k) = L_0k_B T$  and covariance  $Cov(q_k, p_k) = 0$ , where  $k_B$  is the Boltzmann constant. By the central limit effect we see that the pair  $(Q^{(N)}, P^{(N)})$  converge to independent

Wiener processes with temperature dependent variances (which we can always absorb). The result is a limit symplectic noise obtained as a limit of thermalized lossless oscillator circuits.

There is a related result for quantum stochastic evolutions. We start with the Schrödinger equation

$$i\hbar \frac{d}{dt} \hat{U}^{(N)} = \hat{\Upsilon}^{(N)}(t) \hat{U}^{(N)}(t)$$

with the time dependent Hamiltonian

$$\hat{\Upsilon}^{(N)}(t) = \hat{E} \otimes \hat{a}^{(N)}(t)^* + \hat{E}^* \otimes \hat{a}^{(N)}(t) + \hat{H},$$

where we have regular reservoir field operators satisfying commutation relations

$$[\hat{a}^{(N)}(t), \hat{a}^{(N)}(s)^*] = g^{(N)}(t-s).$$

We fix the vacuum state  $|\Omega\rangle$  for the reservoir so that  $\hat{a}^{(N)}(t)|\Omega\rangle = 0$ . In the limit  $N \rightarrow \infty$  we assume that  $g^{(N)}(\tau) \rightarrow \delta(\tau)$  in distribution with  $\int_0^\infty g^{(N)}(\tau) d\tau = \frac{1}{2}$ .

Then we find the limit

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \phi_1 \otimes e^{\int f_1(u) \hat{a}^{(N)}(u)^* du} \Omega | \hat{U}^{(N)}(t) \phi_2 \otimes e^{\int f_2(v) \hat{a}^{(N)}(v)^* dv} \Omega \rangle \\ = \langle \phi_1 \otimes e^{\int f_1(u) dA^*(u)} \Omega | \hat{U}(t) \phi_2 \otimes e^{\int f_2(v) dA^*(v)} \Omega \rangle \end{aligned} \quad (56)$$

for arbitrary  $\phi_1, \phi_2$  in the system Hilbert space  $\mathfrak{h}$  and  $L^2$  functions  $f_1, f_2$  where  $\hat{U}$  is the solution to the quantum stochastic differential equation (QSDE)

$$d\hat{U}(t) = \left\{ \hat{L} d\hat{A}^*(t) - \hat{L}^* d\hat{A}(t) - \left( \frac{1}{2} \hat{L}^* \hat{L} + \frac{i}{\hbar} \hat{H} \right) dt \right\} \hat{U}(t), \quad (57)$$

with  $\hat{L} = \hat{E}/i\hbar$ . (In both cases we start with the initial condition that the unitary is the identity.) This limit is clearly the single noise channel quantum stochastic evolution considered in Section IV. A similar result holds for the Heisenberg equations, as well as generalizations to thermal states of the reservoir.

## VI. CONCLUSION

In this paper we have proposed stochastic models for electric circuits that may contain memristors, both the classical and quantum versions of noisy dynamics have been obtained. Preservation of the canonical structure was used as a guiding principle and the resulting theory allows for approximations schemes using Hamiltonian systems. Future research includes the application of the proposed stochastic models to more general memristive electric circuits and making deeper connections with the underlying statistical mechanical derivations.

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## REFERENCES

- [1] J.C. Willems, *Archive Rational Mech. Anal.*, vol. 45, pp. 321-351 (1972).
- [2] J.C. Willems, *Archive Rat. Mech. Anal.*, vol. 45, pp. 352-393 (1972).
- [3] R.W. Brockett and J.C. Willems, "Stochastic control and the second law of thermodynamics," In *Proc. IEEE Conf. Decision Control*, San Diego, CA, pp. 1007-1011 (1978).
- [4] L.O. Chua, *IEEE Trans. Circuits and Systems I: Fundamental Theory and Applications*, vol. 46, no. 1, pp. 71-82 (1999).

- [5] A. van der Schaft, *L<sup>2</sup> Gain and Passivity Techniques in Nonlinear Control*, (Communications and Control Engineering Series, Springer, London 2000).
- [6] A.M. Bloch, “Dissipative dynamics in conservative classical and quantum systems,” In *Mathematical Systems Theory in Biology, Communications, Computation, and Finance*, (Volume 134 of the series The IMA Volumes in Mathematics and its Applications, pp. 121-155, 2003).
- [7] B. Brogliato, R. Lozano, B. Maschke, and O. Egeland, *Dissipative Systems Analysis and Control. Theory and Applications*, (Springer-Verlag London, second edition, 576-581 pages, 2007).
- [8] F. Verstraete, M.M. Wolf, and J. I. Cirac, *Nature Physics*, vol. 5, pp. 633-636 (2009).
- [9] M.R. James & J.E. Gough, *IEEE Trans. Automat. Contrl.*, vol. 55, pp. 1806-1821 (2010).
- [10] G. Zhang and M.R. James, *IEEE Trans. Automat. Contrl.*, vol. 56, no. 7, pp. 1535-1550 (2011)
- [11] J. Delvenne and H. Sandberg, “Towards a thermodynamics of control: Entropy, energy and Kalman filtering,” In *Proc. IEEE Conf. Decision Control*, pp. 3109-3114 (2013).
- [12] R. Zwanzig, *J. Stat. Phys.*, vol. 9, no. 3, pp. 215-220 (1973).
- [13] H. Spohn, *J.L. Lebowitz, Adv. Chem. Phys.*, vol. 38, pp. 109-142 (1978).
- [14] G. Ford and M. Kac, *J. Stat. Phys.*, vol. 46, no. 5/6, pp. 803-810 (1987).
- [15] A. Caldeira and A. Leggett, *Physic A*, vol. 121, no. 3, pp. 587-616 (1983).
- [16] H. Spohn, *Large Scale Dynamics of Interacting Particles*, (Texts and Monographs in Physics. Springer, Berlin, 1991).
- [17] A.M. Bloch, P.S. Krishnaprasad, J.E. Marsden, T.S. Ratiu, *Commun. Math. Phys.*, vol. 175, no. 1, pp. 1-42 (1996).
- [18] J. L. Lebowitz, *Physica A*, vol. 263, pp. 516-527 (1999).
- [19] M. Le Bellac, F. Mortessagne, and G.G. Batrouni, *Equilibrium and Non-Equilibrium Statistical Thermodynamics*, Cambridge University Press, 1 edition, 2010.
- [20] H. Sandberg, J. Delvenne, and J.C. Doyle, “The Statistical Mechanics of Fluctuation- Dissipation Theorem and Measurement Back Actions”, in *Proceedings of the 2007 American Control Conference*, (New York City, New York, pp. 1033-038, 2007).
- [21] C. Gardiner, *Stochastic Methods: A Handbook for the Natural and Social Sciences, 4th ed*, Springer Series in Synergetics, Springer-Verlag Berlin Heidelberg, 2009.
- [22] H. Sandberg, J. Delvenne, and J.C. Doyle, *IEEE Trans. Aut. Control*, vol. 56, no. 2, pp. 293-308 (2011).
- [23] R.L. Hudson and K.R. Parthasarathy, *Commun. Math. Phys.*, vol. 93, pp. 301-323 (1984).
- [24] G. Lindblad, *Commun. Math. Phys.*, vol. 48, no. 2, pp. 119-130 (1976).
- [25] V. Gorini, A. Kossakowski, and E.C.G. Sudarshan, *J. Math. Phys.*, vol. 17, no. 5, pp. 821-825 (1976).
- [26] C.W. Gardiner and P. Zoller, *Quantum Noise*, (Springer, Berlin, 2004).
- [27] L.O. Chua, *IEEE Trans. Circuit Theory*, vol.18, no.5, pp. 507-519 (1971).
- [28] L.O. Chua and S.M. Kang, *Proceedings of IEEE*, vol. 64, pp. 209-223 (1976).
- [29] D. Strukov, G.S. Snider, D.R. Stewart, R.S. Williams, *Nature*, vol. 453, pp. 80-83 (2008).
- [30] D. Jeltsema, “Memory Elements: A Paradigm Shift in Lagrangian Modeling of Electrical Circuits”, In *the 7th Vienna International Conference on Mathematical Modelling (MATHMOD 2012)*, vol. 7, pp. 445-450 (2012).
- [31] P.M. Sheridan, F. Cai, C. Du, W. Ma, Z. Zhang, W.D. Lu *Nature Nanotechnology*, doi:10.1038/nnano.2017.83 (2017).
- [32] M. Di Ventra, Y., Pershin, and L.O. Chua, *Proceedings of the IEEE*, vol. 97, no. 10, pp. 1717-724 (2009).
- [33] H.H.C. Iu, D. Yu, A.L. Fitch, V. Sreeram and H. Chen, *IEEE Trans. on Circuits and Systems Part I*, vol. 58, no. 6, pp. 1337-1344 (2011).
- [34] G.Z. Cohen, Y.V. Pershin, and M. Di Ventra, *Phys. Rev. B*, vol. 85, 165428 (2012).
- [35] D.S. Yu, Y. Liang, H.H.C. Iu and L.O. Chua, *IEEE Trans. Circuits and Systems Part II*, vol. 61, no.10, pp. 758-762 (2014).
- [36] J.-M. Bismut, *Mechanique Aléatoire*, Lecture Notes in Mathematics, vol. 866, Springer-Verlag (1981).
- [37] J.E. Gough, *Journal of Mathematical Physics*, vol. 40, no. 6, pp. 2805-2815 (1999).
- [38] J.E. Gough, *Journ. Stat. Physics*, vol. 160, pp. 1709-1720 (2015).
- [39] L. Accardi, A. Frigerio, Y.G. Lu, *Comm. Math. Phys.* 131, no. 3, 53770 (1990).
- [40] L. Accardi, J. Gough, Y.G. Lu, *Rep. Math. Phys.* 36, no. 2-3, 15587 (1995)
- [41] B. Oksendal, *Stochastic Differential Equations: An Introduction with Applications*, (Universitext, Springer; 6th edition, 2010).
- [42] C.W. Gardiner and M.J. Collett, *Phys. Rev. A*, vol. 31, no. 6, pp. 3761-3774 (1985).
- [43] K. Parthasarathy, *An Introduction to Quantum Stochastic Calculus*, (Berlin, Germany: Birkhauser, 1992).
- [44] E. Wong and M. Zakai, *Int.. J. Eng. Sci.*, vol. 3, pp. 213-229, 1965.
- [45] J. Gough, *Commun. Math. Phys.*, vol. 254, pp. 489-512 (2005).
- [46] J. Gough, *Journ. Math. Phys.*, vol. 47, 113509 (2006).