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McDonough, Thomas; Mavron, V. C.; Tonchev, V. D.

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On affine designs and Hadamard designs with line spreads

V.C. Mavron^a T.P. McDonough^a V.D. Tonchev^{b,1}

^a*Department of Mathematics, The University of Wales Aberystwyth, SY23 3BZ, U.K.*

^b*Department of Mathematical Sciences, Michigan Technological University, Houghton, Michigan 49931, U.S.A.*

Abstract

Rahilly [10] described a construction that relates any Hadamard design H on $4^m - 1$ points with a line spread to an affine design having the same parameters as the classical design of points and hyperplanes in $AG(m, 4)$. Here it is proved that the affine design is the classical design of points and hyperplanes in $AG(m, 4)$ if, and only if, H is the classical design of points and hyperplanes in $PG(2m - 1, 2)$ and the line spread is of a special type. Computational results about line spreads in $PG(5, 2)$ are given. One of the affine designs obtained has the same 2-rank as the design of points and planes in $AG(3, 4)$, and provides a counter-example to a conjecture of Hamada [6].

Dedicated to Jennifer Seberry on her 60th birthday

1 Introduction

The connection between Hadamard matrices and symmetric or affine designs is well-known, see [12] for example. In this paper, we describe two constructions, based on one of Rahilly [10], that relates affine 2-designs of class number 4 with symmetric Hadamard 2-designs possessing spreads of lines where each line has size 3. In Section 2, we show that the affine design is the classical design of points and hyperplanes in the affine geometry $AG(m, 4)$ of dimension m over the field of 4 elements if, and only if, the Hadamard design is the classical design of points and hyperplanes in the projective geometry $PG(2m - 1, 2)$

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of dimension $2m - 1$ over the field of 2 elements and the line spread is of a special type which we call *normal* and define in 2.5 below. In Section 3, we give an indication of the variety of affine designs produced by this construction using line spreads from the projective geometry $PG(5, 2)$ by summarizing computational results. In particular, we establish the falsity of Hamada's conjecture that, among the 2-designs with the same parameters as the 2-design of points and t -subspaces of a projective or affine geometry over a field of characteristic p , the designs whose incidence matrices are of minimum p -rank are isomorphic to the given design of points and t -subspaces of a projective or affine geometry, by exhibiting a non-geometric affine 2-(64,16,5) design, whose incidence matrix has 2-rank 16. Although it has not yet been established that 16 is the minimum 2-rank of the incidence matrices of 2-(64,16,5) designs, any 2-(64,16,5) designs of 2-rank less than 16 which might be discovered in the future will necessarily be non-geometric.

The basic design theory needed for this paper may be found, for example, in [1], [3], [11]. We give an outline here.

Let $\Pi = (\mathcal{P}, \mathcal{B}, I)$ be a design with point set \mathcal{P} , block set \mathcal{B} and incidence relation $I \subseteq \mathcal{P} \times \mathcal{B}$. Where convenient, as is customary, we may identify a block with the subset of points incident with it, and regard incidence as set-theoretic inclusion. Π is a t -(v, k, λ) design if \mathcal{P} and \mathcal{B} are finite, $|\mathcal{P}| = v$ and $|B| = k$ for all $B \in \mathcal{B}$ and any t -subset of \mathcal{P} is contained in λ blocks. A design is *symmetric* if $|\mathcal{P}| = |\mathcal{B}|$. A t -(v, k, λ) design is *resolvable* if \mathcal{B} has a partition, called a *parallelism*, into *parallel classes* of blocks such that two distinct blocks in the same parallel class are always disjoint and every point belongs to exactly one block from each parallel class. If, further, any two non-parallel blocks (i.e. blocks from different parallel classes) meet in a constant number $\mu > 0$ of points, then Π is *affine resolvable* or simply, *affine*. It is easy to see that each parallel class consists of $m = v/k$ blocks, where we call m the *class number* of the affine design, and $\mu = k/m$. From the definition, it follows that the parallelism in an affine design is unique.

The dual design Π^* of a design $\Pi = (\mathcal{P}, \mathcal{B}, I)$ is defined to be the design $\Pi^* = (\mathcal{B}, \mathcal{P}, I^*)$, where $(x, y) \in I$ if and only if $(y, x) \in I^*$. The *line* joining two distinct points P and Q in a t -design is the intersection of all blocks which contain both P and Q . If $t \geq 2$, the maximum size of a line is $(v - 1)/k + 1$ and a line has this maximal size if, and only if, every block which does not contain it meets it in exactly one point. The set of blocks that contain the intersection of two distinct given blocks in Π forms a line in the dual design Π^* .

The parameters of a symmetric 2-(v, k, λ) design satisfy the equation $\lambda(v - 1) = k(k - 1)$. A symmetric 2-design is said to be *Hadamard* if $v = 2k + 1$. It is well-known that a Hadamard design exists if, and only if, a Hadamard matrix

of order $2k + 2$ exists or, equivalently, a 3 - $(2k + 2, k + 1, \frac{1}{2}(k - 1))$ design, which is necessarily affine, exists. The size of a line in a Hadamard 2-design is at most $(v - 1)/k + 1 = 3$ since $v = 2k + 1$. So, a line of size 3 has maximum size and any block either contains it or meets it in exactly one point.

A set \mathcal{L} of non-empty point subsets of a design is a *spread* if it partitions the point set of the design. In a resolvable design, a parallel class of blocks is a spread of blocks.

2 Affine designs and spreads in symmetric designs

Rahilly [10] established a connection between affine 2-designs with class number 4 (the size of a parallel class) and Hadamard 2-designs with line spreads. This construction is generalized in Al-Kenani and Mavron [9]. Here, we present Rahilly's construction in a different but simpler and more transparent form that is suitable for the exposition of the results of this paper.

Construction 2.1 Let Γ be an affine 2 - $(16\mu, 4\mu, \frac{1}{3}(4\mu - 1))$ design, where $\mu \equiv 1 \pmod{3}$. Define a design Π as follows.

Choose any point w of Γ . The points of Π are all the points of Γ except w . To define a general block of Π , consider any parallel class \mathcal{C} . Then \mathcal{C} has four blocks. Let B_0 be the block of \mathcal{C} on w . For any $B \in \mathcal{C}$ with $B \neq B_0$, we define $B \cup B_0 - \{w\}$ to be a block of Π .

It is not difficult to verify that Π is a symmetric 2 - $(16\mu - 1, 8\mu - 1, 4\mu - 1)$ design and that, for any parallel class \mathcal{C} , the three blocks $B \cup B_0 - \{w\}$, with $B \in \mathcal{C}$ and $B \neq B_0$, form a line in the dual Π^* of Π . The set of all such lines is a spread of lines, each of size 3, in Π^* .

Construction 2.2 Let $\Pi = (\mathcal{P}, \mathcal{B}, I)$ be a symmetric 2 - $(16\mu - 1, 8\mu - 1, 4\mu - 1)$ design whose dual Π^* has a spread \mathcal{L} of lines of size 3, that is, a set of lines of Π^* which partitions \mathcal{B} . Define an incidence structure Γ as follows.

The point set of Γ is $\mathcal{P} \cup \{w\}$ where w is a new point. The block set of Γ is $\mathcal{B} \cup \mathcal{L}$. We define the incidence relation I_Γ for Γ in two parts. Firstly, $w I_\Gamma L$ for all $L \in \mathcal{L}$. Secondly, let $P \in \mathcal{P}$ and $L \in \mathcal{L}$. If P is on exactly one (say B) of the three blocks of L in Π , then $P I_\Gamma B$. If P is on all three blocks of L in Π , then $P I_\Gamma L$.

It is routine to verify that Γ is an affine 2 - $(16\mu, 4\mu, \frac{1}{3}(4\mu - 1))$ design. A typical parallel class consists of L, B_1, B_2, B_3 , where $L \in \mathcal{L}$ and the B_i are the three blocks of L in Π .

The verifications for both of the above constructions may be found in Al-Kenani and Mavron [9] in a more general setting.

The constructions are, in an obvious sense, inverses of one another. However, it should be noted that the choice of spread in the second construction is important. Different choices of spread may result in non-isomorphic designs (see Section 3).

It is also interesting to observe that the symmetric design in both constructions is Hadamard, and therefore constructible from a Hadamard matrix. The constructions relate Hadamard matrices to affine 2-designs with class number 4. Their relationship with affine 2-designs with class number 2 is, of course, well-known.

We shall need the following results due to Kantor (see [3, pp.839]) and Dembowski and Wagner (see [3, pp.812]).

Result 2.3 *An affine 2-design of class number m is isomorphic to the design of points and hyperplanes of some affine geometry $AG(n, m)$ or to the design of points and lines of an affine plane of order m if, and only if, the intersection of any two non-parallel blocks is contained in $m + 1$ blocks.*

Result 2.4 *A symmetric 2- (v, k, λ) design is isomorphic to the design of points and hyperplanes of some projective geometry $PG(n, m)$ or to the design of points and lines of a projective plane of order m , where $m = (v - 1)/k$, if, and only if, the intersection of any two distinct blocks is contained in exactly $m + 1$ distinct blocks (or, dually, every line has exactly $m + 1$ points).*

Given any set $X = \{a, b, c\}$ of three distinct mutually skew lines in $PG(3, 2)$, there are exactly three transversals to the three lines of X (a *transversal* of X is a line meeting each line of X in a point). The three lines of X form a *regulus* and the three transversals form the *opposite regulus*. Thus, any line spread \mathcal{L} in $PG(3, 2)$ is *regular* in the sense that any three lines of \mathcal{L} form a regulus. Moreover, any regulus in $PG(3, 2)$ is contained in a unique line spread of five lines. See Hirschfeld [8] for details of results concerning reguli.

Definition 2.5 *A spread \mathcal{L} of lines in $PG(n, 2)$, $n \geq 3$, is normal if for any two distinct lines in \mathcal{L} the intersection of all hyperplanes containing both lines contains three further lines of \mathcal{L} .*

In Constructions 2.1 and 2.2, the affine design has $m = 4$, while the symmetric design has $m = 2$ and is therefore Hadamard. In what follows, we use the notation $AG_t(n, q)$ (resp. $PG_t(n, q)$) for the design having as points and blocks the points and t -dimensional subspaces of $AG(n, q)$ (resp. $PG(n, q)$). The aim of this section is to prove the following theorem.

Theorem 2.6 *With the notation of Constructions 2.1 and 2.2, the following statements are equivalent:*

- (a) Γ is isomorphic to the design of points and hyperplanes $AG_{n-1}(n, 4)$ of the affine geometry $AG(n, 4)$, where $n \geq 2$.
- (b) Π is isomorphic to the design of points and hyperplanes $PG_{2n-2}(2n-1, 2)$ of the projective geometry $PG(2n-1, 2)$, where $n \geq 2$. The line spread \mathcal{L} in the dual Π^* of Π is normal.

The proof proceeds through a series of lemmas.

Lemma 2.7 *Let Γ be the design $AG_{n-1}(n, 4)$, where $n \geq 2$. Let $A = \{A_1, A_2, A_3, A_4\}$ and $B = \{B_1, B_2, B_3, B_4\}$ be distinct parallel classes of Γ . Let Δ be the design whose points are the sixteen affine subspaces $A_i \cap B_j$ ($i, j = 1, 2, 3, 4$) and whose blocks are the hyperplanes of Γ parallel to these subspaces. Then Δ is isomorphic to the affine plane $AG_1(2, 4)$ of order 4.*

Moreover, if $i, j \in \{2, 3, 4\}$, the points $A_1 \cap B_1$, $A_i \cap B_1$, $A_1 \cap B_j$ and $A_i \cap B_j$ are the four points of a subplane Δ_0 of order 2 whose parallelism is induced by that of Γ .

PROOF. The proof is straightforward and is omitted. ■

Lemma 2.8 *Let Γ be the design $AG_{n-1}(n, 4)$, where $n \geq 2$ and let Π be constructed as in Construction 2.1. Then Π , and hence Π^* also, is isomorphic to $PG_{2n-2}(2n-1, 2)$. The spread \mathcal{L} is a normal line spread of Π^* . Furthermore, given any two distinct lines of \mathcal{L} , the intersection of all hyperplanes of Π^* containing them is a 3-dimensional subspace on which \mathcal{L} induces a spread of five lines.*

PROOF. With the notation of Lemma 2.7, given the two non-parallel blocks $A \cup A_i$ and $B \cup B_j$ of Π , where $w \in A \cap B$, let $C \cup C_k$ be the unique block of Γ containing the two points $A_1 \cap B_1$ and $A_i \cap B_j$ of Δ_0 , where $w \in C$. Then C contains $A \cap B$ and C_k contains $A_i \cap B_j$.

It follows that in Π the intersection of any two distinct blocks is contained in a third block. So $\Pi \cong PG_{2n-2}(2n-1, 2)$ by Result 2.4. This proves the first part of the lemma.

Let L_A and L_B be distinct lines of the line spread \mathcal{L} , where $L_A = \{A \cup A_1 - \{w\}, A \cup A_2 - \{w\}, A \cup A_3 - \{w\}\}$, $L_B = \{B \cup B_1 - \{w\}, B \cup B_2 - \{w\}, B \cup B_3 - \{w\}\}$, $\{A, A_1, A_2, A_3\}$ and $\{B, B_1, B_2, B_3\}$ are parallel classes of Γ and $w \in A \cap B$.

A point $P \neq w$ in Γ is on L_A and L_B , considered as blocks of Γ , if, and only if, P , considered as a hyperplane of Π^* , contains both L_A and L_B , considered as lines of the line spread \mathcal{L} of Π^* or, equivalently, P , as a point of Π , is on

all blocks of L_A and L_B . That is, $P \in A \cap B$ in Γ and $P \neq w$.

Now, from Lemma 2.7, since Δ is an affine plane of order 4, there are exactly five hyperplanes X of Γ containing $A \cap B$. A and B are two of the five hyperplanes. So, in Π^* , the five lines L_X are in the unique subspace S which is the intersection of all hyperplanes containing L_A and L_B . Hence, the line spread \mathcal{L} of Π^* is normal. ■

Lemma 2.9 *Let Π be the design $PG_{2n-2}(2n-1, 2)$ and suppose that \mathcal{L} is a normal line spread of Π^* . Let A and B be points of Π^* on different lines of \mathcal{L} . Then the intersection of all hyperplanes of Π^* that contain A and B but neither of the lines of \mathcal{L} on A and B consists of exactly five points of Π^* .*

PROOF. Let A and B be points of Π on lines a and b , respectively, of \mathcal{L} , where $a \neq b$. Let A also represent the homogeneous coordinates of A in $\Pi^* \cong PG_{2n-2}(2n-1, 2)$ and similarly for the other points of Π^* .

Then $C = A + B$ is the third point of the line AB and C is on the line c of \mathcal{L} , say. Let S be the 3-dimensional subspace of Π^* generated by the lines a and b .

Thus, a , b and c form a regulus in $S \cong PG_2(3, 2)$. Let the opposite regulus of three lines be $\{A, B, C\}$, $\{A', B', C'\}$ and $\{A'', B'', C''\}$. Now, if d and e denote the other two lines in the line spread of S induced by \mathcal{L} containing a , b and c (see Lemma 2.8), then it is not difficult to see, without loss of generality, that $d = \{A + B', B + C', C + A'\}$, $e = \{A + C', B + A', C + B'\}$, and the plane containing the lines c and AB meets d at $B + C'$ and meets e at $A + C'$.

Any hyperplane H of Π^* containing the line AB but neither a nor b must meet S in a plane containing A , B and C but neither A' nor B' . The intersection of H with d is the point $B + C'$; for suppose for instance that $A + B'$ is on H . Then H will contain $A + A + B' = B'$ as well as B . H will therefore contain b , which is a contradiction. The other cases are dealt with similarly. By a similar argument, we can show that H meets e at the point $A + C'$.

It follows that the intersection of all hyperplanes of Π^* containing A and B but neither a nor b consists of the five points A , B , C , $A + C'$ and $B + C'$. This completes the proof. ■

Remark 2.10 With the notation of the proof of Lemma 2.9, it follows that H contains the point $A + (A + C') = C'$. Since C is also on H , H must contain the line c . Therefore, $A + C'$ and $B + C'$ are the points where the plane containing the lines AB and c meets e and d , respectively.

Now we can prove Theorem 2.6.

PROOF OF THEOREM 2.6. That (a) implies (b) follows immediately from Lemma 2.8.

We prove that (b) implies (a). Assume that Π is isomorphic to $PG_{2n-2}(2n-1, 2)$ with $n \geq 2$ and that \mathcal{L} is a normal line spread of Π^* .

We show that in the corresponding affine design Γ , the intersection of two non-parallel blocks is contained in exactly five blocks. Hence, by Result 2.3, it will follow that (b) implies (a).

Let A and B be distinct non-parallel blocks of Γ .

Case 1: $A, B \in \mathcal{L}$ and $A \neq B$.

From Lemma 2.8 and the fact that \mathcal{L} is normal, it follows that the intersection of all hyperplanes of Π^* containing A and B contains three further lines of \mathcal{L} . Thus, in Γ these correspond to three further blocks (apart from A and B) containing $A \cap B$.

Case 2: $A \in \mathcal{L}$, $B \in \mathcal{B}$ and, in Π , $B \notin A$.

Then a point P in Γ is on A and B if, and only if, P as a hyperplane in Π^* contains A and meets the line L of \mathcal{L} on B in B only.

In Π^* , let z be the plane containing the line A and the point B . Then, z cannot contain L , for otherwise A and L would meet. So, z meets L only at B .

There are five lines of \mathcal{L} in the 3-dimensional subspace of Π^* generated by A and L ; denote them by A , L , L_1 , L_2 and L_3 . For each i , the plane z does not contain L_i since A and L_i do not meet. Hence, z meets L_i in a single point Y_i , say, ($i = 1, 2, 3$).

Any hyperplane H of Π^* containing the line A and the point B must contain the plane z . If H does not contain L then H cannot contain any of L_1 , L_2 or L_3 , by the definition of a normal line spread since H contains A .

It follows that the intersection of all hyperplanes containing A and meeting L only at B consists of the line A and the four points B , Y_1 , Y_2 and Y_3 . Hence, in Γ there are five blocks containing the intersection of A and B .

Case 3: $A, B \in \mathcal{B}$ and $A \neq B$.

From Lemma 2.9, it follows that there are exactly five blocks of Γ containing the intersection of A and B . This completes the proof of Theorem 2.6. ■

Next we consider the issue of isomorphism arising from the constructions. Recall that a *dilatation* α of an affine design is an automorphism fixing every parallel class (as a set). It is *central* if it fixes a point, and such a fixed point is called a *centre* of α . If α is central and is not the trivial automorphism, then

α fixes a unique point. The proofs of the following theorem and corollary are routine and are omitted.

Theorem 2.11 *With the notation of Constructions 2.1 and 2.2, extended in the obvious way, let Γ and Γ' be affine 2 - $(16\mu, 4\mu, \frac{1}{3}(4\mu - 1))$ designs and let Π and Π' be the corresponding symmetric Hadamard 2 - $(16\mu - 1, 8\mu - 1, 4\mu - 1)$ designs. Then there is an isomorphism $\Gamma \rightarrow \Gamma'$ mapping w to w' if, and only if, there is an isomorphism $\Pi \rightarrow \Pi'$ mapping the spread \mathcal{L} onto the spread \mathcal{L}' (as sets).*

Corollary 2.12 *Any automorphism of Γ fixing the point w induces a unique automorphism α of Π fixing \mathcal{L} as a set, and conversely, α is a central dilatation of Γ with centre w if, and only if, α fixes each line in \mathcal{L} .*

3 Line spreads in $PG(5, 2)$ and related affine designs

In this section, we give some computational results concerning line spreads in $PG(5, 2)$ and the related affine 2 - $(64, 16, 5)$ designs.

As proved in the preceding section, a normal line spread in $PG(5, 2)$ yields an affine 2 - $(64, 16, 5)$ design that is isomorphic to the classical design $AG_2(3, 4)$ of points and planes in $AG(3, 4)$. However, there are many projectively inequivalent line spreads that produce non-isomorphic affine designs with these parameters.

The most interesting among these designs is a non-geometric design \mathcal{D} that has incidence matrix of the same 2 -rank as the classical design $AG_2(3, 4)$. This design provides a counter-example to the well-known Hamada conjecture [6], which states that the design of the points and subspaces of a given dimension in $AG(n, p^m)$ (p a prime) is characterized as having the minimum p -rank of its incidence matrix amongst those designs of the same parameters (see [1, p.134]). This design \mathcal{D} was originally discovered recently in [7] as a design that contains a symmetric subnet invariant under an elementary Abelian group of order four. However, this new construction by means of spreads suggests that the same method may be used to find more such examples and to provide some insight into the nature of affine designs of minimum rank.

Here is a short outline of the algorithm used for the computations. The points of $PG(5, 2)$ are the 6-bit nonzero $(0, 1)$ -vectors ordered lexicographically:

$$\mathbf{1} = 000001, \mathbf{2} = 000010, \dots, \mathbf{63} = 111111.$$

A line in $PG(5, 2)$ is a set of three linearly dependent points (as vectors in

$GF(2)^6$). There are 651 lines that are listed explicitly in Table 3.2. We define a graph G having the 651 lines as vertices, where two vertices are adjacent if the corresponding lines are disjoint. A line spread in $PG(5, 2)$ is just a 21-clique in G . Using a clique finding program written by the third author, over 30,000 different line spreads were found. Automorphism group considerations suggest that this collection of line spreads is just a small portion of the total number of line spreads in $PG(5, 2)$. Because of their huge number, we did not attempt to classify these 30,000 spreads up to a projective equivalence. Instead, we classified the resulting affine 2-(64, 16, 5) designs according to the 2-rank of their incidence matrices (clearly, affine designs of different 2-rank correspond to projectively inequivalent spreads).

Affine designs were found of all 2-ranks in the range from 16 to 22. Examples of line spreads that yield designs for each rank between 16 and 22 are listed in Table 3.3. In that table, a line spread is a set of 21 labels of lines as in Table 3.2.

The line spread No. 1 is a normal spread, and the corresponding affine design is isomorphic to the classical design $AG_2(3, 4)$. The spread No. 2 yields a non-geometric (i.e. not isomorphic to $AG_2(3, 4)$) affine design of (supposedly) minimum 2-rank 16. This design can be distinguished from the design $AG_2(3, 4)$ by the order of its full automorphism group, 368640. This exceptional design is isomorphic to the design obtained from net 36 in [7], with a complete list of blocks available at www.math.mtu.edu/~tonchev/Z2Z2nets.

Remark 3.1 In [7], two exceptional non-geometric affine 2-(64, 16, 5) designs of 2-rank 16 were found. One of these two designs yields a Hadamard design which is not isomorphic to the classical design $PG_4(5, 2)$. It cannot therefore be obtained from line spreads in $PG(5, 2)$.

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The authors also wish to acknowledge an observation by W. M. Kantor that Corollary 2.12 implies an elementary embedding of $\Gamma L(n, 4)$ into $GL(2n, 2)$ and that this can be used to show that the line spread of Theorem 2.6 is unique up to projective equivalence. He also informed us of a nice construction for the normal line spread in the dual of the symmetric design Π of the points and hyperplanes of $PG(2n - 1, 2)$, which is due to R. C. Bose (see [4], for example).

Consider the n -dimensional vector space $V_n(4)$ over $GF(4)$. Then $V_n(4)$ can also be considered as a $2n$ -dimensional vector space $V_{2n}(2)$ over $GF(2)$. To each hyperplane H of $V_{2n}(2)$ (i.e. a $(2n - 1)$ -dimensional subspace) we can associate a triple of hyperplanes aH , $a \in GF(4)^*$, whose intersection is a

hyperplane of $V_n(4)$. Conversely, any hyperplane of $V_n(4)$ arises in this way.

Considering Π as the design whose points and blocks are the 1-dimensional and $(2n - 1)$ -dimensional subspaces, respectively, of $V_{2n}(2)$, it is easy to see that each of the triples of hyperplanes described above is a line of size 3 in Π^* and the set of such lines is a normal line spread in Π^* .

Kantor also drew our attention to a paper by Lunardon [5] on normal spreads, which gives a very good account of their applications, and a paper by Barlotti and Cofman [2]. A proof that statement (b) implies statement (a) in Theorem 2.6 is implicit in [2] and uses the fact that $PG(2m - 1, q)$ may be regarded as a hyperplane of $PG(2m, q)$.

Our approach in this paper and our proofs, which are relatively self-contained, are in contrast to those of [2] and [5] more synthetic in character. Moreover, we have shown that the Rahilly construction generalizes to a construction in design theory which, in the geometric case, is essentially the Bose construction.

Table 3.2 *Lines in $PG(5, 2)$*

| | | | | | | | | | | | | | | | | | | | | | | | |
|-----|---|----|----|-----|---|----|----|-----|---|----|----|-----|---|----|----|-----|---|----|----|-----|---|----|----|
| 1 | 1 | 2 | 3 | 2 | 1 | 4 | 5 | 3 | 1 | 6 | 7 | 4 | 1 | 8 | 9 | 5 | 1 | 10 | 11 | 6 | 1 | 12 | 13 |
| 7 | 1 | 14 | 15 | 8 | 1 | 16 | 17 | 9 | 1 | 18 | 19 | 10 | 1 | 20 | 21 | 11 | 1 | 22 | 23 | 12 | 1 | 24 | 25 |
| 13 | 1 | 26 | 27 | 14 | 1 | 28 | 29 | 15 | 1 | 30 | 31 | 16 | 1 | 32 | 33 | 17 | 1 | 34 | 35 | 18 | 1 | 36 | 37 |
| 19 | 1 | 38 | 39 | 20 | 1 | 40 | 41 | 21 | 1 | 42 | 43 | 22 | 1 | 44 | 45 | 23 | 1 | 46 | 47 | 24 | 1 | 48 | 49 |
| 25 | 1 | 50 | 51 | 26 | 1 | 52 | 53 | 27 | 1 | 54 | 55 | 28 | 1 | 56 | 57 | 29 | 1 | 58 | 59 | 30 | 1 | 60 | 61 |
| 31 | 1 | 62 | 63 | 32 | 2 | 4 | 6 | 33 | 2 | 5 | 7 | 34 | 2 | 8 | 10 | 35 | 2 | 9 | 11 | 36 | 2 | 12 | 14 |
| 37 | 2 | 13 | 15 | 38 | 2 | 16 | 18 | 39 | 2 | 17 | 19 | 40 | 2 | 20 | 22 | 41 | 2 | 21 | 23 | 42 | 2 | 24 | 26 |
| 43 | 2 | 25 | 27 | 44 | 2 | 28 | 30 | 45 | 2 | 29 | 31 | 46 | 2 | 32 | 34 | 47 | 2 | 33 | 35 | 48 | 2 | 36 | 38 |
| 49 | 2 | 37 | 39 | 50 | 2 | 40 | 42 | 51 | 2 | 41 | 43 | 52 | 2 | 44 | 46 | 53 | 2 | 45 | 47 | 54 | 2 | 48 | 50 |
| 55 | 2 | 49 | 51 | 56 | 2 | 52 | 54 | 57 | 2 | 53 | 55 | 58 | 2 | 56 | 58 | 59 | 2 | 57 | 59 | 60 | 2 | 60 | 62 |
| 61 | 2 | 61 | 63 | 62 | 3 | 4 | 7 | 63 | 3 | 5 | 6 | 64 | 3 | 8 | 11 | 65 | 3 | 9 | 10 | 66 | 3 | 12 | 15 |
| 67 | 3 | 13 | 14 | 68 | 3 | 16 | 19 | 69 | 3 | 17 | 18 | 70 | 3 | 20 | 23 | 71 | 3 | 21 | 22 | 72 | 3 | 24 | 27 |
| 73 | 3 | 25 | 26 | 74 | 3 | 28 | 31 | 75 | 3 | 29 | 30 | 76 | 3 | 32 | 35 | 77 | 3 | 33 | 34 | 78 | 3 | 36 | 39 |
| 79 | 3 | 37 | 38 | 80 | 3 | 40 | 43 | 81 | 3 | 41 | 42 | 82 | 3 | 44 | 47 | 83 | 3 | 45 | 46 | 84 | 3 | 48 | 51 |
| 85 | 3 | 49 | 50 | 86 | 3 | 52 | 55 | 87 | 3 | 53 | 54 | 88 | 3 | 56 | 59 | 89 | 3 | 57 | 58 | 90 | 3 | 60 | 63 |
| 91 | 3 | 61 | 62 | 92 | 4 | 8 | 12 | 93 | 4 | 9 | 13 | 94 | 4 | 10 | 14 | 95 | 4 | 11 | 15 | 96 | 4 | 16 | 20 |
| 97 | 4 | 17 | 21 | 98 | 4 | 18 | 22 | 99 | 4 | 19 | 23 | 100 | 4 | 24 | 28 | 101 | 4 | 25 | 29 | 102 | 4 | 26 | 30 |
| 103 | 4 | 27 | 31 | 104 | 4 | 32 | 36 | 105 | 4 | 33 | 37 | 106 | 4 | 34 | 38 | 107 | 4 | 35 | 39 | 108 | 4 | 40 | 44 |
| 109 | 4 | 41 | 45 | 110 | 4 | 42 | 46 | 111 | 4 | 43 | 47 | 112 | 4 | 48 | 52 | 113 | 4 | 49 | 53 | 114 | 4 | 50 | 54 |
| 115 | 4 | 51 | 55 | 116 | 4 | 56 | 60 | 117 | 4 | 57 | 61 | 118 | 4 | 58 | 62 | 119 | 4 | 59 | 63 | 120 | 5 | 8 | 13 |
| 121 | 5 | 9 | 12 | 122 | 5 | 10 | 15 | 123 | 5 | 11 | 14 | 124 | 5 | 16 | 21 | 125 | 5 | 17 | 20 | 126 | 5 | 18 | 23 |
| 127 | 5 | 19 | 22 | 128 | 5 | 24 | 29 | 129 | 5 | 25 | 28 | 130 | 5 | 26 | 31 | 131 | 5 | 27 | 30 | 132 | 5 | 32 | 37 |
| 133 | 5 | 33 | 36 | 134 | 5 | 34 | 39 | 135 | 5 | 35 | 38 | 136 | 5 | 40 | 45 | 137 | 5 | 41 | 44 | 138 | 5 | 42 | 47 |
| 139 | 5 | 43 | 46 | 140 | 5 | 48 | 53 | 141 | 5 | 49 | 52 | 142 | 5 | 50 | 55 | 143 | 5 | 51 | 54 | 144 | 5 | 56 | 61 |
| 145 | 5 | 57 | 60 | 146 | 5 | 58 | 63 | 147 | 5 | 59 | 62 | 148 | 6 | 8 | 14 | 149 | 6 | 9 | 15 | 150 | 6 | 10 | 12 |
| 151 | 6 | 11 | 13 | 152 | 6 | 16 | 22 | 153 | 6 | 17 | 23 | 154 | 6 | 18 | 20 | 155 | 6 | 19 | 21 | 156 | 6 | 24 | 30 |
| 157 | 6 | 25 | 31 | 158 | 6 | 26 | 28 | 159 | 6 | 27 | 29 | 160 | 6 | 32 | 38 | 161 | 6 | 33 | 39 | 162 | 6 | 34 | 36 |

| | | | | | | | | | | | |
|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|
| 163 | 6 35 37 | 164 | 6 40 46 | 165 | 6 41 47 | 166 | 6 42 44 | 167 | 6 43 45 | 168 | 6 48 54 |
| 169 | 6 49 55 | 170 | 6 50 52 | 171 | 6 51 53 | 172 | 6 56 62 | 173 | 6 57 63 | 174 | 6 58 60 |
| 175 | 6 59 61 | 176 | 7 8 15 | 177 | 7 9 14 | 178 | 7 10 13 | 179 | 7 11 12 | 180 | 7 16 23 |
| 181 | 7 17 22 | 182 | 7 18 21 | 183 | 7 19 20 | 184 | 7 24 31 | 185 | 7 25 30 | 186 | 7 26 29 |
| 187 | 7 27 28 | 188 | 7 32 39 | 189 | 7 33 38 | 190 | 7 34 37 | 191 | 7 35 36 | 192 | 7 40 47 |
| 193 | 7 41 46 | 194 | 7 42 45 | 195 | 7 43 44 | 196 | 7 48 55 | 197 | 7 49 54 | 198 | 7 50 53 |
| 199 | 7 51 52 | 200 | 7 56 63 | 201 | 7 57 62 | 202 | 7 58 61 | 203 | 7 59 60 | 204 | 8 16 24 |
| 205 | 8 17 25 | 206 | 8 18 26 | 207 | 8 19 27 | 208 | 8 20 28 | 209 | 8 21 29 | 210 | 8 22 30 |
| 211 | 8 23 31 | 212 | 8 32 40 | 213 | 8 33 41 | 214 | 8 34 42 | 215 | 8 35 43 | 216 | 8 36 44 |
| 217 | 8 37 45 | 218 | 8 38 46 | 219 | 8 39 47 | 220 | 8 48 56 | 221 | 8 49 57 | 222 | 8 50 58 |
| 223 | 8 51 59 | 224 | 8 52 60 | 225 | 8 53 61 | 226 | 8 54 62 | 227 | 8 55 63 | 228 | 9 16 25 |
| 229 | 9 17 24 | 230 | 9 18 27 | 231 | 9 19 26 | 232 | 9 20 29 | 233 | 9 21 28 | 234 | 9 22 31 |
| 235 | 9 23 30 | 236 | 9 32 41 | 237 | 9 33 40 | 238 | 9 34 43 | 239 | 9 35 42 | 240 | 9 36 45 |
| 241 | 9 37 44 | 242 | 9 38 47 | 243 | 9 39 46 | 244 | 9 48 57 | 245 | 9 49 56 | 246 | 9 50 59 |
| 247 | 9 51 58 | 248 | 9 52 61 | 249 | 9 53 60 | 250 | 9 54 63 | 251 | 9 55 62 | 252 | 10 16 26 |
| 253 | 10 17 27 | 254 | 10 18 24 | 255 | 10 19 25 | 256 | 10 20 30 | 257 | 10 21 31 | 258 | 10 22 28 |
| 259 | 10 23 29 | 260 | 10 32 42 | 261 | 10 33 43 | 262 | 10 34 40 | 263 | 10 35 41 | 264 | 10 36 46 |
| 265 | 10 37 47 | 266 | 10 38 44 | 267 | 10 39 45 | 268 | 10 48 58 | 269 | 10 49 59 | 270 | 10 50 56 |
| 271 | 10 51 57 | 272 | 10 52 62 | 273 | 10 53 63 | 274 | 10 54 60 | 275 | 10 55 61 | 276 | 11 16 27 |
| 277 | 11 17 26 | 278 | 11 18 25 | 279 | 11 19 24 | 280 | 11 20 31 | 281 | 11 21 30 | 282 | 11 22 29 |
| 283 | 11 23 28 | 284 | 11 32 43 | 285 | 11 33 42 | 286 | 11 34 41 | 287 | 11 35 40 | 288 | 11 36 47 |
| 289 | 11 37 46 | 290 | 11 38 45 | 291 | 11 39 44 | 292 | 11 48 59 | 293 | 11 49 58 | 294 | 11 50 57 |
| 295 | 11 51 56 | 296 | 11 52 63 | 297 | 11 53 62 | 298 | 11 54 61 | 299 | 11 55 60 | 300 | 12 16 28 |
| 301 | 12 17 29 | 302 | 12 18 30 | 303 | 12 19 31 | 304 | 12 20 24 | 305 | 12 21 25 | 306 | 12 22 26 |
| 307 | 12 23 27 | 308 | 12 32 44 | 309 | 12 33 45 | 310 | 12 34 46 | 311 | 12 35 47 | 312 | 12 36 40 |
| 313 | 12 37 41 | 314 | 12 38 42 | 315 | 12 39 43 | 316 | 12 48 60 | 317 | 12 49 61 | 318 | 12 50 62 |
| 319 | 12 51 63 | 320 | 12 52 56 | 321 | 12 53 57 | 322 | 12 54 58 | 323 | 12 55 59 | 324 | 13 16 29 |
| 325 | 13 17 28 | 326 | 13 18 31 | 327 | 13 19 30 | 328 | 13 20 25 | 329 | 13 21 24 | 330 | 13 22 27 |
| 331 | 13 23 26 | 332 | 13 32 45 | 333 | 13 33 44 | 334 | 13 34 47 | 335 | 13 35 46 | 336 | 13 36 41 |
| 337 | 13 37 40 | 338 | 13 38 43 | 339 | 13 39 42 | 340 | 13 48 61 | 341 | 13 49 60 | 342 | 13 50 63 |
| 343 | 13 51 62 | 344 | 13 52 57 | 345 | 13 53 56 | 346 | 13 54 59 | 347 | 13 55 58 | 348 | 14 16 30 |
| 349 | 14 17 31 | 350 | 14 18 28 | 351 | 14 19 29 | 352 | 14 20 26 | 353 | 14 21 27 | 354 | 14 22 24 |
| 355 | 14 23 25 | 356 | 14 32 46 | 357 | 14 33 47 | 358 | 14 34 44 | 359 | 14 35 45 | 360 | 14 36 42 |
| 361 | 14 37 43 | 362 | 14 38 40 | 363 | 14 39 41 | 364 | 14 48 62 | 365 | 14 49 63 | 366 | 14 50 60 |
| 367 | 14 51 61 | 368 | 14 52 58 | 369 | 14 53 59 | 370 | 14 54 56 | 371 | 14 55 57 | 372 | 15 16 31 |
| 373 | 15 17 30 | 374 | 15 18 29 | 375 | 15 19 28 | 376 | 15 20 27 | 377 | 15 21 26 | 378 | 15 22 25 |
| 379 | 15 23 24 | 380 | 15 32 47 | 381 | 15 33 46 | 382 | 15 34 45 | 383 | 15 35 44 | 384 | 15 36 43 |
| 385 | 15 37 42 | 386 | 15 38 41 | 387 | 15 39 40 | 388 | 15 48 63 | 389 | 15 49 62 | 390 | 15 50 61 |
| 391 | 15 51 60 | 392 | 15 52 59 | 393 | 15 53 58 | 394 | 15 54 57 | 395 | 15 55 56 | 396 | 16 32 48 |
| 397 | 16 33 49 | 398 | 16 34 50 | 399 | 16 35 51 | 400 | 16 36 52 | 401 | 16 37 53 | 402 | 16 38 54 |
| 403 | 16 39 55 | 404 | 16 40 56 | 405 | 16 41 57 | 406 | 16 42 58 | 407 | 16 43 59 | 408 | 16 44 60 |
| 409 | 16 45 61 | 410 | 16 46 62 | 411 | 16 47 63 | 412 | 17 32 49 | 413 | 17 33 48 | 414 | 17 34 51 |
| 415 | 17 35 50 | 416 | 17 36 53 | 417 | 17 37 52 | 418 | 17 38 55 | 419 | 17 39 54 | 420 | 17 40 57 |
| 421 | 17 41 56 | 422 | 17 42 59 | 423 | 17 43 58 | 424 | 17 44 61 | 425 | 17 45 60 | 426 | 17 46 63 |
| 427 | 17 47 62 | 428 | 18 32 50 | 429 | 18 33 51 | 430 | 18 34 48 | 431 | 18 35 49 | 432 | 18 36 54 |
| 433 | 18 37 55 | 434 | 18 38 52 | 435 | 18 39 53 | 436 | 18 40 58 | 437 | 18 41 59 | 438 | 18 42 56 |
| 439 | 18 43 57 | 440 | 18 44 62 | 441 | 18 45 63 | 442 | 18 46 60 | 443 | 18 47 61 | 444 | 19 32 51 |
| 445 | 19 33 50 | 446 | 19 34 49 | 447 | 19 35 48 | 448 | 19 36 55 | 449 | 19 37 54 | 450 | 19 38 53 |

| | | | | | | | | | | | |
|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|-----|----------|
| 451 | 19 39 52 | 452 | 19 40 59 | 453 | 19 41 58 | 454 | 19 42 57 | 455 | 19 43 56 | 456 | 19 44 63 |
| 457 | 19 45 62 | 458 | 19 46 61 | 459 | 19 47 60 | 460 | 20 32 52 | 461 | 20 33 53 | 462 | 20 34 54 |
| 463 | 20 35 55 | 464 | 20 36 48 | 465 | 20 37 49 | 466 | 20 38 50 | 467 | 20 39 51 | 468 | 20 40 60 |
| 469 | 20 41 61 | 470 | 20 42 62 | 471 | 20 43 63 | 472 | 20 44 56 | 473 | 20 45 57 | 474 | 20 46 58 |
| 475 | 20 47 59 | 476 | 21 32 53 | 477 | 21 33 52 | 478 | 21 34 55 | 479 | 21 35 54 | 480 | 21 36 49 |
| 481 | 21 37 48 | 482 | 21 38 51 | 483 | 21 39 50 | 484 | 21 40 61 | 485 | 21 41 60 | 486 | 21 42 63 |
| 487 | 21 43 62 | 488 | 21 44 57 | 489 | 21 45 56 | 490 | 21 46 59 | 491 | 21 47 58 | 492 | 22 32 54 |
| 493 | 22 33 55 | 494 | 22 34 52 | 495 | 22 35 53 | 496 | 22 36 50 | 497 | 22 37 51 | 498 | 22 38 48 |
| 499 | 22 39 49 | 500 | 22 40 62 | 501 | 22 41 63 | 502 | 22 42 60 | 503 | 22 43 61 | 504 | 22 44 58 |
| 505 | 22 45 59 | 506 | 22 46 56 | 507 | 22 47 57 | 508 | 23 32 55 | 509 | 23 33 54 | 510 | 23 34 53 |
| 511 | 23 35 52 | 512 | 23 36 51 | 513 | 23 37 50 | 514 | 23 38 49 | 515 | 23 39 48 | 516 | 23 40 63 |
| 517 | 23 41 62 | 518 | 23 42 61 | 519 | 23 43 60 | 520 | 23 44 59 | 521 | 23 45 58 | 522 | 23 46 57 |
| 523 | 23 47 56 | 524 | 24 32 56 | 525 | 24 33 57 | 526 | 24 34 58 | 527 | 24 35 59 | 528 | 24 36 60 |
| 529 | 24 37 61 | 530 | 24 38 62 | 531 | 24 39 63 | 532 | 24 40 48 | 533 | 24 41 49 | 534 | 24 42 50 |
| 535 | 24 43 51 | 536 | 24 44 52 | 537 | 24 45 53 | 538 | 24 46 54 | 539 | 24 47 55 | 540 | 25 32 57 |
| 541 | 25 33 56 | 542 | 25 34 59 | 543 | 25 35 58 | 544 | 25 36 61 | 545 | 25 37 60 | 546 | 25 38 63 |
| 547 | 25 39 62 | 548 | 25 40 49 | 549 | 25 41 48 | 550 | 25 42 51 | 551 | 25 43 50 | 552 | 25 44 53 |
| 553 | 25 45 52 | 554 | 25 46 55 | 555 | 25 47 54 | 556 | 26 32 58 | 557 | 26 33 59 | 558 | 26 34 56 |
| 559 | 26 35 57 | 560 | 26 36 62 | 561 | 26 37 63 | 562 | 26 38 60 | 563 | 26 39 61 | 564 | 26 40 50 |
| 565 | 26 41 51 | 566 | 26 42 48 | 567 | 26 43 49 | 568 | 26 44 54 | 569 | 26 45 55 | 570 | 26 46 52 |
| 571 | 26 47 53 | 572 | 27 32 59 | 573 | 27 33 58 | 574 | 27 34 57 | 575 | 27 35 56 | 576 | 27 36 63 |
| 577 | 27 37 62 | 578 | 27 38 61 | 579 | 27 39 60 | 580 | 27 40 51 | 581 | 27 41 50 | 582 | 27 42 49 |
| 583 | 27 43 48 | 584 | 27 44 55 | 585 | 27 45 54 | 586 | 27 46 53 | 587 | 27 47 52 | 588 | 28 32 60 |
| 589 | 28 33 61 | 590 | 28 34 62 | 591 | 28 35 63 | 592 | 28 36 56 | 593 | 28 37 57 | 594 | 28 38 58 |
| 595 | 28 39 59 | 596 | 28 40 52 | 597 | 28 41 53 | 598 | 28 42 54 | 599 | 28 43 55 | 600 | 28 44 48 |
| 601 | 28 45 49 | 602 | 28 46 50 | 603 | 28 47 51 | 604 | 29 32 61 | 605 | 29 33 60 | 606 | 29 34 63 |
| 607 | 29 35 62 | 608 | 29 36 57 | 609 | 29 37 56 | 610 | 29 38 59 | 611 | 29 39 58 | 612 | 29 40 53 |
| 613 | 29 41 52 | 614 | 29 42 55 | 615 | 29 43 54 | 616 | 29 44 49 | 617 | 29 45 48 | 618 | 29 46 51 |
| 619 | 29 47 50 | 620 | 30 32 62 | 621 | 30 33 63 | 622 | 30 34 60 | 623 | 30 35 61 | 624 | 30 36 58 |
| 625 | 30 37 59 | 626 | 30 38 56 | 627 | 30 39 57 | 628 | 30 40 54 | 629 | 30 41 55 | 630 | 30 42 52 |
| 631 | 30 43 53 | 632 | 30 44 50 | 633 | 30 45 51 | 634 | 30 46 48 | 635 | 30 47 49 | 636 | 31 32 63 |
| 637 | 31 33 62 | 638 | 31 34 61 | 639 | 31 35 60 | 640 | 31 36 59 | 641 | 31 37 58 | 642 | 31 38 57 |
| 643 | 31 39 56 | 644 | 31 40 55 | 645 | 31 41 54 | 646 | 31 42 53 | 647 | 31 43 52 | 648 | 31 44 51 |
| 649 | 31 45 50 | 650 | 31 46 49 | 651 | 31 47 48 | | | | | | |

Table 3.3 *Affine 2-(64, 16, 5) designs from line spreads*

| No. | Line spread | 2-rank | No. | Line spread | 2-rank |
|-----|---|--------|-----|---|--------|
| 1 | 1 92 122 151 177 396 414 431 445 468 486 503 517 536 554 571 585 592 610 627 641 | 16 | 5 | 1 92 122 151 177 396 414 431 445 468 486 503 517 536 555 569 586 593 611 626 640 | 19 |
| 2 | 1 92 122 151 177 396 414 431 445 469 487 502 516 538 552 569 587 595 609 624 642 | 16 | 6 | 1 92 122 151 177 396 414 431 445 468 486 503 517 536 555 569 586 592 611 625 642 | 20 |
| 3 | 1 92 122 151 177 396 414 431 445 468 486 503 517 536 554 571 585 593 611 626 640 | 17 | 7 | 1 92 122 151 177 396 414 431 445 468 486 503 520 537 554 560 587 593 611 626 645 | 21 |
| 4 | 1 92 122 151 177 396 414 431 445 468 486 503 517 536 554 571 585 592 611 625 642 | 18 | 8 | 1 92 122 151 177 396 414 431 445 468 486 503 520 537 554 560 587 594 609 627 645 | 22 |

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