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# Squeezing Components in Linear Quantum Feedback Networks

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The aim of this paper is to extend linear quantum dynamical network theory to include static Bogoliubov components (such as squeezers). Within this integrated quantum network theory we provide general methods for cascade or series connections, as well as feedback interconnections using linear fractional transformations. In addition, we define input-output maps and transfer functions for representing components and describing convergence. We also discuss the underlying group structure in this theory arising from series interconnection. Several examples illustrate the theory.

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## I. INTRODUCTION

The aim of this paper is to develop systematic methods for describing and manipulating a class of quantum feedback networks. Quantum networks are important because of the fundamental role they play in quantum technology, and quantum information, computing, and control in particular. Knowledge of how to interconnect quantum (and classical) components in a network is a pre-requisite for feedback control.

A *quantum feedback network* (QFN) [8], [9] is a physical network whose nodes (components) are open quantum systems and whose branches (wires) are quantum fields. Quantum dynamical components have linear relations between their input and output fields, but in general their internal physical variables may evolve nonlinearly. Furthermore, delays may be present due to the non-zero physical length of the branches and the finite speed of light. The network theory developed in [8], [9] is based on quantum stochastic models of open quantum systems (Hudson-Parthasarathy [10], Gardiner-Collett [3]), and provides methods and tools for QFN modeling including series (or cascade) connections, and feedback loops. The series connection defines a group operation in the class of open quantum dynamical systems. In this paper we will consider the subclass of networks consisting of dynamical components whose internal variables evolve linearly. In [15] the theory was presented for components based on unitary transformations such as beamsplitters and phase shift modulators, but does not include static components that require an external source of quanta for their operation as such amplifiers and squeezers.

In quantum optics one encounters the class of quantum linear networks with components implementing static linear transformations, called *Bogoliubov transformations*,

see [14]. Examples of Bogoliubov components include devices capable of creating squeezed states of field (“squeezers”) out of a vacuum, a non-unitary process. Series connection of static Bogoliubov components is given by matrix multiplication of representations of the Bogoliubov transformations, and defines a group product for this class of systems.

These two classes of quantum networks, which are characterized by the nature of their components, are distinct, but have elements in common. The beamsplitter and phase shift are components in both classes. However, the squeezer does not belong to the class of open dynamical quantum systems (in the framework of Hudson-Parthasarathy [10] and Gardiner-Collett [3]), though it may be approximated by systems that are in this class.

The purpose of this paper is to merge together these two classes of linear networks in a unified, *multivariable* algebraic framework. By ‘multivariable’ we mean that the framework allows for systems comprised of multiple oscillator modes and multiple field channels; accordingly, a vector-matrix notation is used. The new class of linear QFNs (LQFNs) we consider in this paper are therefore assembled from components of the following two types: (i) dynamical components, with linear evolution of physical variables, and (ii) static components characterized by Bogoliubov transformations. An example of such a network is shown in Figure 1, [19], [8].

To this end, we consider linear open quantum components that are in general a series connection of a linear dynamical part, and a static Bogoliubov part. We define series connections of these components, and extend linear fractional transformation (LFT) methods for describing feedback loops. The series connection defines a group structure for this new class of systems, which includes the linear dynamic and static Bogoliubov classes as subgroups. This group structure is interesting from a physical as well as a systems and control theoretic point of view.

Open quantum systems have a natural input-output structure. We define and make use of input-output maps for our class of linear open quantum systems, and discuss

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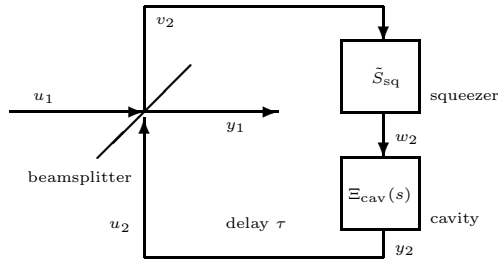


FIG. 1: Linear quantum feedback network (LQFN) consisting of a beamsplitter, squeezer, and cavity. The time delay around the optical loop is  $\tau$ .

convergence of systems in these terms. This input-output notion of convergence is important for applications, and is weaker than stronger notions of convergence involving all system variables.

We begin in section II by describing Bogoliubov transformations, which is followed in section III by a discussion of open linear dynamical models of Hudson-Parthasarathy type. In section V we discuss quantum components involving Bogoliubov transformations, both static and dynamic. Linear quantum feedback networks are described in section VI. Several examples are discussed in sections IV and VII.

## II. BOGOLIUBOV TRANSFORMATIONS

In this section we present models for the quantum components considered in this paper. Before this can be done, some notation is needed.

### A. Notation

Let  $X = (X_{jk})$ ,  $j, k = 1, \dots, n$ , denote a matrix whose entries  $X_{jk}$  are operators on a Hilbert space  $\mathfrak{H}$ , or are complex numbers. We define the matrices

$$X^\# = (X_{jk}^*), \quad X^\top = (X_{kj}), \quad X^\dagger = (X_{kj}^*).$$

Here, the asterisk  $*$  indicates Hilbert space adjoint or complex conjugation.

For a column vector  $x$  of operators of length  $k$ , we shall introduce the *doubled-up column vector*

$$\check{x} \triangleq \begin{bmatrix} x \\ x^\# \end{bmatrix} \quad (1)$$

of length  $2k$ , so that  $\check{x}^\dagger = (x^\dagger, x^\top)$ .

Given a linear transformation of the form

$$y = E_- x + E_+ x^\#$$

where  $x$  and  $y$  are vectors of operators of lengths  $k$  and  $r$  respectively, and  $E_\pm \in \mathbb{C}^{r \times k}$ , we define the transformation  $y^\# = E_-^\# x^\# + E_+^\# x$ , and in doubled-up notation we

have

$$\check{y} = \Delta(E_-, E_+) \check{x},$$

where we introduce the  $2r \times 2k$  *doubled-up matrix*

$$\Delta(E_-, E_+) \triangleq \begin{bmatrix} E_- & E_+ \\ E_+^\# & E_-^\# \end{bmatrix}. \quad (2)$$

We note that  $\Delta(E_-, E_+)^\dagger = \Delta(E_-^\dagger, E_+^\dagger)$ , and when the dimensions are compatible,  $\Delta(E_-, E_+) \Delta(F_-, F_+) = \Delta(E_- F_- + E_+ F_+^\#, E_- F_+ + E_+ F_-^\#)$ . In the examples we consider, the linear transformations will be between vectors of equal dimensions and so the matrices  $E_\pm$ , etc., will be square.

For a  $2n \times 2m$  matrix  $X$ , we define an involution  $\flat$  by

$$X^\flat \triangleq J_m X^\dagger J_n, \quad (3)$$

where

$$J_n \triangleq \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \quad (4)$$

with  $I_n$  the  $n \times n$  identity matrix. When understood, we shall often drop the dimension index and just write  $I$  and  $J$ . For doubled-up matrices, we then have

$$\Delta(E_-, E_+)^\flat = \Delta(E_-^\dagger, -E_+^\dagger). \quad (5)$$

### B. Canonical Commutation Relations

We consider a collection of  $m$  harmonic oscillators, whose behavior is characterized by independent annihilation  $a_j$  and creation  $a_j^*$  operators ( $j = 1, \dots, m$ ) satisfying the canonical commutation relations  $[a_j, a_k^*] = \delta_{jk}$ , with  $[a_j, a_k] = 0 = [a_j^*, a_k^*]$ . The commutation relations may be written compactly as

$$[\check{a}_j, \check{a}_k^\#] = J_{jk},$$

where  $(J_{jk}) = J_m$  is the matrix defined in (4).

Systems consisting of  $m$  oscillator modes are equivalent, for fixed  $m$ , and it is convenient to consider just the category  $\mathcal{S}(m)$  of such systems with representative described by column vector  $a = (a_1, \dots, a_m)^\top$ .

### C. The Bogoliubov Matrix Lie Group, $\text{Sp}(\mathbb{C}^m)$

**Definition** A  $2m \times 2m$  complex matrix  $\tilde{S}$  is said to be  $\flat$ -unitary if it is invertible and

$$\tilde{S}^\flat \tilde{S} = \tilde{S} \tilde{S}^\flat = I_{2m}.$$

The group of Bogoliubov matrices  $\text{Sp}(\mathbb{C}^m)$  is the subgroup of  $\flat$ -unitary matrices that are of doubled-up form, that is  $\tilde{S} = \Delta(S_-, S_+)$  for suitable  $S_-, S_+ \in \mathbb{C}^{m \times m}$ . This is also known as the symplectic group [12].

The transformation  $a' = S_- a + S_+ a^\#$  is called a *Bogoliubov transformation* for  $a \in \mathcal{S}(m)$ . In doubled-up notation this takes the simpler form

$$\check{a}' = \tilde{S}\check{a}. \quad (6)$$

Note that  $a' \in \mathcal{S}(m)$  and in particular, the transformation preserves the canonical commutation relations.

A Bogoliubov matrix  $\tilde{S} \in \text{Sp}(\mathbb{C}^m)$  admits a *Shale decomposition* [14]

$$\tilde{S} = \Delta(S_{\text{out}}^\dagger, 0)\Delta(\cosh R, \sinh R)\Delta(S_{\text{in}}, 0), \quad (7)$$

where  $S_{\text{in}}, S_{\text{out}}$  are  $m \times m$  unitary matrices, and  $R$  a real diagonal  $m \times m$  matrix. Note that  $\Delta(\cosh R, \sinh R) = \exp \Delta(0, R)$ . The middle term in (7) corresponds to *squeezing*, an important characteristic widely exploited in applications of quantum optics. To see what this means, suppose  $\tilde{S} = \Delta(\cosh R, \sinh R)$ . Define the quadratures  $a^x = \frac{1}{2}(a + a^\#)$  and  $a^y = \frac{1}{2i}(a - a^\#)$ , and similarly for  $a'$ . Then

$$(a')^x = e^R a^x, \quad (a')^y = e^{-R} a^y, \quad (8)$$

which shows that if the  $y$  quadrature is scaled by less than unity, the  $x$  quadrature must correspondingly be expanded by an amount greater than unity. Also, note that the unitary group  $U(m)$  of unitary  $m \times m$  matrices can be viewed as a subgroup of  $\text{Sp}(\mathbb{C}^m)$  via the correspondence  $\Delta(S, 0) \in \text{Sp}(\mathbb{C}^m)$  whenever  $S \in U(m)$ .

The Bogoliubov transformation (6) defined by a fixed  $\tilde{S} \in \text{Sp}(\mathbb{C}^m)$  corresponds to the action of a physical device acting on a vector  $a \in \mathcal{S}(m)$ . By Shale's theorem [14] the Bogoliubov transformation may be unitarily implemented, that is, there exists a unitary operator  $U$  such that

$$\tilde{S}\check{a} = U^* \check{a}' U. \quad (9)$$

#### D. The Bogoliubov Lie Algebra, $\mathfrak{sp}(\mathbb{C}^m)$

We remark that the Lie algebra  $\mathfrak{sp}(\mathbb{C}^m)$  consists of matrices  $-i\tilde{\Omega} \in \mathbb{C}^{2m \times 2m}$  that are of doubled-up form (in order to generate doubled-up matrices  $\tilde{S} = e^{-i\tilde{\Omega}}$ ) and satisfy  $\tilde{\Omega}^b = \tilde{\Omega}$ . The second condition can be written as  $J\tilde{\Omega} - \tilde{\Omega}^\dagger J = 0$ . We therefore deduce that the infinitesimal generators take the form

$$-i\tilde{\Omega} = -\Delta(i\Omega_-, i\Omega_+), \quad (10)$$

with complex matrices  $\Omega_-$  and  $\Omega_+$  having the symmetries  $\Omega_-^\dagger = \Omega_-$  and  $\Omega_+^\dagger = \Omega_+$ . We remark that we may construct a hermitean operator  $H$  on the oscillator Hilbert space as

$$H = \sum_{\alpha, \beta=1}^m (a_\alpha^* \omega_{\alpha\beta}^- a_\beta + \frac{1}{2} a_\alpha^* a_\beta^* \omega_{\alpha\beta}^+ + \frac{1}{2} a_\alpha a_\beta \omega_{\alpha\beta}^{+*}), \quad (11)$$

where the coefficients are the entries of the matrices  $\Omega_\pm = (\omega_{\alpha\beta}^\pm) \in \mathbb{C}^{m \times m}$ . From the familiar quantum mechanical point of view,  $H$  is the Hamiltonian generating the canonical transformation (6), that is, in (9) we have  $U = e^{-iH}$ .

It is instructive to look at the  $m = 1$  case. Here the Hamiltonian is  $H = \omega_- a^* a + \frac{1}{2} \omega_+ a^{*2} + \frac{1}{2} \omega_+^* a^2$  with  $\omega_-$  real and  $\omega_+ = |\omega_+| e^{i\theta}$  complex. The corresponding element  $-i\tilde{\Omega} \in \mathfrak{sp}(\mathbb{C})$  can be written in terms of Pauli matrices as as

$$\begin{aligned} -i\tilde{\Omega} &= -\Delta(i\omega_-, i\omega_+) \\ &= \omega_{+y} \sigma_x + \omega_{+x} \sigma_y - i\omega_- \sigma_z, \end{aligned}$$

where  $\omega_+ = \omega_{+x} + i\omega_{+y}$ , or

$$\tilde{\Omega} = \begin{bmatrix} \omega_- & \omega_{+x} + i\omega_{+y} \\ \omega_{+x} - i\omega_{+y} & -\omega_- \end{bmatrix}.$$

The eigenvalues of  $-i\tilde{\Omega}$  are  $\pm\sqrt{\zeta}$  where

$$\zeta = |\omega_+|^2 - \omega_-^2. \quad (12)$$

We note that Heisenberg dynamical equations will be trigonometric type for  $\zeta < 0$ , and hyperbolic type for  $\zeta > 0$ . Let us try and diagonalize the Hamiltonian by introducing the Bogoliubov transformation  $e^{i\theta} a = \cosh r a' - \sinh r a'^*$ , (this is the original purpose of Bogoliubov transformations!). For  $\zeta < 0$  we may choose  $\tanh 2r = |\omega_+|/\omega_-$  to get  $H \equiv \omega_- \sqrt{1 - |\frac{\omega_+}{\omega_-}|^2} a'^* a'$ . While for  $\zeta < 0$  we may choose  $\tanh 2r = \omega_-/|\omega_+|$  to get  $H \equiv \frac{1}{2} |\omega_+| \sqrt{1 - |\frac{\omega_-}{\omega_+}|^2} (a'^{*2} + a'^2)$ . That is, for  $\zeta < 0$  we may diagonalize  $\Omega$  as  $\tilde{S}^b \Omega \tilde{S} = \Delta(\omega'_-, 0)$  using a Bogoliubov matrix  $\tilde{S}$ , but for  $\zeta > 0$  the best we can do is to put into the form  $\Delta(0, \omega'_+)$ .

We say that  $H$  is *passive* if  $\tilde{\Omega}$  has only real eigenvalues. In this case we may find an  $\tilde{S}$  in  $\text{Sp}(\mathbb{C})$  such that  $\tilde{S}^b \tilde{\Omega} \tilde{S} = \Delta(\Omega'_-, 0)$  for some  $\Omega'_-$ . The term “passive” means that such Hamiltonians do not describe energy flow into the system from an external pumping source, and that the dynamical equations are always trigonometric type.

(For  $m = 1$ ,  $H$  is passive if and only if the parameter  $\zeta \leq 0$  in (12) as the eigenvalues are  $\pm\sqrt{-\zeta}$ .)

The group  $\text{Sp}(\mathbb{C}^m)$  is a non-compact group and, in fact, is not covered by the exponential mapping on its Lie algebra  $\mathfrak{sp}(\mathbb{C}^m)$ . We see this in the case  $m = 1$ , where the form of  $e^{-i\tilde{\Omega}}$  depends on the sign of  $\zeta$ . We have  $e^{-i\tilde{\Omega}}$  given respectively by ( $\zeta > 0$ )

$$\begin{bmatrix} \cosh \sqrt{\zeta} - i\omega_- \frac{\sinh \sqrt{\zeta}}{\sqrt{\zeta}} & \omega_+ \frac{\sinh \sqrt{\zeta}}{\sqrt{\zeta}} \\ \omega_+^* \frac{\sinh \sqrt{\zeta}}{\sqrt{\zeta}} & \cosh \sqrt{\zeta} + i\omega_- \frac{\sinh \sqrt{\zeta}}{\sqrt{\zeta}} \end{bmatrix},$$

and ( $\zeta < 0$ )

$$\begin{bmatrix} \cos \sqrt{-\zeta} - i\omega_- \frac{\sin \sqrt{-\zeta}}{\sqrt{-\zeta}} & \omega_+ \frac{\sin \sqrt{-\zeta}}{\sqrt{-\zeta}} \\ \omega_+^* \frac{\sin \sqrt{-\zeta}}{\sqrt{-\zeta}} & \cos \sqrt{-\zeta} + i\omega_- \frac{\sin \sqrt{-\zeta}}{\sqrt{-\zeta}} \end{bmatrix}.$$

Also  $e^{-i\tilde{\Omega}} = 1 - i\tilde{\Omega}$  if  $\zeta = 0$  (note that  $\tilde{\Omega}^2 = 0$  in this case). As observed in [16], we must have  $\text{tr } e^{-i\tilde{\Omega}} \geq -2$  so that there exist matrices in  $\tilde{S} \in \text{Sp}(\mathbb{C})$  which do not possess a logarithm in  $\mathfrak{sp}(\mathbb{C})$ , for example,  $\tilde{S} = -\Delta(\cosh u, \sinh u) \equiv \exp(-\Delta(i\pi, 0))\exp(-\Delta(0, -u))$ . In particular, such Bogoliubov transformations are not generated by a single Hamiltonian  $H$ . The best that can be done is to write the unitary  $U$  in (9) as  $U = U_1 \cdots U_k$  where each  $U_i$  has a logarithm in  $\mathfrak{sp}(\mathbb{C}^m)$ , see [16] for higher-order cases.

### E. Gaussian States

A state on  $\mathcal{S}(m)$  is said to be Gaussian if we have

$$\langle \exp i \{ \check{u}^\dagger \check{a} \} \rangle = \exp \left\{ -\frac{1}{2} \check{u}^\dagger F \check{u} + i \check{u}^\dagger \check{\alpha} \right\},$$

where  $F \geq 0$ . For simplicity we consider mean zero states ( $\alpha = 0$ ). In particular, we have that  $F = \langle \check{a} \check{a}^\dagger \rangle$  takes the specific form

$$F = \begin{bmatrix} \langle a a^\dagger \rangle & \langle a a^\top \rangle \\ \langle a^\# a^\dagger \rangle & \langle a^\# a^\top \rangle \end{bmatrix} = \begin{bmatrix} I + N^\top & M \\ M^\dagger & N \end{bmatrix} \quad (13)$$

with

$$N_{jk} = \langle a_j^* a_k \rangle, \quad M_{jk} = \langle a_j a_k \rangle \quad (14)$$

and we note that  $N = N^\dagger$  and  $M = M^\top$ . In particular, positivity of  $F$  implies that  $N \geq 0$ . The vacuum state is the special state determined by the choice  $N = 0, M = 0$ , for

$$F_{\text{vac}} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \quad (15)$$

For fixed  $N \geq 0$  the choice of  $M$  is constrained by the requirement that  $F$  be positive. For the  $n = 1$  case,  $N$  and  $M$  are scalars and the positivity condition is easily seen to be  $N \geq 0$  with  $|M|^2 \leq N(N+1)$ . More generally we should have a diagonalization  $V^\dagger N V = \text{diag}(N_1, \dots, N_n)$  for unitary  $V$  in which case we could consider new fields  $a' = V a$ . Here  $N_j$  can be interpreted as the average number of quanta in the mode  $a'_j$ . In general we cannot expect to simultaneously diagonalize  $N$  and  $M$ .

#### 1. Generalized Araki-Woods Representation

Given a Gaussian state determined by  $F$  in equation (13), we now show that we can construct modes having that state through canonical transformations of vacuum modes. That is, given a state for which  $a \in \mathcal{S}(m)$  has covariance  $F$  given by (13), there exists a  $2m \times 4m$  matrix  $\tilde{S}_0$  such that

$$\check{a} = \tilde{S}_0 \check{a}_0 \quad (16)$$

where  $a_0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathcal{S}(m+m)$  has vacuum statistics, and

$$\tilde{S}_0 \tilde{S}_0^\dagger = I. \quad (17)$$

Indeed, we will construct  $\tilde{S}_0 = \Delta(E_-^0, E_+^0)$  for some  $m \times 2m$  matrices  $E_-^0, E_+^0$ . This generalizes a construction originally due to Araki and Woods [21] for non-squeezed thermal states, see [22, 23].

#### 2. Construction of Araki-Woods Vacuum Representation

*Step 1: Diagonalize  $N$ .* We may find a unitary matrix  $V \in \mathbb{C}^{m \times m}$  such that  $V^\dagger N V = \text{diag}(N_1, \dots, N_m)$ . The eigenvalues are assumed to be ordered such that  $N_1 \geq \dots \geq N_m \geq 0$ . Therefore we can restrict our attention to the case where  $N$  is diagonalized in this way.

*Step 2: Ignore zero eigenvalues.* Take the first  $m_+$  eigen values to be strictly positive, with the remaining  $m_0 = m - m_+$  to be zero. We respect to the eigen decomposition  $\mathbb{C}^m = \mathbb{C}^{m_+} \oplus \mathbb{C}^{m_0}$  we decompose  $F$  as

$$F = \begin{bmatrix} I + N_{++} & 0 & M_{++} & M_{+0} \\ 0 & I & M_{0+} & M_{00} \\ M_{++}^\top & M_{+0}^\top & N_{++} & 0 \\ M_{0+}^\top & M_{00}^\top & 0 & 0 \end{bmatrix}.$$

However, we observe that if a positive matrix has a zero on a diagonal, then every entry on the corresponding row and column must vanish [26] so that actually

$$F \equiv \begin{bmatrix} I + N_{++} & 0 & M_{++} & 0 \\ 0 & I & 0 & 0 \\ M_{++}^\top & 0 & N_{++} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we can restrict our attention to the case where  $N$  is diagonal and strictly positive, and in particular invertible.

*Step 3: Explicit Construction.* We begin by noting the constraint  $I + N \geq M \frac{1}{N} M^\dagger$ , which follows from noting the positivity of

$$\begin{bmatrix} I & -M \frac{1}{N} \\ 0 & 0 \end{bmatrix} F \begin{bmatrix} I & -M \frac{1}{N} \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} I + N - M \frac{1}{N} M^\dagger & 0 \\ 0 & 0 \end{bmatrix}.$$

We then introduce the following matrices:

$$\begin{aligned} X &= \sqrt{I + N - M \frac{1}{N} M^\dagger}, \\ Y &= \sqrt{N} = \text{diag}(\sqrt{N_1}, \dots, \sqrt{N_m}), \\ Z &= M Y^{-1}. \end{aligned}$$

Note that  $Y = Y^\top$  and from  $Z = M Y^{-1}$  we have that  $Y Z^\top = M^\top = M = Z Y = Z Y^\top$ . These matrices satisfy the conditions

$$X X^\dagger - Y Y^\dagger + Z Z^\dagger = I \text{ and } Y Z^\top = Z Y^\top. \quad (18)$$

Now take  $b_1$  and  $b_2$  to be independent (commuting) modes in  $\mathcal{S}(m)$ . We fix the state to be the joint vacuum state for both of these modes. Then we may represent  $a$  as

$$a = Xb_1 + Yb_2^\# + Zb_2; \quad (19)$$

indeed, it is straightforward to check that

$$\begin{aligned} \langle a^\# a^\top \rangle &= Y^2 = N \\ \langle a a^\top \rangle &= ZY^\top = ZY = M. \end{aligned}$$

Therefore we have obtained the representation (16) with  $\tilde{S}_0 = \Delta(E_-^0, E_+^0)$  and

$$E_-^0 = (X \ 0), \quad E_+^0 = (Z \ Y).$$

Property (17) follows from (18).

### III. QUANTUM OPEN LINEAR DYNAMICS

In this section we consider the general class of open linear dynamical models arising from a unitary model for the joint system and field. The system will be a collection  $\mathcal{S}(m)$  of  $m$  harmonic modes with representative  $a = (a_1, \dots, a_m)^\top$ .

#### A. Boson Fields: Vacuum States

The open quantum systems to be described below are driven by  $n$  quantum noise fields (input processes) represented by annihilation  $b_j(t)$  and creation  $b_j^*(t)$  operators ( $j = 1, \dots, n$ ) satisfying canonical commutation relations  $[b_j(t), b_k^*(t')] = \delta_{jk}\delta(t-t')$ , with  $[b_j(t), b_k(t')] = 0$ . This may be written compactly as

$$[\check{b}_j(t), \check{b}_k^\#(t')] = J_{jk} \delta(t-t'). \quad (20)$$

We shall denote the class of  $m$  independent input processes by  $\mathcal{F}(n)$  with representative described by column vector  $b = (b_1, \dots, b_n)^\top$ .

The vacuum state for the field is characterized by

$$\begin{aligned} \left\langle \exp i \int_0^\infty \{u(t) b^\dagger(t) + u^\dagger(t) b(t)\} dt \right\rangle_{\text{vac}} \\ = \exp -\frac{1}{2} \int_0^\infty u^\dagger(t) u(t) dt. \end{aligned}$$

It is convenient to introduce the integrated fields

$$B_j(t) = \int_0^t b_j(r) dr,$$

and in the vacuum representation their future-pointing (It $\bar{o}$ ) increments satisfy the quantum It $\bar{o}$  table

$$\begin{array}{c|cc} \times & dB_k^\dagger & dB_k \\ \hline dB_j & \delta_{jk} dt & 0 \\ \hline dB_j^\dagger & 0 & 0 \end{array}.$$

We may write the above more compactly as

$$\begin{aligned} \left\langle \exp i \int_0^\infty \check{u}^\dagger(t) \check{b}(t) dt \right\rangle_{\text{vac}} = \\ \exp -\frac{1}{2} \int_0^\infty \check{u}^\dagger(t) F_{\text{vac}} \check{u}(t) dt. \end{aligned}$$

The vacuum state is then the Gaussian state for which  $\langle \check{b}(t) \check{b}^\dagger(t') \rangle_{\text{vac}} = F_{\text{vac}} \delta(t-t')$  and the It $\bar{o}$  table may be summarized by

$$d\check{B}d\check{B}^\dagger = F_{\text{vac}} dt.$$

Here  $B = (B_1, B_2, \dots, B_n)^\top$ . In the vacuum case we may also define the counting process

$$\Lambda_{jk}(t) = \int_0^t b_j^*(r) b_k(r) dr,$$

which may be included in the It $\bar{o}$  table [10]. The additional non-trivial products of differentials are

$$d\Lambda_{jk} dB_l^\dagger = \delta_{kl} dB_j^\dagger, \quad dB_j d\Lambda_{kl} = \delta_{jk} dB_l, \quad d\Lambda_{jk} d\Lambda_{li} = \delta_{kl} d\Lambda_{ji}.$$

#### B. Boson Fields: Gaussian Field States

We may generalize the situation in III A to the case where the input fields are in Gaussian states with zero mean, but with the correlation functions

$$\begin{aligned} \langle b_j^*(t) b_k(t') \rangle &= N_{jk} \delta(t-t'), \\ \langle b_j(t) b_k(t') \rangle &= M_{jk} \delta(t-t'), \end{aligned}$$

with  $N$  and  $M$  as in (14). That is,

$$\langle \check{b}(t) \check{b}^\dagger(t') \rangle \equiv F \delta(t-t'), \quad (21)$$

where  $F$  has the same form encountered in the case of a finite number of modes in equation (13). The extended It $\bar{o}$  table is then

$$\begin{array}{c|cc} \times & dB_k^\dagger & dB_k \\ \hline dB_j & (\delta_{jk} + N_{kj}) dt & M_{jk} dt \\ \hline dB_j^\dagger & M_{kj}^* dt & N_{jk} dt \end{array}.$$

Generalized Araki-Woods representations for arbitrary Gaussian field states may be obtained based on a straightforward lifting of the constructions in II E, [22, 23].



### C. Quantum Linear Dynamical Models

The dynamical behavior of a system comprised of  $m$  oscillators, interacting with  $n$  input fields (vacuum state) is given in terms of the Hudson-Parthasarathy Schrödinger equation (in Itô form) [10]

$$dU(t) = \left\{ \sum_{i,j=1}^n (S_{ij} - \delta_{ij}) d\Lambda_{ij}(t) + \sum_{j=1}^n dB_j^*(t) L_j - \sum_{j,k=1}^n L_j^* S_{jk} dB_k(t) - \left( \frac{1}{2} \sum_{j=1}^n L_j^* L_j + iH \right) dt \right\} U(t), \quad (22)$$

for a unitary operator  $U(t)$ , with  $U(0) = I$ . To obtain a unitary evolution leading to linear dynamics we must take  $S \in \mathbb{C}^{n \times n}$  to be unitary,  $H$  to be of the form encountered in (11), while the coupling of the system modes to the fields is to be of the form

$$L_j = \sum_{\alpha=1}^m (C_{j\alpha}^- a_{\alpha} + C_{j\alpha}^+ a_{\alpha}^*), \quad (23)$$

where  $C_{\pm} = (C_{j\alpha}^{\pm}) \in \mathbb{C}^{n \times m}$ .

The oscillator variables evolve unitarily  $a_j(t) = U^*(t) a_j U(t)$ , and likewise the output field is  $B_{\text{out}}(t) = U^*(t) B(t) U(t)$ . The dynamical equations are

$$\begin{aligned} \dot{a}(t) &= C_+^{\top} S^{\#} b^{\#}(t) - C_-^{\dagger} S b(t) + A_- a(t) + A_+ a^{\#}(t), \\ b_{\text{out}}(t) &= S b(t) + C_- a(t) + C_+ a^{\#}(t), \end{aligned} \quad (24)$$

where

$$A_{\mp} = -\frac{1}{2}(C_-^{\dagger} C_{\mp} - C_+^{\top} C_{\pm}^{\#}) - i\Omega_{\mp}. \quad (25)$$

Note that  $-\frac{1}{2i}(A_- - A_-^{\dagger}) = \Omega_-$  and  $-\frac{1}{2i}(A_+ + A_+^{\top}) = \Omega_+$ , but in general  $A_- \neq A_-^{\dagger}$  and  $A_+ \neq A_+^{\top}$ . Here and below differential equations are expressed in terms of quantum noise fields, and may be interpreted in the Stratonovich or Itô sense, moreover the evolution preserves the commutation relations of the oscillator variables.

The linear dynamical equations can be written in doubled-up form as

$$\begin{aligned} \dot{\check{a}}(t) &= \Delta(A_-, A_+) \check{a}(t) - \Delta(C_-, C_+)^{\flat} \Delta(S, 0) \check{b}(t), \\ \check{b}_{\text{out}}(t) &= \Delta(C_-, C_+) \check{a}(t) + \Delta(S, 0) \check{b}(t). \end{aligned} \quad (26)$$

Let us introduce the doubled-up matrices  $\tilde{A} = \Delta(A_-, A_+)$ ,  $\tilde{C} = \Delta(C_-, C_+)$ , and  $-i\tilde{\Omega} = -\Delta(i\Omega_-, i\Omega_+)$ , then we have the identities

$$2\text{Re}_b(A) = \tilde{A} + \tilde{A}^{\flat} = -\tilde{C}^{\flat} \tilde{C}, \quad \tilde{\Omega}^{\flat} = \tilde{\Omega}. \quad (27)$$

These are readily established by noting

$$\begin{aligned} &\Delta(A_-, A_+) + \Delta(A_-, A_+)^{\flat} = \Delta(A_- + A_-^{\dagger}, A_+ - A_+^{\top}) \\ &= -\Delta(C_-^{\dagger} C_- - C_+^{\top} C_+^{\#}, C_-^{\dagger} C_+ - C_+^{\top} C_-^{\#}) \\ &= -\Delta(C_-, C_+)^{\flat} \Delta(C_-, C_+), \end{aligned}$$

and

$$\Delta(i\Omega_-, i\Omega_+)^{\flat} = \Delta(-i\Omega_-^{\dagger}, -i\Omega_+^{\top}) = -\Delta(i\Omega_-, i\Omega_+).$$

The dynamical equations can then be recast as

$$\begin{aligned} \dot{\check{a}}(t) &= \tilde{A} \check{a}(t) + \tilde{B} \check{b}(t), \\ \check{b}_{\text{out}}(t) &= \tilde{C} \check{a}(t) + \tilde{D} \check{b}(t), \end{aligned} \quad (28)$$

where

$$\tilde{D} = \Delta(S, 0), \quad (29)$$

and

$$\tilde{B} = -\tilde{C}^{\flat} \tilde{D}, \quad \tilde{A} = -\frac{1}{2} \tilde{C}^{\flat} \tilde{C} - i\tilde{\Omega}. \quad (30)$$

We denote this class of linear Hudson-Parthasarathy systems with  $n$  input fields and  $m$  oscillators by  $\mathcal{L}^{\text{HP}}(n, m)$ , and write  $\mathcal{L}^{\text{HP}}(n) = \cup_m \mathcal{L}^{\text{HP}}(n, m)$ . Systems  $G \in \mathcal{L}^{\text{HP}}(n, m)$  may be parameterized in several ways. In terms of the scattering matrix,  $S$ , vector of coupling operators  $L$  and Hamiltonian  $H$  we may write

$$G = (S, L, H). \quad (31)$$

Since these physical parameters are determined by the matrices given above, we may also write

$$G = (S, \tilde{C}, \tilde{\Omega}). \quad (32)$$

Alternatively, we may use the matrices appearing in equations (28),

$$G = \left[ \begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right], \quad (33)$$

a notation commonly used in linear systems and control theory. We remark that an arbitrary quadruple of matrices  $\tilde{A}$ ,  $\tilde{B}$ ,  $\tilde{C}$ ,  $\tilde{D}$  need not necessarily correspond to a quantum open system, [11], [18].

### D. Stability

In linear systems theory, a system  $G$  of the form (33) is said to be *Hurwitz stable* if the matrix  $\tilde{A}$  has all eigenvalues having strictly negative real part. Now if the joint system-field state is one in which the inputs are mean-zero, then  $\frac{d}{dt} \langle \check{a}(t) \rangle = \tilde{A} \langle \check{a}(t) \rangle$  and so Hurwitz stability implies that  $\langle \check{a}(t) \rangle \rightarrow 0$  as  $t \rightarrow \infty$ .

When  $A_+ = 0$ , we have  $\tilde{A} = \Delta(A_-, 0)$  with  $A_- \equiv -\frac{1}{2} C_-^{\dagger} C_- - i\Omega_-$  and  $\Omega_-^* = \Omega_-$ . Since  $X^{\dagger} X$  is non-negative definite it is easy to determine whether  $A_-$  is Hurwitz. For instance, it is sufficient to have  $C_-$  invertible. However, expressions such as  $X^{\flat} X$  are indefinite due to the presence of the matrix  $J$ . There may be non-passive contributions to  $\tilde{A}$  from both  $C_+$  and  $\Omega_+$ .

As an illustration let us consider how the eigenvalues of  $\tilde{A}$  depend on the physical parameters in the simplest

case  $n = 1 = m$ . We have seen that the most general parameterization is

$$\tilde{C} = \Delta(\sqrt{\gamma_-}e^{i\phi_-}, \sqrt{\gamma_+}e^{i\phi_+}) \text{ and } \tilde{\Omega} = \Delta(\omega_-, \omega_+),$$

with  $\gamma_{\pm}, \phi_{\pm}$  and  $\omega_{\pm}$  real and  $\omega_{\pm} \in \mathbb{C}$ . In this case

$$\tilde{A} = -\Delta\left(\frac{1}{2}(\gamma_- - \gamma_+) + i\omega_-, i\omega_+\right)$$

which has eigenvalues  $\frac{1}{2}(\gamma_- - \gamma_+) \pm \sqrt{\zeta}$  where we recall the parameter  $\zeta = |\omega_+|^2 - \omega_-^2$  from (12). The plant is Hurwitz if

1.  $\zeta \leq 0$  and  $\gamma_- > \gamma_+$ ;
2. or  $\zeta > 0$  and  $\sqrt{\zeta} < \frac{1}{2}(\gamma_- - \gamma_+)$ .

In situation 1 the system has a passive Hamiltonian and the damping rate is greater than the pumping rate. However situation 2 shows that if the damping is sufficiently large then the system may still be stable even if the Hamiltonian is not passive. In general, as one expects, stability will depend on the relative flows of energy into and out of the system.

### E. Series Connections

Open linear dynamical systems  $G_1 = (S_1, L_1, H_1)$  and  $G_2 = (S_2, L_2, H_2)$  in  $\mathcal{L}^{\text{HP}}(n)$  (recall the parameterization (31)) may be connected in series by passing the output of system  $G_1$  into the input of system  $G_2$ , [5], [2], [9]. The system formed from this connection in the zero-delay limit is an open system  $G = G_2 \triangleleft G_1$ , which in terms of the parameters (31) is given by

$$G_2 \triangleleft G_1 = (S_2 S_1, L_2 + S_2 L_1, H_1 + H_2 + \text{Im}\{L_2^\dagger S_2 L_1\}). \quad (34)$$

We refer to  $\triangleleft$  as the *series product* of  $G_1$  and  $G_2$  and gives the cascaded system. The set  $\mathcal{L}^{\text{HP}}(n)$  forms a group with respect to the series product, with inverse  $G^{-1} = (S^\dagger, -S^\dagger L, -H)$  (where  $G = (S, L, H)$ ).

Given an open system  $G = (S, \tilde{C}, \tilde{\Omega})$  (now we use the parameterization (31) in anticipation of later use), it follows from properties of the series product that

$$G = (I, \tilde{C}, \tilde{\Omega}) \triangleleft (S, 0, 0). \quad (35)$$

This factorization says that an open system with scattering  $S$  is equivalent (in the zero-delay limit) to a dynamic open system without scattering  $(I, \tilde{C}, \tilde{\Omega})$  connected in series with a non-dynamic or static open system  $(S, 0, 0)$ .

### F. Input-Output Maps

In classical systems and control theory [20] the input-output map is a basic tool, which in the case of linear

systems may be expressed explicitly in the time and frequency domains. Input-output maps for the open quantum linear system may be defined in the same way; in terms of the doubled-up parameters (33), we have

$$\check{b}_{\text{out}}(t) = \tilde{C}e^{\tilde{A}t}\check{a}(0) + \tilde{\Sigma}_G[t; \check{b}_{\text{in}}] \quad (36)$$

where

$$\tilde{\Sigma}_G[t; \check{b}] = -\int_0^t \tilde{C}e^{\tilde{A}(t-r)}\tilde{C}^b\tilde{D}\check{b}(r)dr + \tilde{D}\check{b}(t). \quad (37)$$

Here the input  $b_{\text{in}}$  is understood as the input field  $b$  and we shall often use the subscript for emphasis. The impulse response associated with the term  $\tilde{\Sigma}_G[t; \check{b}]$  is  $\tilde{\sigma}_G(t) = -\tilde{C}e^{\tilde{A}t}\tilde{C}^b\tilde{D} + \tilde{D}\delta(t)$ , from which we have the transfer function (the Laplace transform of  $\tilde{\Sigma}_G[t; \check{b}]$ , in which  $s$  is a complex variable):

$$\tilde{\Xi}_G(s) = \left[ \begin{array}{c|c} \tilde{A} & -\tilde{C}^b\tilde{D} \\ \hline \tilde{C} & \tilde{D} \end{array} \right] (s) = -\tilde{C}(sI - \tilde{A})^{-1}\tilde{C}^b\tilde{D} + \tilde{D}. \quad (38)$$

Let us introduce the transformed

$$\check{b}_{\text{in}}[s] \triangleq \int_0^\infty e^{-st}\check{b}_{\text{in}}(t)dt, \quad (39)$$

that is,  $b_{\text{in}}[s] = \int_0^\infty e^{-st}b_{\text{in}}(t)dt$  and  $b_{\text{in}}^\# [s] = b_{\text{in}}[s^*]^\# = \int_0^\infty e^{-st}b_{\text{in}}^\#(t)dt$ .

Adopting a similar convention for the outputs, we then obtain an input-output relation of the form  $b_{\text{out}}[s] = \Xi_{G,-}(s)b_{\text{in}}[s] + \Xi_{G,+}(s)b_{\text{in}}^*[s]$ , or

$$\check{b}_{\text{out}}[s] = \tilde{\Xi}_G(s)\check{b}_{\text{in}}[s] \quad (40)$$

where

$$\tilde{\Xi}_G(s) = \left[ \begin{array}{cc} \Xi_{G,-}(s) & \Xi_{G,+}(s) \\ \Xi_{G,+}(s^*)^\# & \Xi_{G,-}(s^*)^\# \end{array} \right], \quad (41)$$

and we have ignored the initial value contribution of the system modes.

Note that while the transfer function  $\tilde{\Xi}_G(s)$  is uniquely determined by  $G$ , the transfer function does not uniquely determine the system  $G$  - many systems may have the same transfer function.

## IV. EXAMPLES

### A. Annihilation Systems

A system  $\tilde{G} = (S, \tilde{C}, \tilde{\Omega})$  with  $C_+ = 0$ ,  $\Omega_+ = 0$  has dynamics and output relations that depend only on the annihilation operators and annihilation fields, [15, 17, 24]. For a physically motivated reason, since neither the Hamiltonian (passive) nor the coupling operator of the system contain terms that would require an external source of quanta (i.e., a classical pump beam) to



implement (this follows from the synthesis theory of [18]; see also [24, Section 7] for a discussion), they are also referred to as passive systems [24]. This type of system often arises in applications, and includes optical cavities. Transfer functions for this class of systems takes a simpler form, as we now describe.

We have  $A_- \equiv -\frac{1}{2}C_-^\dagger C_- + i\Omega_-$  and  $A_+ = 0$ . Then the matrices  $\tilde{C} = \Delta(C_-, 0)$  and  $\tilde{A} = \Delta(A_-, 0)$  are block diagonal and the transfer function takes the form

$$\tilde{\Xi}_G(s) = \begin{bmatrix} \Xi_{G,-}(s) & 0 \\ 0 & \Xi_{G,-}(s^*)^\# \end{bmatrix}, \quad (42)$$

with

$$\Xi_{G,-}(s) = \left[ \frac{A_- | -C_-^\dagger S}{C_- | S} \right] (s) = -C_- (sI - A_-)^{-1} C_-^\dagger S + S. \quad (43)$$

In this situation, we have the input-output relation  $b_{\text{out}}(t) = Ce^{At}a(0) + \Sigma_G[t; b]$  with  $\Sigma_G[t; b] = -\int_0^t Ce^{A(t-r)}C^b Db(r)dr + Db(t)$ . In comparison with (36) and (37), the output field depends affinely on  $b$ , but not the conjugate  $b^\#$ .

## B. Cavity

In a rotating reference frame, a model for a detuned cavity is characterized by the parameters  $G_{\text{cav}} = (1, \sqrt{\gamma}a, \omega a^*a)$ ; i.e.  $\Omega_- = \omega$ ,  $\Omega_+ = 0$ ,  $C_- = \sqrt{\gamma}$ ,  $C_+ = 0$ ,  $S = I$ .

This corresponds to an annihilation-form system

$$\begin{aligned} \dot{a} &= -\left(\frac{\gamma}{2} + i\omega\right)a - \sqrt{\gamma}b_{\text{in}} \\ b_{\text{out}} &= \sqrt{\gamma}a + b_{\text{in}} \end{aligned} \quad (44)$$

when driven by vacuum input  $b$ . The transfer function for this system may readily be computed to be

$$\Xi_{\text{cav},-}(s) = \frac{s - \frac{\gamma}{2} + i\omega}{s + \frac{\gamma}{2} + i\omega}, \quad (45)$$

which in doubled-up form is

$$\tilde{\Xi}_{\text{cav}}(s) = \begin{bmatrix} \frac{s - \frac{\gamma}{2} + i\omega}{s + \frac{\gamma}{2} + i\omega} & 0 \\ 0 & \frac{s - \frac{\gamma}{2} - i\omega}{s + \frac{\gamma}{2} - i\omega} \end{bmatrix}. \quad (46)$$

Thus this system is

$$G_{\text{cav}} = (I, \Delta(\sqrt{\gamma}I, 0), -i\Delta(i\omega, 0)) \in \mathcal{L}^{\text{HP}}(1).$$

## C. Degenerate Parametric Amplifier

We consider the model for a degenerate parametric amplifier (DPA) [6, sec. 7.2], which corresponds to a single oscillator  $G$  coupled to a single field with  $S = I$ ,  $\omega_- = 0$ ,

$\omega_+ = \frac{i}{2}\epsilon$ ,  $\epsilon > 0$ ,  $C_- = \sqrt{\kappa}$ , and  $C_+ = 0$ . The Hamiltonian will not be passive, however, the system is stable in the sense of Hurwitz if take  $\epsilon \leq \kappa$  as we shall do from now on. Using (38) we find that the doubled-up transfer function, in agreement with [6], is

$$\tilde{\Xi}_{\text{DPA}}(s) = \frac{1}{P(s)} \begin{bmatrix} s^2 - \frac{\kappa^2 + \epsilon^2}{4} & -\frac{1}{2}\epsilon\kappa \\ -\frac{1}{2}\epsilon\kappa & s^2 - \frac{\kappa^2 + \epsilon^2}{4} \end{bmatrix},$$

where  $P(s) = (s + \frac{1}{2}\kappa)^2 - \frac{1}{4}\epsilon^2$ . The poles of the transfer function therefore occur at the zeros of  $P$ , namely  $s = \pm\frac{\epsilon}{2} - \frac{\kappa}{2}$ . In the frequency domain, the output field is

$$b_{\text{out}}(s) = \frac{1}{P(s)}(s^2 - \frac{\kappa^2 + \epsilon^2}{4})b(s) - \frac{1}{2P(s)}\epsilon\kappa b^*(s).$$

(Here we ignore the initial condition contribution which is justified by the stability of the system.) In terms of quadratures  $b^x = \frac{1}{2}(b + b^*)$  and  $b^y = \frac{1}{2i}(b - b^*)$ , we find that

$$b_{\text{out}}^x(s) = \Xi_{\text{DPA}}^x(s)b^x(s), \quad b_{\text{out}}^y(s) = \Xi_{\text{DPA}}^y(s)b^y(s)$$

where (in agreement with [6, eq. (7.2.26)])

$$\Xi_{\text{DPA}}^x(s) = \frac{s - \frac{\kappa + \epsilon}{2}}{\kappa - \epsilon + \frac{s}{2}} = \frac{1}{\Xi_{\text{DPA}}^y(s)}.$$

The DPA can be implemented in a single-ended cavity and a case that is of our main interest in this paper is the idealized one (for a full discussion see [6, 10.2.1.g]) where  $\kappa, \epsilon \rightarrow \infty$  (in practice to be taken large) such that the ratio  $\frac{\epsilon}{\kappa}$  is constant. Rescaling  $\kappa = k\kappa_0$  and  $\epsilon = k\epsilon_0$  is equivalent to replacing  $\kappa$  by  $\kappa_0$  and  $\epsilon$  by  $\epsilon_0$  and rescaling  $s$  as  $\frac{s}{k}$ :

$$\tilde{\Xi}_{\text{DPA}}(s, \kappa = k\kappa_0, \epsilon = k\epsilon_0) = \tilde{\Xi}_{\text{DPA}}\left(\frac{s}{k}, \kappa_0, \epsilon_0\right).$$

The limit  $k \rightarrow \infty$  is the appropriate limit and here the cavity has an instantaneous response and the internal cavity dynamics are essentially eliminated by adiabatic elimination. This results in  $b_{\text{out}}(s)$  being given as the following Bogoliubov transformation of the input:

$$b_{\text{b}}(s) = -\cosh(r_0)b(s) - \sinh(r_0)b^\dagger(s),$$

where

$$r_0 = \ln \frac{\kappa_0 + \epsilon_0}{\kappa_0 - \epsilon_0}.$$

The output is then an ideal squeezed white noise process satisfying the quantum Itô rule discussed in section III A, where

$$\begin{aligned} N &= \sinh^2 r_0 = \frac{4\kappa_0\epsilon_0}{(\kappa_0^2 - \epsilon_0^2)^2}, \\ M &= \cosh r_0 \sinh r_0 = \frac{2\kappa_0\epsilon_0(\kappa_0^2 - \epsilon_0^2)}{(\kappa_0^2 - \epsilon_0^2)^2}. \end{aligned}$$

Note here that  $M$  and  $N$  satisfy the relation  $|M|^2 = N(N+1)$ . In this limit the DPA device behaves like a static device that instantaneously outputs a squeezed white noise field from a vacuum white noise field source, and the transfer function has a constant Bogoliubov matrix value across *all* frequencies. That is

$$\begin{aligned}\tilde{\Xi}_{\text{DPA static}}(s) &= \lim_{k \rightarrow \infty} \tilde{\Xi}_{\text{DPA}}\left(\frac{s}{k}, \kappa_0, \epsilon_0\right) \\ &= -\Delta(\cosh r_0, \sinh r_0), \forall s \in \mathbb{C},\end{aligned}$$

and the quadrature transfer functions are

$$\Xi_{\text{DPA static}}^x(s) = -e^{r_0}, \quad \Xi_{\text{DPA static}}^y(s) = -e^{-r_0}.$$

For a DPA device with a sufficiently wide bandwidth, one may approximately model it as a static DPA device with the ideal characteristics described above. Clearly,  $\mathcal{L}^{\text{HP}}(n)$  is not closed with respect to this type of approximation.

## V. COMPONENTS INVOLVING BOGOLIUBOV TRANSFORMATIONS

In section IV C we obtained the constant transfer function  $\tilde{\Xi}_{\text{DPA static}} \in \text{Sp}(\mathbb{C})$  for a static approximation to a DPA. Such static approximations afford useful simplifications, though in reality the DPA is a dynamical physical device. The idealized DPA therefore yields outputs that are a squeezing of the inputs.

Motivated by this, in section V A we consider Bogoliubov matrices acting on boson fields, thereby extending the class of static components beyond unitary scattering devices. These components will be combined with linear dynamics in section V C to form a general class of quantum linear systems; such models may be useful when the time scales of the dynamical parts are slower than the time scales of the systems represented by static Bogoliubov matrices. These components will be combined with linear dynamics in section V C to form a general class of quantum linear systems; such models may be useful when the time scales of the dynamical parts are slower than the time scales of the systems represented by static Bogoliubov matrices.

### A. Bogoliubov Static Components

More generally we could consider a static component which performs a Bogoliubov transformation of the input field  $b_{\text{in}} \in \mathcal{F}(n)$ :

$$\check{b}_{\text{out}}(t) = \tilde{S} \check{b}_{\text{in}}(t). \quad (47)$$

where now  $\tilde{S} \in \text{Sp}(\mathbb{C}^n)$ . This transformation, of course, preserves the canonical commutation relations so that  $b_{\text{out}} \in \mathcal{F}(n)$ .

Some caution should be applied here as we are now using the symbol  $\tilde{S}$  in (47) in a purely algebraic manner as an element of  $\text{Sp}(\mathbb{C}^n)$  when we strictly mean the second-quantization of the Bogoliubov matrix as an operator on the fields. Despite its formal similarity to (6), the relation (47) is of a different character as the fields carry a continuous time-variable. Moreover, since such a transformation in general form linear combinations of field annihilation operator and creation operators, the transformation  $\check{b}_{\text{out}}(t) = \tilde{S} \check{b}_{\text{in}}(t)$  cannot be described by the usual Hudson-Parthasarathy QSDE for open Markov systems (cf. Section III C). Such a QSDE can only model linear combinations of field annihilation operators of the form  $\check{b}_{\text{out}}(t) = \Delta(S, 0) \check{b}_{\text{in}}(t)$  for a unitary matrix  $S$  that appears as one of the parameters of the QSDE (here we set the other parameters to  $L = 0$  and  $H = 0$ ). As such, in the transformation of fields with a non-unitary Bogoliubov matrix we do not have an analogue of (9) in the form of  $\check{b}_{\text{out}}(t) = U(t)^* \check{b}_{\text{in}}(t) U(t)$  for some unitary process  $U(t)$  on the system and noise Hilbert space. At present we do not know whether a unitary transformation exists and if it exists what kind of dynamical equations it would satisfy. Since unitary evolution is a fundamental postulate of quantum mechanics, the situation is somewhat unsatisfactory and is the subject of continuing research. However, the relation (47) is nevertheless a useful idealization for certain devices used in quantum optics, such as what we have seen with the static DPA in Section IV C, and has formally been employed up to now (see, for instance, the discussion in Chapter 7 of [6] on various quantum optical amplifiers). The physical meaning of the Bogoliubov transformation (47) is correctly interpreted as a limiting situation.

### B. Bogoliubov Static Components as Limits of Dynamical Components

The class of linear dynamical components described in section III C is not closed under input-output convergence. We now show how arbitrary static Bogoliubov components may arise as limits of unitary models. The idea is to exploit the Shale decomposition (7). Thus any given Bogoliubov matrix  $\tilde{S}$  has the decomposition  $\tilde{S} = \Delta(\tilde{S}_{\text{out}}^\dagger, 0) \Delta(\cosh R, \sinh R) \Delta(\tilde{S}_{\text{in}}, 0)$ , where  $\tilde{S}_{\text{in}}$  and  $\tilde{S}_{\text{out}}$  are some unitary matrices and  $R$  is some real diagonal matrix. We note that the end terms  $\Delta(\tilde{S}_{\text{out}}^\dagger, 0)$  and  $\Delta(\tilde{S}_{\text{in}}, 0)$  can each be realized as a static passive network made of beamsplitters, mirrors and phase shifters. The middle term of course describes squeezing but this arises from a straightforward construction involving  $n$  independent static-limit DPAs acting as ideal squeezing devices. (Each DPA corresponding to a diagonal entry of  $R$  providing a degree of squeezing (cf. Section IV C) as determined by that entry.) Then we note that we may approximate each DPA with a corresponding dynamic (non-ideal) DPA with appropriate parameters (see the

discussion of the DPA in Subsection IV C).

### C. Dynamical Bogoliubov Components

We introduce an extension of the class of dynamical linear models  $\mathcal{L}^{\text{HP}}(n)$  considered up to now to accommodate the notion of squeezing. This extension is inspired by the factorization (35) for open linear systems of  $\mathcal{L}^{\text{HP}}(n)$  type, suggesting that we consider a new class of dynamical components of the form

$$G = (\tilde{S}, \tilde{C}, \tilde{\Omega}) \triangleq (I, \tilde{C}, \tilde{\Omega}) \triangleleft \tilde{S}, \quad (48)$$

where  $(I, \tilde{C}, \tilde{\Omega}) \in \mathcal{L}^{\text{HP}}(n, m)$  and  $\tilde{S} = \Delta(S_-, S_+) \in \text{Sp}(\mathbb{C}^n)$ .

A system  $G = (\tilde{S}, \tilde{C}, \tilde{\Omega})$  is defined by equations (28), where  $\tilde{A}$ ,  $\tilde{B}$  and  $\tilde{C}$  are as before (section III C), but now  $\tilde{D} = \tilde{S}$ . We use the notation  $\mathcal{L}^{\text{Bog.}}(n, m)$  to denote this class of systems and write  $\mathcal{L}^{\text{Bog.}}(n) = \cup_m \mathcal{L}^{\text{Bog.}}(n, m)$ . The class  $\mathcal{L}^{\text{Bog.}}(n)$  includes  $\mathcal{L}^{\text{HP}}(n)$  as a special case (with  $\tilde{S} = \Delta(S, 0)$ ). The justification for the cascade expression (48) will be given in section VI F, where we shall extend the series product for cascaded systems in  $\mathcal{L}^{\text{Bog.}}(n)$ .

The doubled up input-output map is of the form (36), where now  $\tilde{D} = \tilde{S}$ . The transfer function is explicitly

$$\tilde{\Xi}_G(s) = \left[ \frac{\tilde{A}}{C} \middle| \frac{-\tilde{C}^b \tilde{S}}{\tilde{S}} \right] (s) = -\tilde{C}(sI - \tilde{A})^{-1} \tilde{C}^b \tilde{S} + \tilde{S}. \quad (49)$$

The transfer function has the following properties:

1.  $\tilde{\Xi}_G \equiv \left[ \frac{\tilde{A}}{C} \middle| \frac{-\tilde{C}^b}{I} \right] \tilde{S}$ ;
2. Whenever its value exists, we have  $\tilde{\Xi}_G(i\omega) \in \text{Sp}(\mathbb{C}^n)$ , for  $\omega \in \mathbb{R}$ .

Property 1 follows directly from (49) while Property 2 follows *mutatis mutandis* from the proof of [15, Lemma 2] by the replacing  $^\dagger$  with  $^b$  and  $(A, B, C, D)$  with  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ .

Physically, the meaning of (48) is that to process an input signal, the Bogoliubov transformation  $\tilde{S}$  is applied and the result is fed into the dynamical subsystem. As we have argued in section V A, Shale's theorem precludes a unitary stochastic dynamical model giving rise to a system  $\tilde{G} \in \mathcal{L}^{\text{Bog.}}(n)$ . Nevertheless, as we have also seen that there will exist a sequences  $\tilde{G}_L \in \mathcal{L}^{\text{HP}}(n)$  such that pointwise

$$\lim_{L \rightarrow \infty} \Xi_{\tilde{G}_L}(s) = \Xi_{\tilde{G}}(s),$$

with  $\tilde{G} \in \mathcal{L}^{\text{Bog.}}(n)$  but not in  $\mathcal{L}^{\text{HP}}(n)$ . One might envisage other modes of convergence of transfer function, however, we shall restrict to pointwise convergence for the purposes of this paper. It is interesting to note that

the class of Hudson-Parthasarathy models is not closed in the above sense of convergence in the input-output sense, but may be extended to include Bogoliubov transformations.

We remark that in many cases where a boson field is in a squeezed state (recall section III B), this field may be regarded as the output of a static Bogoliubov component  $\tilde{S}$  driven by vacuum inputs. This means, for example, that a dynamical component  $(1, \tilde{C}, \tilde{\Omega})$  driven by squeezed fields may be represented as a dynamical Bogoliubov component  $(\tilde{S}, \tilde{C}, \tilde{\Omega})$ .

### D. Example: Cavity with Squeezed Input

Consider the cavity discussed in section IV B, where now we suppose that the cavity input is given by the output of a squeezer  $G_{\text{sq}}$ , described by the Bogoliubov transformation

$$\tilde{S}_{\text{sq}} = \Delta(\cosh \lambda, \sinh \lambda) = \begin{bmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{bmatrix}. \quad (50)$$

That is,

$$G_{\text{sq}} = (\Delta(\cosh \lambda, \sinh \lambda), 0, 0).$$

The squeezed-input cavity  $G_{\text{cav, sq}} = \tilde{G}_{\text{cav}} \triangleleft \tilde{G}_{\text{sq}}$  has transfer function

$$\begin{aligned} \tilde{\Xi}_{\text{cav, sq}}(s) &= \tilde{\Xi}_{\text{cav}}(s) \tilde{S}_{\text{sq}} \\ &= \begin{bmatrix} \frac{s - \frac{\gamma}{2} + i\omega}{s + \frac{\gamma}{2} + i\omega} \cosh r & \frac{s - \frac{\gamma}{2} + i\omega}{s + \frac{\gamma}{2} + i\omega} \sinh r \\ \frac{s - \frac{\gamma}{2} - i\omega}{s + \frac{\gamma}{2} - i\omega} \sinh r & \frac{s - \frac{\gamma}{2} - i\omega}{s + \frac{\gamma}{2} - i\omega} \cosh r \end{bmatrix}. \end{aligned} \quad (51)$$

This corresponds to the equations

$$\begin{aligned} \begin{bmatrix} \dot{a} \\ \dot{a}^* \end{bmatrix} &= \begin{bmatrix} -(\frac{\gamma}{2} + i\omega) & 0 \\ 0 & -(\frac{\gamma}{2} - i\omega) \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} \\ &+ \begin{bmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{bmatrix} \begin{bmatrix} b \\ b^* \end{bmatrix} \\ \begin{bmatrix} b_{\text{out}} \\ b_{\text{out}}^* \end{bmatrix} &= \begin{bmatrix} \sqrt{\gamma} & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} \\ &+ \begin{bmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{bmatrix} \begin{bmatrix} b \\ b^* \end{bmatrix}. \end{aligned} \quad (52)$$

The physical parameters for the squeezed-input cavity are  $\Omega_{\text{cav, sq}-} = \omega$ ,  $\Omega_{\text{cav, sq}+} = 0$ ,  $C_{\text{cav, sq}-} = \sqrt{\gamma}$ ,  $C_{\text{cav, sq}+} = 0$ ,  $\tilde{S}_{\text{cav, sq}-} = \tilde{S}_{\text{sq}} = \Delta(\cosh r, \sinh r)$ , and so

$$G_{\text{cav, sq}} = (\Delta(\cosh \lambda, \sinh \lambda), \Delta(\sqrt{\gamma} I, 0), -i\Delta(i\omega, 0)).$$

This system is a member of  $\mathcal{L}^{\text{Bog.}}(1)$  but not of  $\mathcal{L}^{\text{HP}}(1)$ .

## VI. LINEAR QUANTUM FEEDBACK NETWORKS

We are now in a position to describe feedback networks constructed from Bogoliubov dynamical components as nodes, and boson fields as links. The general

form of such a *linear quantum feedback network* (LQFN) is shown in Figure 2. The fundamental algebraic tool for describing such networks in subsequent sections is the linear fractional transformation (LFT).

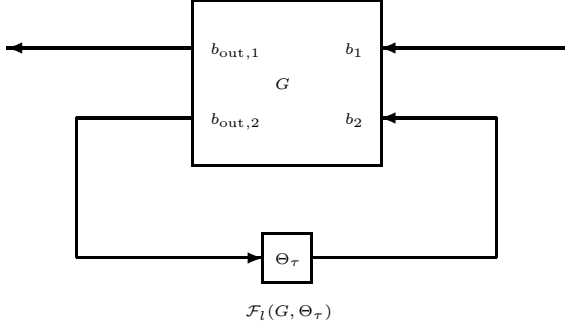


FIG. 2: General form of a *linear quantum feedback network* (LQFN), with the time delay due to the spatial extent of the feedback connection represented by  $\Theta_\tau$  (see text).

### A. Linear Fractional Transformations

Linear fractional transformations (LFTs) arise naturally when dealing with feedback networks, and a formal notation has been developed in classical linear systems theory, [20]. Consider a classical transfer function  $\Xi(s)$  partitioned as

$$\Xi(s) = \begin{bmatrix} \Xi_{11}(s) & \Xi_{12}(s) \\ \Xi_{21}(s) & \Xi_{22}(s) \end{bmatrix}$$

corresponding to a partition  $u = (u_1, u_2)^\top$ ,  $y = (y_1, y_2)^\top$  of the input and output signals. If the system is placed in a feedback arrangement defined by  $u_2 = K(s)y_2$ , then the closed-loop system is described by the transfer function

$$\mathfrak{F}(\Xi(s), K(s)) = \Xi_{11}(s) + \Xi_{12}(s)K(s)[I - \Xi_{22}(s)K(s)]^{-1}\Xi_{21}(s).$$

The arrangement is said to be well-posed whenever the inverse  $[I - \Xi_{22}(s)K(s)]^{-1}$  exist. This transfer function is obtained by eliminating the in-loop variables.

In what follows we generalize this type of representation to our class of LQFNs (see also [9], [15]).

### B. Fractional Linear Transformations

In this section we provide some technical results needed for the network theory described in subsequent sections.

**Lemma 1** Let  $\tilde{S} = \Delta(S_-, S_+) \in \text{Sp}(\mathbb{C}^{n_1+n_2})$  with block decomposition

$$S_\mp = \begin{bmatrix} S_{11}^\mp & S_{12}^\mp \\ S_{21}^\mp & S_{22}^\mp \end{bmatrix},$$

where  $S_{jk}^\mp \in \mathbb{C}^{n_j \times n_k}$ . Setting  $\hat{S}_{jk} = \Delta(S_{jk}^-, S_{jk}^+) \in \mathbb{C}^{2n_j \times 2n_k}$ , we have that

$$\sum_{k=1,2} \hat{S}_{ki}^b \hat{S}_{kj} = \sum_{k=1,2} \hat{S}_{ik} \hat{S}_{jk}^b = \delta_{ij}. \quad (53)$$

**Proof.** The relation  $\tilde{S}^b \tilde{S} = I$  implies that  $\Delta(S_-^\dagger, -S_+^\top) \Delta(S_-, S_+) = \Delta(I, 0)$ , and so  $S_-^\dagger S_- - S_+^\top S_+^\# = I$ ,  $S_-^\dagger S_+ - S_+^\top S_-^\# = 0$ . These may be written as  $\sum_{k=1,2} (S_{ki}^\dagger S_{kj}^- - S_{ki}^\top S_{kj}^\#) = \delta_{ij}$ , and  $\sum_{k=1,2} (S_{ki}^\dagger S_{kj}^+ - S_{ki}^\top S_{kj}^\#) = 0$ . Therefore

$$\begin{aligned} \sum_{k=1,2} \hat{S}_{ki}^b \hat{S}_{kj} &= \sum_{k=1,2} \Delta(S_{ki}^\dagger, -S_{ki}^\top) \Delta(S_{kj}^-, S_{kj}^+) \\ &= \sum_{k=1,2} \Delta(S_{ki}^\dagger S_{kj}^- - S_{ki}^\top S_{kj}^\#, S_{ki}^\dagger S_{kj}^+ - S_{ki}^\top S_{kj}^\#) \end{aligned}$$

which equals  $\Delta(\delta_{ij}, 0) = \delta_{ij}$ . The second identity similarly follows from  $\tilde{S} \tilde{S}^b = I$ .  $\square$

**Theorem 2** Let  $\tilde{S} \in \text{Sp}(\mathbb{C}^{n_1+n_2})$  and define the fractional linear (Möbius) transformation  $\Psi_{\tilde{S}}^{2 \rightarrow 1} : \text{dom}(\Psi_{\tilde{S}}^{2 \rightarrow 1}) \in \mathbb{C}^{n_2 \times n_2} \mapsto \mathbb{C}^{n_1 \times n_1}$  by

$$\Psi_{\tilde{S}}^{2 \rightarrow 1}(X) \triangleq \hat{S}_{11} + \hat{S}_{12}X(I - \hat{S}_{22}X)^{-1}\hat{S}_{21}, \quad (54)$$

with  $X \in \text{dom}(\Psi_{\tilde{S}}^{2 \rightarrow 1})$  if and only if the inverse  $(I - \hat{S}_{22}X)^{-1}$  exists. Then  $\Psi_{\tilde{S}}^{2 \rightarrow 1}$  maps  $\text{Sp}(\mathbb{C}^{n_2}) \cap \text{dom}(\Psi_{\tilde{S}}^{2 \rightarrow 1})$  into  $\text{Sp}(\mathbb{C}^{n_1})$ .

**Proof.** We first note the Siegel-type identities

$$\begin{aligned} \Psi_{\tilde{S}}^{2 \rightarrow 1}(X)^b \Psi_{\tilde{S}}^{2 \rightarrow 1}(Y) &= I - \hat{S}_{21}^b (I - X^b \hat{S}_{22}^b)^{-1} (I - X^b Y) (I - \hat{S}_{22} Y)^{-1} \hat{S}_{21}, \\ \Psi_{\tilde{S}}^{2 \rightarrow 1}(X) \Psi_{\tilde{S}}^{2 \rightarrow 1}(Y)^b &= I - \hat{S}_{12} (I - X \hat{S}_{22})^{-1} (I - XY^b) (I - \hat{S}_{22}^b Y^b)^{-1} \hat{S}_{12}^b. \end{aligned} \quad (55)$$

These are structurally the same as the standard Siegel identities based on partitioning a unitary  $\hat{S}$  but the in-

volution  $^b$  replacing the usual Hermitian involution  $^\dagger$ ,

see Theorem 21.16 and Corollary 21.17 of reference [7]. The identities rely on the unitary analogue of the identities (53) and so follow *mutatis mutandis*. Evidently, if  $X \in \text{Sp}(\mathbb{C}^{n_2}) \cap \text{dom}(\Psi_{\hat{S}}^{2 \rightarrow 1})$  then  $\Psi_{\hat{S}}^{2 \rightarrow 1}(X)^b \Psi_{\hat{S}}^{2 \rightarrow 1}(X) = \Psi_{\hat{S}}^{2 \rightarrow 1}(X) \Psi_{\hat{S}}^{2 \rightarrow 1}(X)^b = I$ .  $\square$

**Corollary 3** *If  $\tilde{K}(i\omega)$  is an  $\text{Sp}(\mathbb{C}^{n_2})$ -valued transfer matrix function taking values in  $\text{dom}(\Psi_{\hat{S}}^{2 \rightarrow 1})$  for all  $\omega$  real, then the fractional linear transformation  $\Psi_{\hat{S}}^{2 \rightarrow 1}(\tilde{K}(i\omega))$  will be  $\text{Sp}(\mathbb{C}^{n_1})$ -valued function of  $\omega$ .*

*In particular, if  $I \in \text{dom}(\Psi_{\hat{S}}^{2 \rightarrow 1})$  then*

$$\Psi_{\hat{S}}^{2 \rightarrow 1}(I) = \hat{S}_{11} + \hat{S}_{12}(1 - \hat{S}_{22})^{-1} \hat{S}_{21} \in \text{Sp}(\mathbb{C}^{n_1}). \quad (56)$$

### C. Finite Time-Delay LQFNs

A general LQFN is a network of linear quantum components  $G_v \in \mathcal{L}^{\text{Bog.}}(n)$ , labeled by the vertices  $v$  of the network, with quantum fields traveling along the edges. The edges are directed so that we distinguish inputs and outputs, and the multiplicity of input fields equals the multiplicity of outputs for each component.

In a physical LQFN we will have time delays associated with each internal edge due to the finite time taken by light to travel from an output port to an input port. In fact, we may lump the individual components as one single global component  $\tilde{G}$  with all external inputs going into a collective input port 1 and coming out from a collective output port 1, as in Figure 2. Likewise, all the internal fields can be viewed as traveling from the collective output port 2 to the collective input port 2. The effect of the (multichannel) time-delay can be described by the operator  $\Theta_\tau$  defined by

$$\Theta_\tau(f_1(t), \dots, f_n(t))^\top = (f_1(t - \tau_1), \dots, f_n(t - \tau_n))^\top,$$

where  $\tau_1 > 0, \dots, \tau_n > 0$  are the time delays of each channel. Here  $f_k(t)$  denotes the quantum stochastic process propagating along channel  $k$  in doubled-up form. For instance,  $f_k(t)$  could be  $\check{y}_k(t)$ , the doubled-up output quantum output processes propagating along channel  $k$ . In a slight abuse of notation, we also occasionally over-load the notation  $\Theta_\tau$  to denote the delayed version of a quantum process that is not in doubled-up form, such as when  $f_k(t)$  is taken to be  $y_k(t)$  for all  $k$ . Note that  $[\Theta_\tau(i\omega)]_{jk} = e^{i\omega\tau_j} \delta_{jk}$ . Extending the standard notation recalled in section VIA), we denote this by

$$\tilde{N}_\tau = \mathfrak{F}(\tilde{G}, \Theta_\tau).$$

A Hamiltonian for a LQFN with squeezing components could be constructed approximately by replacing Bogoliubov components  $\tilde{S}$  with dynamical components  $G_{\tilde{S}}^\epsilon$ . This would then fit into the QFN framework of [8].

### D. Parameters for Network Model

We now suppose that the LQFN of Figure 2 is described by field channels  $b_1, b_{\text{out},1}$  and  $b_2, b_{\text{out},2}$  having lengths  $n_1$  and  $n_2$ , respectively, so that the total number is  $n_1 + n_2 = n$ . The system is parameterized by  $G = (\tilde{S}, L, H)$ , with  $\tilde{S} = \Delta(S_-, S_+)$  and we partition the matrices as

$$C_\mp = \begin{bmatrix} C_1^\mp \\ C_2^\mp \end{bmatrix}, \quad S_\mp = \begin{bmatrix} S_{11}^\mp & S_{12}^\mp \\ S_{21}^\mp & S_{22}^\mp \end{bmatrix}.$$

The field-field component of the input-output relations can be now written as

$$\check{b}_{\text{out},i} = \sum_{j=1,2} \hat{G}_{ij}(s) \check{b}_j,$$

with transfer matrix function

$$\begin{aligned} \hat{\Xi}_G(s) &= \left[ \begin{array}{c|c} \tilde{A} & -[\tilde{C}_1^b, \tilde{C}_2^b] \hat{S} \\ \hline \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & \hat{S} \end{array} \right] (s) \\ &= - \begin{bmatrix} \tilde{C}_1 \\ \tilde{C}_2 \end{bmatrix} (sI - \tilde{A})^{-1} [\tilde{C}_1^b, \tilde{C}_2^b] \hat{S} + \hat{S} \end{aligned} \quad (57)$$

where

$$\hat{S}_{jk} = \Delta(S_{jk}^-, S_{jk}^+), \quad \tilde{C}_j = \Delta(C_j^-, C_j^+).$$

The network  $N_\tau$  is given by the linear fractional transformation

$$\tilde{N}_\tau = \mathfrak{F}(\tilde{G}, \Theta_\tau) = \left[ \begin{array}{c|c} \tilde{A}_\tau & -\tilde{C}_\tau^b \tilde{S}_\tau \\ \hline \tilde{C}_\tau & \tilde{S}_\tau \end{array} \right] \quad (58)$$

where

$$\tilde{S}_\tau = \hat{S}_{11} + \hat{S}_{12} \Theta_\tau (I - \hat{S}_{22} \Theta_\tau)^{-1} \hat{S}_{21} \quad (59)$$

$$\tilde{C}_\tau = \tilde{C}_1 + \hat{S}_{12} \Theta_\tau (I - \hat{S}_{22} \Theta_\tau)^{-1} \tilde{C}_2 \quad (60)$$

$$\tilde{A}_\tau = \tilde{A} - \sum_{j=1,2} \tilde{C}_j^b \hat{S}_{j2} \Theta_\tau (I - \hat{S}_{22} \Theta_\tau)^{-1} \tilde{C}_2. \quad (61)$$

Due to the non-zero delay, the network model  $N_\tau$  is non-Markovian.

### E. Zero Delay Limit Models

Of particular interest are the simpler models that arise in the zero-delay limit  $\Theta_\tau \rightarrow I$  ( $\tau \rightarrow 0$ ). Assume that  $I - \hat{S}_{11}$  is invertible. From above we have

$$\tilde{N}_0 = \mathfrak{F}(\tilde{G}, I) = \left[ \begin{array}{c|c} \tilde{A}_0 & -\tilde{C}_0^b \tilde{S}_0 \\ \hline \tilde{C}_0 & \tilde{S}_0 \end{array} \right] \quad (62)$$

where

$$\tilde{S}_0 = \hat{S}_{11} + \hat{S}_{12} (I - \hat{S}_{22})^{-1} \hat{S}_{21} \quad (63)$$

$$\tilde{C}_0 = \tilde{C}_1 + \hat{S}_{12} (I - \hat{S}_{22})^{-1} \tilde{C}_2 \quad (64)$$

$$\tilde{A}_0 = \tilde{A} - \sum_{j=1,2} \tilde{C}_j^b \hat{S}_{j2} (I - \hat{S}_{22})^{-1} \tilde{C}_2. \quad (65)$$



We note that

$$\tilde{A}_0 = -\frac{1}{2}\tilde{C}_0^\dagger\tilde{C}_0 - i\tilde{\Omega}_0, \quad (66)$$

where

$$\tilde{\Omega}_0 = \tilde{\Omega} + \text{Im}_b \sum_{j=1,2} \tilde{C}_j^\dagger \hat{S}_{j2} (I - \hat{S}_{22})^{-1} \tilde{C}_2. \quad (67)$$

Here,  $\text{Im}_b X$  means  $\frac{1}{2i}(X - X^\dagger)$ . The matrix  $\tilde{S}_0$  defined by (63) is a Bogoliubov matrix, as it corresponds to the matrix in (56). Therefore the zero-delay limit  $N_0$  is a Markovian system belongs to  $\mathcal{L}^{\text{Bog.}}(n)$  with parameters

$$\begin{aligned} N_0 = & \left( \hat{S}_{11} + \hat{S}_{12}(I - \hat{S}_{22})^{-1}\hat{S}_{21}, \right. \\ & \tilde{C}_1 + \hat{S}_{12}(I - \hat{S}_{22})^{-1}\tilde{C}_2, \\ & \left. \tilde{\Omega} + \text{Im}_b \sum_{j=1,2} \tilde{C}_j^\dagger \hat{S}_{j2} (I - \hat{S}_{22})^{-1} \tilde{C}_2 \right). \end{aligned} \quad (68)$$

Thus  $\mathcal{L}^{\text{Bog.}}(n)$  is closed with respect to this zero-delay limit network construction.

Other types of limits are also considered in applications (see, e.g. [1, sec. 2.3]). Suppose that the system  $G^\epsilon$  and the delay  $\tau^\epsilon$  depend on a small parameter  $\epsilon > 0$ , defining a physical regime of operation. Then one may obtain a limit model  $\lim_{\epsilon \rightarrow 0} \mathfrak{F}(G^\epsilon, \Theta_{\tau^\epsilon})$ . An example of this is considered in section VII B below.

## F. Series Product

The series product  $G_2 \triangleleft G_1$  of two systems  $\tilde{G}_1 = (\tilde{S}_1, \tilde{C}_1, \tilde{\Omega}_1)$  and  $\tilde{G}_2 = (\tilde{S}_2, \tilde{C}_2, \tilde{\Omega}_2)$  follows from the zero-delay limit (62). For the series product, we interchange the index 1 and 2 (this simply means interchanging the role of  $G_1$  and  $G_2$  in the LQFN) and then set  $\hat{S}_{12} = \tilde{S}_1$ ,  $\hat{S}_{21} = \tilde{S}_2$ ,  $\hat{S}_{22} = 0$  and  $\hat{S}_{11} = 0$  (note that without the interchange we would be computing  $G_1 \triangleleft G_2$  instead  $G_2 \triangleleft G_1$ ).

Substituting into (63,64,65) we find

$$\tilde{\Xi}_{\text{series}} = \left[ \frac{\tilde{A} - \tilde{C}_2^\dagger \tilde{S}_2 \tilde{C}_1}{\tilde{C}_2 + \tilde{S}_2 \tilde{C}_1} \middle| - \frac{(\tilde{C}_2^\dagger \tilde{S}_2 + \tilde{C}_1^\dagger) \tilde{S}_1}{\tilde{S}_2 \tilde{S}_1} \right]. \quad (69)$$

The matrices for  $G_2 \triangleleft G_1 = (\tilde{S}_{\text{series}}, \tilde{C}_{\text{series}}, \tilde{\Omega}_{\text{series}})$  are given by

$$\begin{aligned} \tilde{S}_{\text{series}} &= \tilde{S}_2 \tilde{S}_1, \\ \tilde{C}_{\text{series}} &= \tilde{C}_2 + \tilde{S}_2 \tilde{C}_1 = \Delta(C_{\text{series-}}, C_{\text{series+}}), \\ \tilde{\Omega}_{\text{series}} &= \tilde{\Omega}_1 + \tilde{\Omega}_2 + \text{Im}_b \tilde{C}_2^\dagger \tilde{S}_2 \tilde{C}_1 \\ &\equiv \Delta(\Omega_{\text{series-}}, \Omega_{\text{series+}}), \end{aligned}$$

where

$$\begin{aligned} C_{\text{series-}} &= C_{2-} + S_{2-}C_{1-} + S_{2+}C_{1+}^\#, \\ C_{\text{series+}} &= C_{2+} + S_{2-}C_{1+} + S_{2+}C_{1-}^\#. \end{aligned}$$

From

$$\begin{aligned} \tilde{C}_2^\dagger \tilde{S}_2 \tilde{C}_1 &= \Delta(C_{2-}^\dagger, -C_{2+}^\dagger) \Delta(S_{2-}, S_{2+}) \Delta(C_{1-}, C_{1+}) \\ &= \Delta(X_-, X_+) \end{aligned}$$

with

$$X_{\mp} = (C_{2-}^\dagger S_{2-} - C_{2+}^\dagger S_{2+}^\#) C_{1\mp} + (C_{2-}^\dagger S_{2+} - C_{2+}^\dagger S_{2-}^\#) C_{1\pm}^\#,$$

we see that

$$\begin{aligned} \Omega_{\text{series-}} &= \Omega_{1-} + \Omega_{2-} + \frac{1}{2i}(X_- - X_-^\dagger), \\ \Omega_{\text{series+}} &= \Omega_{1+} + \Omega_{2+} + \frac{1}{2i}(X_+ + X_+^\dagger). \end{aligned}$$

Succinctly, the series product in  $\mathcal{L}^{\text{Bog.}}(n)$  is given by

$$G_2 \triangleleft G_1 = (\tilde{S}_2 \tilde{S}_1, \tilde{C}_2 + \tilde{S}_2 \tilde{C}_1, \tilde{\Omega}_1 + \tilde{\Omega}_2 + \text{Im}_b \{ \tilde{C}_2^\dagger \tilde{S}_2 \tilde{C}_1 \}); \quad (70)$$

cf. (34).

Clearly,  $\mathcal{L}^{\text{Bog.}}(n)$  is a group with respect to the series product, and the classes of components  $\mathcal{L}^{\text{HP}}(n)$ ,  $\text{Sp}(\mathbb{C}^n)$ , and  $U(n)$  are subgroups. If  $G = (\tilde{S}, \tilde{C}, \tilde{\Omega})$  the inverse is given by

$$G^{-1} = (\tilde{S}^\dagger, -\tilde{S}^\dagger \tilde{C}, -\tilde{\Omega}), \quad (71)$$

with transfer function given by

$$\tilde{\Xi}_{G^{-1}}(s) \equiv \tilde{\Xi}_G(s^*)^\dagger. \quad (72)$$

The series product is not limited simply to feedforward, and the above formula apply to the case where the two systems have one or more modes in common. The series product is therefore highly non-trivial [8, 9, 15].

## G. Series Product and Cascaded Transfer Functions

In classical linear systems theory, the transfer function of a cascade of two separate systems is obtained by multiplying the transfer functions (see, e.g., [20]).

We need to emphasize here that two systems will be distinct if they consist of different oscillator modes. Specifically let there be  $m_1$  modes in the first system

and  $m_2$  in the second, and set  $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathcal{S}(m_1 + m_2)$ ,

then we wish to consider  $G_1 = (\tilde{S}_1, \tilde{C}_1, \tilde{\Omega}_1)$  and  $G_2 = (\tilde{S}_2, \tilde{C}_2, \tilde{\Omega}_2)$  with

$$\begin{aligned} \tilde{C}_1 &= \Delta([C_{1-}, 0], [C_{1+}, 0]), \\ -i\tilde{\Omega}_1 &= -\Delta\left(\begin{bmatrix} i\Omega_{1-} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} i\Omega_{1+} & 0 \\ 0 & 0 \end{bmatrix}\right), \\ \tilde{C}_2 &= \Delta([0, C_{2-}], [0, C_{2+}]), \\ -i\tilde{\Omega}_2 &= -\Delta\left(\begin{bmatrix} 0 & 0 \\ 0 & i\Omega_{2-} \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & i\Omega_{2+} \end{bmatrix}\right), \end{aligned}$$



with respect to the decomposition  $\mathbb{C}^m = \mathbb{C}^{m_1} \oplus \mathbb{C}^{m_2}$ . This is simply a statement that the dynamics of  $G_1$  does not depend on the internal variables of  $G_2$ , and vice-versa. Putting this particular form into (69), we then obtain

$$\begin{aligned} \tilde{\Xi}_{G_2 \triangleleft G_1}(s) &= \left[ \begin{array}{c|c} \left[ \begin{array}{cc} \bar{A}_1 & 0 \\ -\bar{C}_2^b \tilde{S}_2 \bar{C}_1 & \bar{A}_2 \end{array} \right] & \left[ \begin{array}{c} -\bar{C}_1^b \tilde{S}_1 \\ -\bar{C}_2^b \tilde{S}_2 \tilde{S}_1 \end{array} \right] \\ \hline \left[ \begin{array}{c} \tilde{S}_2 \bar{C}_1, \bar{C}_2 \end{array} \right] & \tilde{S}_2 \tilde{S}_1 \end{array} \right] (s) \\ &= \tilde{\Xi}_{G_2}(s) \tilde{\Xi}_{G_1}(s), \end{aligned} \quad (73)$$

where  $\bar{C}_j = \Delta(C_{j,-}, C_{j,+})$  and  $\bar{A}_j = -\frac{1}{2}\bar{C}_j^b \bar{C}_j - i\Delta(\Omega_{j,-}, \Omega_{j,+})$ . Here we use the unitary transformation  $\check{a} \mapsto \begin{bmatrix} \check{a}_1 \\ \check{a}_2 \end{bmatrix}$  to present the transfer function in a more convenient form. The algebra is then similar to Section IV A in [15].

If the systems are not separate, then we do not expect such a factorization of the transfer function to hold. The series product [8], [9], [15] is defined quite generally in terms of physical parameters (section III E) which may, for example, depend on the same oscillator mode variables). In the general case, the transfer function can be computed using the general formulas (62)-(65).

Let us remark that the series product inverse  $G^{-1}$  given by (71) may be realized in terms of a physical system that is *not* separate from the original system. Physically, if we pass input fields through a system with parameters  $G$ , then  $G^{-1}$  gives the parameters required to undo the effect by passing the output back through the *same* system for a second pass.

## H. Inverse Transfer Functions

The input-output relation  $\check{b}_{\text{out}} = \tilde{\Xi} \check{b}_{\text{in}} + \tilde{\xi} \check{a}(0)$  may be inverted to yield

$$\check{b}_{\text{in}} = \tilde{\Xi}^{-1} \check{b}_{\text{out}} - \tilde{\Xi}^{-1} \tilde{\xi} \check{a}(0).$$

In particular, we can give the following useful description of  $\tilde{\Xi}^{-1}$ . The linear equations (28) in the time domain may be rearranged algebraically to give

$$\begin{aligned} \frac{d}{dt} \check{a} &= (\tilde{A} - \tilde{B} \tilde{D}^{-1} \tilde{C}) \check{a} + \tilde{B} \tilde{D}^{-1} \check{b}_{\text{out}}, \\ \check{b}_{\text{in}} &= -\tilde{D}^{-1} \tilde{C} \check{a} + \tilde{D}^{-1} \check{b}_{\text{out}}, \end{aligned}$$

with  $\tilde{D}$  invertible, and in the transform domain, we deduce that

$$\left[ \begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array} \right]^{-1} = \left[ \begin{array}{c|c} \tilde{A} - \tilde{B} \tilde{D}^{-1} \tilde{C} & \tilde{B} \tilde{D}^{-1} \\ \hline -\tilde{D}^{-1} \tilde{C} & \tilde{D}^{-1} \end{array} \right].$$

For the model with parameters  $G = (\tilde{S}, \tilde{C}, \tilde{\Omega})$ , we find

$$\begin{aligned} \tilde{\Xi}_G^{-1}(s) &= \left[ \begin{array}{c|c} \tilde{A} & -\tilde{C}^b \tilde{S} \\ \hline \tilde{C} & \tilde{S} \end{array} \right]^{-1} \equiv \left[ \begin{array}{c|c} -\tilde{A}^b & -\tilde{C}^b \\ \hline -\tilde{S}^b \tilde{C} & \tilde{S}^b \end{array} \right] \\ &= \tilde{S}^b \tilde{C} (sI + \tilde{A}^b)^{-1} \tilde{C}^b + \tilde{S}^b, \end{aligned}$$

or

$$\tilde{\Xi}_G(s)^{-1} \equiv \tilde{\Xi}_G(-s^*)^b. \quad (74)$$

We note that  $\tilde{\Xi}_{G^{-1}}(s) = \tilde{\Xi}_G(-s)^{-1} \equiv \tilde{\Xi}_G(s^*)^b$  (recall (72)).

### 1. Example: separate cavity inverse

As a concrete example, consider the single mode cavity  $G = G_{\text{cav}}$  considered in IV B. The transfer function (45) and related functions are given by

$$\begin{aligned} \Xi_{G,-}(s) &= \frac{s - \frac{\gamma}{2} + i\omega}{s + \frac{\gamma}{2} + i\omega}, \\ \Xi_{G^{-1},-}(s) &= \frac{s - \frac{\gamma}{2} - i\omega}{s + \frac{\gamma}{2} - i\omega}, \\ \Xi_{G,-}^{-1}(s) &= \frac{s + \frac{\gamma}{2} + i\omega}{s - \frac{\gamma}{2} + i\omega}. \end{aligned}$$

That is,  $G^{-1}$  is obtained from  $G$  by keeping  $C_- = \sqrt{\gamma}$ ,  $C_+ = 0$  and replacing  $\Omega_- = \omega$  by  $-\omega$ . In what follows we obtain a physical realization  $\hat{G}$  of the transfer function  $\Xi_{G,-}^{-1}(s)$  that is a system distinct from  $G$ , so that

$$\tilde{\Xi}_{\hat{G}}(s) = \tilde{\Xi}_G^{-1}(s). \quad (75)$$

The system  $\hat{G}$  is obtained by setting  $\Omega_- = -\omega, \Omega_+ = 0$  and swapping  $C_- = 0$  and  $C_+ = \sqrt{\gamma}$ .

In this example we can take the two modes to be  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathcal{S}(2)$  and write  $G$  as  $(\tilde{S}_1 = I, \tilde{C}_1, \tilde{\Omega}_1)$  where

$$\begin{aligned} \tilde{C}_1 &= \Delta([\sqrt{\gamma}, 0], [0, 0]), \\ -i\tilde{\Omega}_1 &= \Delta\left(\begin{bmatrix} -i\omega & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right), \end{aligned}$$

and  $\hat{G}$  as  $(\tilde{S}_2 = I, \tilde{C}_2, -\tilde{\Omega}_2)$  where

$$\begin{aligned} \tilde{C}_2 &= \Delta([0, 0], [0, \sqrt{\gamma}]), \\ -i\tilde{\Omega}_2 &= -\Delta\left(\begin{bmatrix} -i\omega & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right). \end{aligned}$$

That is, in terms of scattering matrices, coupling operators and Hamiltonians (31) we have

$$G = (I, \sqrt{\gamma} a_1, \omega a_1^* a_1), \text{ and } \hat{G} = (I, \sqrt{\gamma} a_1^*, -\omega a_2^* a_2).$$

The series product  $\hat{G} \triangleleft G$  is given by  $(I, \tilde{C} = \tilde{C}_1 + \tilde{C}_2, \tilde{\Omega} = \tilde{\Omega}_1 + \tilde{\Omega}_2 + \text{Im}_b[\tilde{C}_2^b \tilde{C}_1])$  (recall (70)) and this corresponds to a system with a nontrivial dynamics, since (by (34))

$$\hat{G} \triangleleft G = (I, \sqrt{\gamma} (a_1 + a_2^*), \omega (a_1^* a_1 - a_2^* a_2) + \text{Im}[a_2 a_1]).$$

Nevertheless, some calculation shows that

$$\tilde{\Xi}_{\hat{G} \triangleleft G}(s) = I,$$

as required by (73) and (75). This continues to hold if there is a term  $\Omega_+ = \omega_+$  added to the cavity Hamiltonian.

### 2. General separate system inverses

We now show how to obtain a physical realization for  $\Xi_G^{-1}(s)$  in the case where  $\tilde{\Omega} = 0$ . We have

$$\begin{aligned} \tilde{\Xi}_G^{-1}(s) &= \tilde{S}^b + \tilde{S}^b \tilde{C} (sI - \frac{1}{2} \tilde{C}^b \tilde{C})^{-1} \tilde{C}^b \\ &= [I + \tilde{S}^b (s - \frac{1}{2} \tilde{C} \tilde{C}^b)^{-1} \tilde{C} \tilde{C}^b] \tilde{S}^b. \end{aligned}$$

Now  $\tilde{C} \tilde{C}^b$  is not definite and in fact we have

$$\begin{aligned} \tilde{C} \tilde{C}^b &= \Delta(C_- C_-^\dagger - C_+ C_+^\dagger, -C_- C_+^\top + C_+ C_-^\top) \\ &= -\Delta(C_+, C_-) \Delta(C_+^\dagger, -C_-^\top) \\ &\equiv -\tilde{K} \tilde{K}^b, \end{aligned}$$

where

$$\tilde{K} = \Delta(C_+, C_-). \quad (76)$$

We therefore obtain

$$\begin{aligned} \tilde{\Xi}_G^{-1}(s) &= [I - \tilde{S}^b (s + \frac{1}{2} \tilde{K} \tilde{K}^b)^{-1} \tilde{K} \tilde{K}^b] \tilde{S}^b \\ &= [I - \tilde{S}^b \tilde{K} (s + \frac{1}{2} \tilde{K}^b \tilde{K})^{-1} \tilde{K}^b] \tilde{S}^b \\ &\equiv \tilde{\Xi}_{\hat{G}}(s), \end{aligned}$$

where

$$\hat{G} = (\tilde{S}^b, \tilde{S}^b \tilde{K}, 0). \quad (77)$$

We note that this choice of  $\hat{G}$  is not unique in producing a transfer function inverse inverse to  $\tilde{\Xi}_G$ . Finding an inverse when  $\tilde{\Omega} \neq 0$  is more involved.

### 3. Comments

We have the mapping  $G \mapsto \tilde{\Xi}_G$  from the group of  $\mathcal{L}^{\text{Bog}}(n)$  of systems parameters with series product to the group of matrix transfer functions. This mapping is not however a group homomorphism. Indeed, we typically have

$$\tilde{\Xi}_{G_2 \triangleleft G_1}(s) \neq \tilde{\Xi}_{G_2}(s) \tilde{\Xi}_{G_1}(s),$$

though equality - the cascade formula (73) - holds when the systems are separate assemblies of oscillators.

We should also caution that, as we have seen in the example in VIH 1, there are solutions for  $G$  other than the trivial  $G = (I, 0, 0)$  to the equation  $\tilde{\Xi}_G(s) = I$ .

## VII. NETWORK EXAMPLES

### A. In-loop Squeezing and Cavity as a Feedback Network

We now describe the LQFN of Figure 1, which contains a cavity and squeezer in a feedback loop resulting from interconnection with a beamsplitter. The total propagation delay around the loop is  $\tau$ , which we take to be small and send  $\tau \rightarrow 0$ . In order to determine an equivalent zero-delay limit model, following the general approach of section VI E, we re-draw the network as shown in Figure 3.

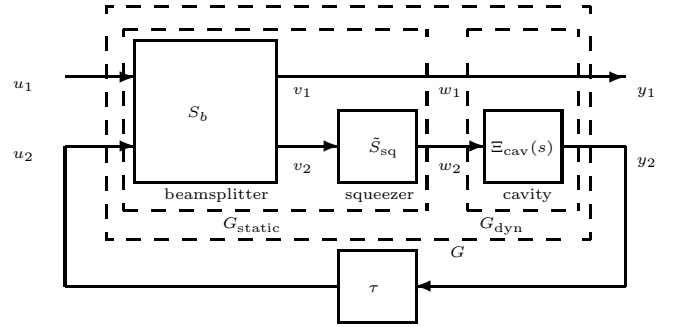


FIG. 3: Linear quantum feedback network of Figure 1 redrawn in standard form as Figure 2.

As indicated in Figure 3, the in-loop system  $G = G_{\text{dyn}} \triangleleft G_{\text{static}}$  is a dynamical Bogoliubov component obtained by cascading the beamsplitter, the (augmented) squeezer (which together form  $G_{\text{static}}$ ), and the (augmented) cavity  $G_{\text{dyn}}$ .

The static part  $G_{\text{static}}$  is described as follows. Because the beamsplitter has two inputs and two outputs, we augment the squeezer  $S_{\text{sq}}$  (given by (50)) by including a direct feed through channel ( $v_1$  to  $w_1$  in Figure 3). Because the squeezer is represented by a static Bogoliubov transformation expressed in doubled-up form, we express the beamsplitter in doubled-up form:  $\tilde{S}_b = \Delta(S_b, 0)$ . To be clear, the beamsplitter is described by

$$\begin{aligned} v_1 &= \alpha u_1 - \beta u_2 \\ v_2 &= \beta u_1 + \alpha u_2, \end{aligned} \quad (78)$$

where  $|\alpha|^2 + |\beta|^2 = 1$ ,  $\alpha^* \beta = \beta^* \alpha$ . Thus we have

$$S_b = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}, \quad (79)$$

and

$$\begin{aligned} \tilde{S}_b &= \Delta(S_b, 0) \\ &= \begin{bmatrix} \alpha & -\beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha^* & -\beta^* \\ 0 & 0 & \beta^* & \alpha^* \end{bmatrix}. \end{aligned} \quad (80)$$

The static component  $G_{\text{static}}$  has inputs  $(u_1, u_2)^\top$  and outputs  $(w_1, w_2)^\top$ , and is given by the Bogoliubov matrix

$$\begin{aligned}\tilde{R} &= \Delta \left( \begin{bmatrix} 1 & 0 \\ 0 & \cosh r \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & \sinh r \end{bmatrix} \right) \tilde{S}_b \\ &= \Delta(R_-, R_+)\end{aligned}\quad (81)$$

where

$$\begin{aligned}R_- &= \begin{bmatrix} \alpha & -\beta \\ \beta \cosh r & \alpha \cosh r \end{bmatrix}, \\ R_+ &= \begin{bmatrix} 0 & 0 \\ \beta^* \sinh r & \alpha^* \sinh r \end{bmatrix}.\end{aligned}\quad (82)$$

The dynamic component  $G_{\text{dyn}}$ , with inputs  $(w_1, w_2)^\top$  and outputs  $(y_1, y_2)^\top$ , is given by

$$\begin{aligned}\begin{bmatrix} \dot{a} \\ \dot{a}^* \end{bmatrix} &= \begin{bmatrix} -(\frac{\gamma}{2} + i\omega) & 0 \\ 0 & -(\frac{\gamma}{2} - i\omega) \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & \sqrt{\gamma} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\gamma} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_1^* \\ w_2^* \end{bmatrix}, \\ \begin{bmatrix} y_1 \\ y_2 \\ y_1^* \\ y_2^* \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \\ 0 & 0 \\ 0 & \sqrt{\gamma} \end{bmatrix} \begin{bmatrix} a \\ a^* \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ w_1^* \\ w_2^* \end{bmatrix}.\end{aligned}\quad (83)$$

Thus  $A_- = -(\frac{\gamma}{2} + i\omega)$ ,  $A_+ = 0$ ,  $\tilde{A} = \Delta(-(\frac{\gamma}{2} + i\omega), 0)$ ,

$$C_- = \begin{bmatrix} 0 \\ \sqrt{\gamma} \end{bmatrix}, \quad C_+ = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \tilde{C} = \Delta\left(\begin{bmatrix} 0 \\ \sqrt{\gamma} \end{bmatrix}, 0\right). \quad (84)$$

Also,  $\Omega_- = \omega$ ,  $\Omega_+ = 0$ .

Now that we have a complete model for the in-loop system  $G$ , we may apply the formulas in section VI E to obtain a zero-delay network model  $N_0$ . This involves first working out the Bogoliubov matrix in partitioned form:

$$\begin{aligned}\hat{S}_{11} &= \Delta(\alpha, 0), \quad \hat{S}_{12} = \Delta(-\beta, 0), \\ \hat{S}_{21} &= \Delta(\beta \cosh r, \beta^* \sinh r), \\ \hat{S}_{22} &= \Delta(\alpha \cosh r, \alpha^* \sinh r).\end{aligned}\quad (85)$$

We set  $\alpha = \sqrt{\epsilon}$  and  $\beta = \sqrt{1-\epsilon}$  to simplify some of the algebra. The network model is given as follows. We now use the equations (63,64,65) to determine the network parameters. The equivalent network Bogoliubov matrix is

$$\tilde{S}_0 = \Delta\left(\sqrt{\epsilon} - \frac{1-\epsilon}{\mu}(\cosh r - \sqrt{\epsilon}), -\frac{(1-\epsilon)\sinh r}{\mu}\right), \quad (86)$$

where

$$\mu = 1 - 2\sqrt{\epsilon} \cosh r + \epsilon. \quad (87)$$

Next,

$$\tilde{C}_0 = -\frac{\sqrt{1-\epsilon}\sqrt{\gamma}}{\mu} \Delta(1 - \sqrt{\epsilon} \cosh r, \sqrt{\epsilon} \sinh r), \quad (88)$$

so that

$$\begin{aligned}C_0^- &= -\frac{\sqrt{1-\epsilon}\sqrt{\gamma}}{\mu}(1 - \sqrt{\epsilon} \cosh r), \\ C_0^+ &= -\frac{\sqrt{1-\epsilon}\sqrt{\gamma}}{\mu}\sqrt{\epsilon} \sinh r.\end{aligned}$$

Now  $\tilde{A}_0 = \Delta(A_0^-, A_0^+)$ , where

$$A_0^- = -\left(\frac{\gamma}{2} + i\omega\right) - \frac{\sqrt{\epsilon}\gamma}{\mu}(\cosh r - \sqrt{\epsilon}), \quad A_0^+ = \frac{\sqrt{\epsilon}\gamma}{\mu} \sinh r \quad (89)$$

From this we compute

$$\Omega_0^- = \omega, \quad \Omega_0^+ = i\frac{\sqrt{\epsilon}\gamma}{\mu} \sinh r. \quad (90)$$

We therefore see that not only does the network model  $N_0 \in \mathcal{L}^{\text{Bog}}(1)$  have a non-trivial static Bogoliubov term, it also has field couplings involving a creation operator  $a^*$ , and Hamiltonian terms involving  $a^2$  and  $(a^*)^2$ .

Stability of the feedback system may be analyzed using the methods of section III D or the small gain theorem [20], [4].

As a possible application, we note that the squeezing parameter of a DPA may be altered by placing it in-loop in a beam-splitter arrangement of this type [25].

## B. Dynamics from Feedback

In this example we give an illustration from quantum optics showing that LQFNs involving only static components may give rise to dynamical behavior. This dynamical behavior is due to a time delay in the feedback loop. We consider the network shown in Figure 1, [19], [8]. This is a special case of the LQFN network of Figure 1, but with no squeezing and no cavity. The beamsplitter  $S_b$  is given by (78) or 79, with  $\alpha = \sqrt{\epsilon}$  and  $\beta = \sqrt{1-\epsilon}$ .

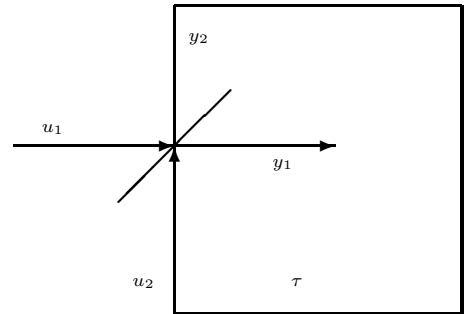


FIG. 4: Beamsplitter feedback network with time delay  $\tau$ .

Feedback in the network is defined by the constraint  $u_2(t) = \Theta_\tau y_2(t) = y_2(t - \tau)$ , where  $\tau > 0$  is the time taken for light to travel from the output to the input.

This network is a LQFN with  $G^\epsilon = (\Delta(S_b, 0), 0, 0) \in U(2)$ . With  $\epsilon > 0$  fixed, the zero-delay network model  $\tau \rightarrow 0$  is the system  $N_0 = \mathfrak{F}_l(G^\epsilon, I) = (\Delta(-I, 0), 0, 0) \in U(1)$ , with transfer function  $N_0(s) = -1$ , a trivial pass-through system with sign change (phase shift).

Now if the reflectivity coefficient  $\epsilon$  and the time delay are comparable, say  $\tau = \epsilon/\gamma$ , where  $\gamma > 0$ , then we obtain a dynamical model as  $\epsilon \rightarrow 0$ , [1, Section 2.3] (recall section VI E above). Indeed, solving (79) and (80) in the frequency domain we find that the transfer function is

$$N^\epsilon(s) = \mathfrak{F}_l(G^\epsilon, \Theta_{\tau\epsilon})(s) = \sqrt{1-\epsilon} - \frac{\epsilon e^{-s\epsilon/\gamma}}{1 - \sqrt{1-\epsilon} e^{-s\epsilon/\gamma}} \quad (91)$$

By L'Hopital's rule, we find that the limit transfer function is

$$N(s) = \lim_{\epsilon \rightarrow 0} N^\epsilon(s) = 1 - \frac{\gamma}{s + \gamma/2} = \frac{s - \gamma/2}{s + \gamma/2}. \quad (92)$$

This transfer function corresponds to a cavity  $N = (I, \sqrt{\gamma}a, 0) = (I, \sqrt{\gamma}I, 0) \in \mathcal{L}^{\text{HP}}(1, 1)$ , where  $a \in \mathcal{S}(1)$ , [6], [1]. Here,  $\gamma$  plays the role of the coupling strength between the trapped cavity mode and the external free field.

This example shows that  $U(2)$  is not closed under this type of physically natural approximation process (since the limit belongs to  $\mathcal{L}^{\text{HP}}(1)$  which is outside  $U(2)$ ),

while  $\mathcal{L}^{\text{Bog}}(2)$  is closed (since it contains both  $U(2)$  and  $\mathcal{L}^{\text{HP}}(1)$ ).

## VIII. CONCLUSION

We have shown how to extend linear quantum dynamical network theory to include static Bogoliubov components (such as squeezers). This unified framework accommodates squeezing components which are important in quantum information applications. We provided tools for describing network connections and feedback using generalizations of linear fractional transformations and the series product, [9], [15], [19], [20, Chapter 10]. We have also defined input output maps and transfer functions within this linear quantum network theory, and shown how they can be used in applications. Finally, we explained the natural group structure arising from the series product.

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- [26] To see this, let  $E \geq 0$  be a complex  $k \times k$  matrix with  $E_{11} = 0$ . Set  $u(t) = (tx_1, x_2, \dots, x_k)^\top$  then we have  $0 \leq u(t)^\dagger E u(t) = 2t \operatorname{Re} \sum_{j>1} x_1 E_{1j} x_j + \sum_{j,k>1} x_j^* E_{jk} x_k$ . However, the only way to guarantee this inequality for all real  $t$  is to require that  $\operatorname{Re} \sum_{j>1} x_1 E_{1j} x_j = 0$ . Replacing  $t$  with  $it$  shows that  $\operatorname{Im} \sum_{j>1} x_1 E_{1j} x_j$  likewise vanishes. As this must be true for arbitrary  $x_j$ , we conclude that  $E_{1j} = E_{j1}^* = 0$  for all  $j > 1$ .