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Evolution-free Hamiltonian parameter estimation through Zeeman markers - Supplementary Material

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The essential equation of this article is Eq. (2) which we derive here. Its derivation is elementary but cumbersome and follows standard arguments for Green's functions of rank one perturbations [1, Chapter 6] up to Eq. (7). The remainder of the derivation is analogous to one of Gladwell's inverse problems in vibration [2, Section 4.5]. For completeness we provide a full derivation.

Consider an eigenvalue e' and eigenvector of $|e'\rangle$ of H'. We have the eigenequation

$$H|e'\rangle + f|1\rangle\langle 1|e'\rangle = e'|e'\rangle. \tag{1}$$

Express $|e'\rangle$ in the eigenbasis $\{|e_n\rangle\}$ of H with eigenvalues e_n as

$$|e'\rangle = \sum_{n=1}^{N} \alpha_n |e_n\rangle, \tag{2}$$

which yields

$$f|1\rangle\langle 1|e'\rangle = \sum_{n=1}^{N} (e' - e_n)\alpha_n |e_n\rangle. \tag{3}$$

Upon multiplication with $\langle e_m|$ we obtain

$$\alpha_m = \frac{f\langle e_m|1\rangle\langle 1|e'\rangle}{(e' - e_m)} \tag{4}$$

where we assumed that the spectrum of H and H' have no overlap (this is true for almost all values of f). From Eq. (2) upon multiplication with $\langle 1|$ we arrive at

$$\langle 1|e'\rangle = \sum_{n=1}^{N} \frac{f\langle e_n|1\rangle\langle 1|e'\rangle}{(e' - e_n)} \langle 1|e_n\rangle. \tag{5}$$

Since $\langle 1|e'\rangle \neq 0$ [3], this is equivalent to

$$0 = 1 - \sum_{n=1}^{N} \frac{f|\langle e_n|1\rangle|^2}{(e' - e_n)} \tag{6}$$

or through expansion with $\prod_{m=1}^{N}(e' - e_m)$

$$0 = \frac{\prod_m (e' - e_m) - \sum_n f|\langle e_n|1\rangle|^2 \prod_{m\neq n}(e' - e_m)}{\prod_m (e' - e_m)}. \tag{7}$$

This holds for any eigenvalue e' of H'. The polynomial in x given by the numerator

$$P(x) = \prod_{m=1}^{N} (x - e_m) - \sum_{n=1}^{N} f|\langle e_n|1\rangle|^2 \prod_{m\neq n} (x - e_m) \tag{8}$$

and leading term x^N must therefore factorise as

$$P(x) = \prod_n (x - e'_n), \tag{9}$$

where e'_n are the eigenvalues of H'. We can thus write

$$\begin{aligned} 1 - \sum_{n=1}^{N} &\frac{f|\langle e_n|1\rangle|^2}{(x - e_n)} \\ &= \frac{\prod_m (x - e_m) - \sum_n f|\langle e_n|1\rangle|^2 \prod_{m\neq n}(x - e_m)}{\prod_m (x - e_m)} \\ &= \frac{\prod_n (x - e'_n)}{\prod_m (x - e_m)}. \end{aligned} \tag{10}$$

Multiply with $(x - e_k)$ to obtain

$$(x - e_k) - \sum_{n=1}^{N} \frac{f|\langle e_n|1\rangle|^2 (x - e_k)}{(x - e_n)} = \frac{(x - e_k)\prod_n (x - e'_n)}{\prod_m (x - e_m)} \tag{11}$$

and perform the limit $x \to e_k$ such that

$$-f|\langle e_k|1\rangle|^2 = \frac{\prod_n (e_k - e'_n)}{\prod_{m\neq k}(e_k - e_m)} = (e_k - e'_k) \prod_{m\neq k} \frac{(e_k - e'_m)}{(e_k - e_m)}. \tag{12}$$

Finally we arrive at

$$|\langle e_k|1\rangle|^2 = (e'_k - e_k)/f \prod_{m\neq k} \frac{(e_k - e'_m)}{(e_k - e_m)} \tag{13}$$

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