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# Continuous measurement of canonical observables and limit Stochastic Schrödinger equations

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## Abstract

We derive the stochastic Schrödinger equation for the limit of continuous weak measurement where the observables monitored are canonical position and momentum. To this end we extend an argument due to Smolianov and Truman from the von Neumann model of indirect measurement of position to the Arthurs and Kelly model for simultaneous measurement of position and momentum. We only require unbiasedness of the detector states and an integrability condition sufficient to ensure a central limit effect. Despite taking a weak interaction, as opposed to weak measurement limit, the resulting stochastic wave equation is of the same form as that derived in a recent paper by Scott and Milburn for the specific case of joint Gaussian states.

## 1 Introduction

The theory of continuous measurements of a quantum system, based on the Ludwig formalism, was originally presented by Barchielli et al. [1]. Since then stochastic Schrödinger equations describing the dynamical evolution of the state of a system conditional on observations of a monitored set of observables  $\{\hat{X}_j\}$  have been developed by several authors [2][3][4][5] [6]. The generally accepted form is an adapted stochastic differential equation for a vector state valued process  $|\Psi\rangle$  of the type (see, for instance [7])

$$|d\Psi\rangle = \left\{ \frac{1}{i\hbar} \hat{H} - \sum_j \kappa_j \left( \hat{X}_j - \langle \hat{X}_j \rangle \right)^2 \right\} |\Psi\rangle dt - \sum_j \sqrt{2\kappa_j} \left( \hat{X}_j - \langle \hat{X}_j \rangle \right) |\Psi\rangle dB_t^{(j)}. \quad (1)$$

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where  $\langle \hat{X}_j \rangle = \langle \Psi | \hat{X}_j | \Psi \rangle$  are current observations (thus making the equation non-linear) and  $\{B^{(j)}\}$  is a multidimensional Wiener process. The constants  $\kappa_j$  are positive and  $\hat{H}$  is the Hamiltonian governing the background evolution.

Note that the stochastic wave function is normalized, since, by the Itô rule  $dB_t^{(j)} dB_t^{(k)} = \delta_{jk} dt$ ,

$$d \|\Psi\|^2 = \langle d\Psi | \Psi \rangle + \langle \Psi | d\Psi \rangle + \langle d\Psi | d\Psi \rangle = 0. \quad (2)$$

If  $\hat{Y}$  is an observable and if we set  $\langle \hat{Y} \rangle = \langle \Psi | \hat{Y} | \Psi \rangle$  then

$$d \langle \hat{Y} \rangle = \langle \mathcal{L}(\hat{Y}) \rangle dt - 2 \sum_j \sqrt{2\kappa_j} \text{cov}(\hat{Y}, \hat{X}_j) dB_t^{(j)}$$

where  $\mathcal{L}$  is the (self-dual) Lindblad generator

$$\mathcal{L}(\hat{Y}) = \frac{1}{i\hbar} [\hat{Y}, \hat{H}] + \sum_j \kappa_j \left\{ [\hat{X}_j, \hat{Y}] \hat{X}_j + \hat{X}_j [\hat{Y}, \hat{X}_j] \right\}$$

and we have the covariance

$$\text{cov}(\hat{Y}, \hat{X}) = \frac{1}{2} \langle \hat{X} \hat{Y} + \hat{Y} \hat{X} \rangle - \langle \hat{X} \rangle \langle \hat{Y} \rangle.$$

Stochastic Schrödinger equations can alternatively be derived [8] from quantum stochastic calculus [9]. Recently, a probabilistic derivation based on limit theorems was presented by Smolianov and Truman [10] which considered repeated position measurements according to the von Neumann model. In this paper, we recap on the calculations in [10] and correct some numerical errors. We then generalize their argument to consider the simultaneous measurement of position and momentum according to the Arthurs and Kelly model [11]: this is non-trivial as we have to work with quantum rather than classical probability distributions. This model has been examined by Scott and Milburn [12] with explicit calculations made for Gaussian states of the measurement apparatus. They obtain the crucial result that the resulting stochastic wave equation for continuous limit of weak measurements will be of the form (1) with  $\hat{X}_1 = \hat{q}$  and  $\hat{X}_2 = \hat{p}$ . Our approach uses a different limit: a weak interactions limit. (Our treatment of the Smolianov and Truman shows that the limit can be interpreted as either as finite interactions with weak measurements (as they consider) or as finite measurements with weak interactions: actually both limits yield the same result.) We consider a general classes of states for the apparatus where the only restriction is unbiasedness for the pointer positions and momenta and a simple integrability condition, that the constants  $\kappa$  given by (13) below are finite, which ensures a central limit effect. The resulting stochastic wave equation is nevertheless of the same form as that obtained by Scott and Milburn.

## 1.1 Measurement Conditioned Wave Functions

We consider a fixed Hilbert space  $\mathfrak{h}_S = L^2(\mathbb{R}^n)$  which describes our systems and consider indirect measurements made on a second system (the apparatus) having Hilbert state space and  $\mathfrak{h}_A = L^2(\mathbb{R}^m)$ . The joint space is  $\mathfrak{h}_S \otimes \mathfrak{h}_A$ . A vector state  $\phi$  can be considered as a function  $\phi = \phi(\mathbf{q}; \mathbf{q}')$  where  $\mathbf{q}$  and  $\mathbf{q}' = (q'_1, \dots, q'_m)$  are appropriate coordinate variables for the system and apparatus respectively. Let  $(\hat{q}'_1, \dots, \hat{q}'_m)$  denote the corresponding position observables for the apparatus: they form a maximal set of commuting, self-adjoint operators on  $\mathfrak{h}_A$ .

Suppose that the system and apparatus are initially prepared independently, so that the joint state is the vector  $\phi_0 = \phi_S \otimes \phi_A$ . Subsequently, the two undergo an interaction described by a unitary operator  $\hat{V}$ . The state immediately prior to measurement is then  $\phi = \hat{V} \phi_0$ . We then measure the observables  $(\hat{q}'_1, \dots, \hat{q}'_m)$ . According to the rules of quantum mechanics, we obtain a family of classical random variables  $\mathbf{Q}' = (Q'_1, \dots, Q'_m)$  describing the measured values. Here  $Q'_j$  is the random variable describing the measured value of the observable  $\hat{q}'_j$ , distributed according to

$$\begin{aligned} \mathbb{E} \left[ \exp \left\{ i \sum_{j=1}^m t_j Q'_j \right\} \right] &\equiv \left\langle \phi \left| \exp \left\{ i \sum_{j=1}^m t_j \hat{q}'_j \right\} \phi \right\rangle \\ &\equiv \int_{\mathbb{R}^m} \rho(\mathbf{q}') \exp \left\{ i \sum_{j=1}^m t_j q'_j \right\} d^m q', \end{aligned} \quad (3)$$

where  $\rho(\mathbf{q}') = \int |\langle \mathbf{q}; \mathbf{q}' | \phi \rangle|^2 d^n q$ .

We then consider the function

$$\psi(\mathbf{q} | \mathbf{q}') := \frac{\langle \mathbf{q}; \mathbf{q}' | \phi \rangle}{\sqrt{\rho(\mathbf{q}')}} \quad (4)$$

and introduce the *stochastic wave-function*  $\Psi$  defined by

$$\langle \mathbf{q} | \Psi \rangle = \psi(\mathbf{q} | \mathbf{Q}') \quad (5)$$

If we do not wish to make the  $\mathbf{q}$ -dependence explicit, we shall write  $\Psi = \Psi_{\mathbf{Q}'}$ . The stochastic wave-function interpretation comes about as the mathematical re-interpretation of the wave-function as the  $\mathfrak{h}_S$ -valued random variable dependent on the measured random variables  $\mathbf{Q}' = (Q'_1, \dots, Q'_m)$ .

Let  $\hat{X}$  is an observable of the system, only. We set  $\bar{X}(\mathbf{Q}') = \langle \Psi_{\mathbf{Q}'} | \hat{X} | \Psi_{\mathbf{Q}'} \rangle = \int \psi^*(\mathbf{q} | \mathbf{Q}') (\hat{X} \psi)(\mathbf{q} | \mathbf{Q}') d^n q$ . Then the average value of the random variable  $\bar{X}(\mathbf{Q}')$  will be

$$\mathbb{E}[\bar{X}(\mathbf{Q}')] = \int \bar{X}(\mathbf{q}') \rho(\mathbf{q}') d^m q' \quad (6)$$

and this is clearly  $\langle \phi | \hat{X} | \phi \rangle = \int \phi^*(\mathbf{q}; \mathbf{q}') (\hat{X} \phi)(\mathbf{q}; \mathbf{q}') d^n q d^m q'$ .

## 1.2 Repeated Measurements

Next suppose that several measurements are made at regular intervals of time  $\tau$ . Let  $A(j)$  denote the apparatus employed for the  $j$ -th measurement and let  $\phi_{A(j)}$  be its state prior to interaction with the system. (We may either consider an assembly of apparatuses  $A(1), A(2), A(3), \dots$  or a single apparatus whose state is forced to be  $\phi_{A(j)}$  somehow just before the  $j$ -th measurement for each  $j$ .) Between measurements, the system is allowed to evolve according to the Hamiltonian  $\hat{H}$  on  $\mathfrak{h}_S$ .

If we denote the conditioned wave-function immediately after the  $n$ -th measurement by  $\psi_n = \psi_n(x|\mathbf{Q}'_1; \dots, \mathbf{Q}'_n)$ , then we have the iterative relation

$$\psi_n(\mathbf{q}|\mathbf{q}'_1; \dots; \mathbf{q}'_n) = \frac{1}{\sqrt{\rho(\mathbf{q}'_n)}} \left\langle \mathbf{q}; \mathbf{q}'_n \left| e^{\tau \hat{H}/i\hbar} \hat{V}_n \psi_{n-1} \otimes \phi_{A(n)} \right. \right\rangle. \quad (7)$$

## 2 Measurement of Position

We begin with the simplest problem of monitoring a single observable. We take this to be a position coordinate  $\hat{q}$ .

As system-apparatus interaction, we take a linear coupling of the system position with the apparatus momentum  $\hat{p}'$ . The interaction is then given by the unitary  $\hat{V} = \exp\{\mu \hat{q} \hat{p}'/i\hbar\}$ . We then measure the position observable,  $\hat{q}'$ , for the apparatus. This is essentially the von Neumann model for indirect measurements. The apparatus state prior to measurement is taken to have a real symmetric wave-function and so its mean position and moment are zero.

The calculations in this section follow the same arguments as in Smolyanov and Truman [10].

### 2.1 Von Neumann's Model

The interaction between the system and apparatus is then given by

$$\phi_S(q) \phi_A(q') \mapsto \phi_S(q) \phi_A(q' - \mu q). \quad (8)$$

The probability density for the apparatus position after interaction is then

$$\rho(q') = \int |\phi_S(q)|^2 |\phi_A(q' - \mu q)|^2 dq \quad (9)$$

which is a convolution of two probability densities.

We choose  $\phi_A$  to have the form

$$\phi_A(q') = \frac{1}{\sqrt{\sigma}} \chi\left(\frac{|q'|^2}{\sigma^2}\right) \quad (10)$$

where  $\chi$  is real-valued and normalized so that  $\int_{-\infty}^{\infty} \chi(y^2) dy = 1 = \int_{-\infty}^{\infty} \chi(y^2) y^2 dy$ . In particular,  $Y = \frac{1}{\sigma} Q'_0$  will be a mean-zero, unit-variance random variable. We

then have the decomposition

$$Q' = \sigma Y + \mu(Z + \langle \hat{q} \rangle) \quad (11)$$

where  $Z = Q_S - \langle \hat{q} \rangle$  is centered.

Assuming that  $\chi$  is analytic, we get an expansion of the form

$$\phi_A(Q' - \mu q) = \frac{1}{\sqrt{\sigma}} \chi(Y^2) \left\{ 1 + \frac{\chi'(Y^2)}{\chi(Y^2)} \varepsilon + \frac{1}{2!} \frac{\chi''(Y^2)}{\chi(Y^2)} \varepsilon^2 + \dots \right\}$$

where  $\varepsilon = \sigma^{-2} |Q' - \mu q|^2 - Y^2$ .

Likewise the factor  $\rho(Q')^{-1/2}$  can be expanded. First of all we observe that

$$\begin{aligned} \int |\phi_S(q)|^2 \varepsilon dq &= 2 \frac{\mu}{\sigma} Z Y + \left( \frac{\mu}{\sigma} \right)^2 (Z^2 + \sigma_q^2) \\ \int |\phi_S(q)|^2 \varepsilon^2 dq &= 4 \left( \frac{\mu}{\sigma} \right)^2 (Z^2 + \sigma_q^2) Y^2 + O\left( \left( \frac{\mu}{\sigma} \right)^3 \right) \end{aligned}$$

where we set  $\sigma_q^2 := \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2$  (the variance of the observable  $\hat{q}$  for state  $\phi_S$ ) and treat  $(\frac{\mu}{\sigma})$  as small parameter.

$$\begin{aligned} \rho(Q') &= \int |\phi_S(q)|^2 \frac{1}{\sigma^2} \chi^2 \left( \frac{|Q' - \mu q|^2}{\sigma^2} \right) dq \\ &= \int |\phi_S(q)|^2 \frac{\chi^2(Y^2)}{\sigma^2} \left\{ 1 + \frac{2\chi'(Y^2)}{\chi(Y^2)} \varepsilon + \frac{1}{2!} \left[ \frac{\chi'(Y^2)^2}{\chi(Y^2)^2} + \frac{\chi''(Y^2)}{\chi(Y^2)} \right] \varepsilon^2 + \dots \right\} dq \\ &= \frac{\chi^2(Y^2)}{\sigma^2} \left\{ 1 + \frac{4\chi'(Y^2)}{\chi(Y^2)} \frac{\mu}{\sigma} Z Y \right. \\ &\quad \left. + 2 \frac{\mu^2}{\sigma^2} \left[ \frac{\chi'(Y^2)}{\chi(Y^2)} + 2 \frac{\chi'(Y^2)^2}{\chi(Y^2)^2} + 2 \frac{\chi''(Y^2)}{\chi(Y^2)} \right] (Z^2 + \sigma_q^2) Y^2 + O\left( \left( \frac{\mu}{\sigma} \right)^3 \right) \right\} \end{aligned}$$

We obtain to lowest orders

$$\begin{aligned} \psi(q|Q') &= \left\{ 1 - 2 \frac{\mu}{\sigma} \frac{\chi'}{\chi} (q - \langle \hat{q} \rangle) Y + \right. \\ &\quad + \frac{\mu^2}{\sigma^2} \left[ \left( \frac{\chi'}{\chi} + 2 \frac{\chi''}{\chi} Y^2 \right) (q - \langle \hat{q} \rangle)^2 - \left( \frac{\chi'}{\chi} + 2 \frac{\chi''}{\chi} Y^2 + 2 \left( \frac{\chi'}{\chi} \right)^2 Y^2 \right) \sigma_q^2 \right. \\ &\quad \left. \left. + 2Z \left( 2 \left( \frac{\chi'}{\chi} \right)^2 Y^2 - \frac{\chi'}{\chi} - 2 \frac{\chi''}{\chi} Y^2 \right) (q - \langle \hat{q} \rangle) \right] + O\left( \left( \frac{\mu}{\sigma} \right)^3 \right) \right\} \phi_S(q). \quad (12) \end{aligned}$$

where  $\chi, \chi'$  and  $\chi''$  are evaluated at  $Y^2$ . Note that the terms involving  $Z^2$  cancel to this order.

## 2.2 Continuous Measurement Limit

We now consider sequential measurements with the successive apparatus states prior to interaction being copies of  $\phi_A$  chosen as above (10).

Adopting the scaling  $(\frac{\mu}{\sigma}) = \sqrt{\tau}$ , we get

$$\begin{aligned} \frac{1}{\tau}[\psi_n - \psi_{n-1}] &= \left\{ \frac{1}{i\hbar} \hat{H} - 2\sqrt{\frac{1}{\tau}} \frac{\chi'_n}{\chi_n} (q - \langle \hat{q} \rangle) Y_n \right. \\ &+ \left[ \left( \frac{\chi'_n}{\chi_n} + 2\frac{\chi''_n}{\chi_n} Y_n^2 \right) (q - \langle \hat{q} \rangle)^2 - \left( \frac{\chi'_n}{\chi_n} + 2\frac{\chi''_n}{\chi_n} Y_n^2 + 2 \left( \frac{\chi'_n}{\chi_n} \right)^2 Y_n^2 \right) \sigma_q^2 \right. \\ &\left. \left. + 2Z_n \left( 2 \left( \frac{\chi'_n}{\chi_n} \right)^2 Y_n^2 - \frac{\chi'_n}{\chi_n} - 2\frac{\chi''_n}{\chi_n} Y_n^2 \right) (q - \langle \hat{q} \rangle) \right] + O(\tau^{1/2}) \right\} \psi_{n-1}. \end{aligned}$$

where  $\chi_n = \chi(Y_n^2)$ , etc.

In the limit  $\tau \rightarrow 0$  we expect from the law of large numbers that

$$\lim_{\tau \rightarrow 0} \tau \sum_{j=1}^{[t/\tau]} \left( \frac{\chi'_j}{\chi_j} + 2\frac{\chi''_j}{\chi_j} Y_j^2 \right) = -\kappa t;$$

$$\lim_{\tau \rightarrow 0} \tau \sum_{j=1}^{[t/\tau]} f(Y_j^2) Z_j = 0;$$

for suitable  $f$ . Here  $\kappa = -\int_{-\infty}^{\infty} \left( \frac{\chi'}{\chi} + 2\frac{\chi''}{\chi} y^2 \right) \chi^2 dy$ , after some algebra and an integration by parts we have that

$$\kappa = -\int_{-\infty}^{\infty} (\chi' \chi + 2\chi \chi'' y^2) dy = 2 \int_{-\infty}^{\infty} (\chi' y)^2 dy > 0. \quad (13)$$

Inspecting the coefficient of  $\sigma_q^2$ , we are lead to consider

$$\lim_{\tau \rightarrow 0} \tau \sum_{j=1}^{[t/\tau]} \left( \frac{\chi'_j}{\chi_j} + 2\frac{\chi''_j}{\chi_j} Y_j^2 + 2 \left( \frac{\chi'_j}{\chi_j} \right)^2 Y_j^2 \right) = \theta t;$$

where  $\theta = \int_{-\infty}^{\infty} (\chi \chi' + 2\chi'^2 y^2 + 2\chi'' \chi y^2) dy$  and comparison with the above integration by parts shows  $\theta = 0$ .

Likewise, from the central limit theorem, we expect that

$$\lim_{\tau \rightarrow 0} \frac{1}{\sqrt{\tau}} \sum_{j=1}^{[t/\tau]} \frac{\chi'(Y_j^2)}{\chi(Y_j^2)} Y_j = \sqrt{\frac{\kappa}{2}} B_t$$

where  $B_t$  is a standard Wiener process.

We therefore are lead to the stochastic differential equation

$$|d\Psi\rangle = \left\{ \frac{1}{i\hbar} \hat{H} - \kappa (\hat{q} - \langle \hat{q} \rangle)^2 \right\} |\Psi\rangle dt - \sqrt{2\kappa} (\hat{q} - \langle \hat{q} \rangle) |\Psi\rangle dB_t. \quad (14)$$

**Remark 1** The limit  $\frac{\mu}{\sigma} \rightarrow 0$  can occur in two distinct ways. We can consider a weak coupling limit  $\mu \rightarrow 0$  where the measurement process is fixed ( $\sigma = 1$ ). Alternatively we can consider a weak measurement limit  $\sigma \rightarrow \infty$  where the interaction is fixed ( $\mu = 1$ ).

**Remark 2** The equation (14) can clearly be generalized to (1) provide that we consider indirect measurement of commuting system observables.

**Remark 3** The result (14) corrects some numerical errors made in [10] with regard to the coefficients.

### 3 Measurement of Position & Momentum

We take the system to have canonical position and momentum observables  $\hat{q}$  and  $\hat{p}$ , respectively. Our apparatus consist of two distinct components,  $A'$  and  $A''$ , which have canonical observables  $\hat{q}', \hat{p}'$  and  $\hat{q}'', \hat{p}''$ , respectively. Our aim is to couple the system to the apparatus by interaction and make inferences about the system based on simultaneous measurements of the position of  $A'$  (that is,  $q'$ ) and the momentum of  $A''$  (that is,  $p''$ ).

Initially we take the system to be prepared in state  $\phi_S$  while the apparatus is prepared in a state  $\phi_A \in \mathfrak{h}_A = \mathfrak{h}_{A'} \otimes \mathfrak{h}_{A''}$ . We assume that the two components of the apparatus are not entangled - that is,  $\phi_A(q', p'') = \phi_{A'}(q') \phi_{A''}(p'')$ .

The interaction between the system and apparatus is described by the unitary operator

$$\hat{V} = \exp \{ (\mu \hat{p}' \hat{q} - \nu \hat{q}'' \hat{p}) / i\hbar \} \quad (15)$$

and we define an operator  $\hat{\mathcal{V}}(q', p'')$  on  $\mathfrak{h}_S$  by

$$\left( \hat{\mathcal{V}}(q', p'') \phi_S \right) (q) = \left\langle q, q', p'' \middle| \hat{V} \phi_S \otimes \phi_A \right\rangle. \quad (16)$$

We see that

$$\begin{aligned} \hat{\mathcal{V}}(q', p'') &= \int_{\Gamma \times \Gamma} \frac{dq'' dp'}{2\pi\hbar} \frac{d\bar{q} d\bar{p}''}{2\pi\hbar} \langle q' | p' \rangle \langle p'' | q'' \rangle \hat{W}(-\nu q'', -\mu p') \langle p' | \bar{q}' \rangle \langle q'' | \bar{p}'' \rangle \langle \bar{q}', \bar{p}'' | \phi_A \rangle \\ &= \int_{\Gamma \times \Gamma} \frac{dq dp}{2\pi\hbar} \frac{d\bar{q} d\bar{p}}{2\pi\hbar} e^{(p\bar{q} - q\bar{p})/i\hbar} \langle q' - \mu \bar{q}, p'' - \nu \bar{p} | \phi_A \rangle \hat{W}(q, p) \end{aligned}$$

and so we obtain  $\hat{\mathcal{V}}(q', p'')$  as a Weyl-quantized operator (see appendix)

$$\hat{\mathcal{V}}(q', p'') \equiv [ \langle q' - \mu \hat{q}, p'' - \nu \hat{p} | \phi_A \rangle ]_{\text{Weyl}}. \quad (17)$$

We next denote the random variables obtained by observing  $\hat{q}'$  and  $\hat{p}''$  by  $Q'$  and  $P''$  respectively. As the observables commute, we are able to assign a joint probability to  $Q', P''$  once a density matrix is prescribed.



A stochastic wave-function dependent on the observed position and momentum variables of the apparatus is

$$\Psi = \frac{\hat{V}(Q', P'')}{\sqrt{\rho(Q', P'')}} \phi_S \quad (18)$$

where  $\rho(q', p'') = \int \left| \langle q, q', p'' | \hat{V} \phi \rangle \right|^2 dq \equiv \left\| \hat{V}(q', p'') \phi_S \right\|^2$ .

Under the action of the unitary  $\hat{V}$  we have the relations

$$\hat{V}^\dagger \hat{q}' \hat{V} = \hat{q}' + \mu \hat{q} - \frac{1}{2} \mu \nu \hat{q}''; \quad \hat{V}^\dagger \hat{p}'' \hat{V} = \hat{p}'' + \nu \hat{p} - \frac{1}{2} \mu \nu \hat{p}'. \quad (19)$$

(The action of  $\hat{V}$  can be understood as that due to the evolution in unit time governed by Hamiltonian  $\mu \hat{p}' \hat{q} - \nu \hat{q}'' \hat{p}$ . This Hamiltonian generates a linear set of equation for the canonical variables which is readily soluble. The variables  $\hat{p}'$  and  $\hat{q}''$  are evidently invariants.)

The joint probability distribution of  $Q'$  and  $P''$  is determined from the characteristic function

$$\mathbb{E} \left[ e^{i(\alpha Q' + \beta P'')} \right] = \langle \phi | e^{i(\alpha \hat{q}' + \beta \hat{p}'')} \phi \rangle$$

where  $\phi = \hat{V}(\phi_S \otimes \phi_A)$  is the post-interaction state. Using the lemma, we can write the characteristic function as

$$\langle \phi_S | e^{i(\alpha \mu \hat{q} + \beta \nu \hat{p})} \phi_S \rangle \times \langle \phi_{A'} | e^{i(\alpha \hat{q}' - \frac{1}{2} \beta \mu \nu \hat{p}')} \phi_{A'} \rangle \times \langle \phi_{A''} | e^{i(\beta \hat{p}'' - \frac{1}{2} \alpha \mu \nu \hat{q}'')} \phi_{A''} \rangle$$

The result is that we have the following decompositions into sums of independent random variables:

$$Q' = Q'_0 + \mu Q_S - \frac{1}{2} \mu \nu Q''_0; \quad P'' = P''_0 + \nu P_S - \frac{1}{2} \mu \nu P'_0. \quad (20)$$

Specifically,

$$\begin{aligned} \Pr[q \leq Q_S < q + dq] &= |\phi_S(q)|^2 dq \\ \Pr[q' \leq Q'_0 < q' + dq'] &= |\phi_{A'}(q')|^2 dq' \\ \Pr[q'' \leq Q''_0 < q'' + dq''] &= |\langle q'' | \phi_{A''} \rangle|^2 dq'' \end{aligned}$$

while a similar set of laws hold for the  $P$ 's.

**Remark 4** *Some caution is needed here: a joint distribution for  $Q', P''$  exists as they correspond to commuting observables, however this is not the case for any of the pairs  $Q_S, P_S$  or  $Q'_0, P'_0$  or  $Q''_0, P''_0$ .*

**Remark 5** *The parameters  $\mu, \nu$  are free. In [12] they are chosen as  $\nu = \mu^{-1}$  so that symplectic area is preserved: the single parameter  $\mu$  is referred to therein as a squeezing parameter.*

### 3.1 Perturbative Expansions

We take the initial states of the apparatus to be

$$\phi_{A'}(q') = \chi(|q'|^2), \quad \phi_{A''}(p'') = \Lambda(|p''|^2)$$

Here we shall adopt the weak coupling scheme and fix  $\sigma = 1$  while  $\mu$  and  $\nu$  are the small parameters. Expanding to lowest orders, we see

$$\begin{aligned} \langle q' - \mu q, p'' - \nu p | \phi_A \rangle &= \chi(|q'|^2) \Lambda(|p''|^2) \\ &= \chi \Lambda \left\{ 1 + \frac{\chi'}{\chi} \varepsilon_q + \frac{1}{2!} \frac{\chi''}{\chi} \varepsilon_q^2 + \dots \right\} \left\{ 1 + \frac{\Lambda'}{\Lambda} \varepsilon_p + \frac{1}{2!} \frac{\Lambda''}{\Lambda} \varepsilon_p^2 + \dots \right\} \end{aligned}$$

where  $\chi = \chi(|q'|^2)$  and  $\Lambda = \Lambda(|p''|^2)$ , etc., and

$$\varepsilon_q = -2\mu q' q + \mu^2 q^2, \quad \varepsilon_p = -2\nu p'' p + \nu^2 p^2.$$

Taking Weyl quantization yields

$$\begin{aligned} \hat{\mathcal{V}}(q', p'') &= \chi \Lambda \left\{ 1 - 2\mu \frac{\chi'}{\chi} q' \hat{q} - 2\nu \frac{\Lambda'}{\Lambda} p'' \hat{p} \right. \\ &+ \mu^2 \left( \frac{\chi'}{\chi} + 2 \frac{\chi''}{\chi} q'^2 \right) \hat{q}^2 + \nu^2 \left( \frac{\Lambda'}{\Lambda} + 2 \frac{\Lambda''}{\Lambda} p''^2 \right) \hat{p}^2 + 2\mu\nu \frac{\chi'}{\chi} \frac{\Lambda'}{\Lambda} q' p'' (\hat{q}\hat{p} + \hat{p}\hat{q}) + \dots \left. \right\} \end{aligned}$$

and the joint probability density for the observations is

$$\begin{aligned} \rho(q', p'') &= \langle \phi_S | \hat{\mathcal{V}}(q', p'')^2 | \phi_S \rangle \\ &= \chi^2 \Lambda^2 \left\{ 1 - 4\mu \frac{\chi'}{\chi} q' \langle \hat{q} \rangle - 4\nu \frac{\Lambda'}{\Lambda} p'' \langle \hat{p} \rangle \right. \\ &+ 2\mu^2 \left[ \left( \frac{\chi'}{\chi} + 2 \frac{\chi''}{\chi} q'^2 \right) + 2 \left( \frac{\chi' q'}{\chi} \right)^2 \right] \langle \hat{q}^2 \rangle + \nu^2 \left[ \left( \frac{\Lambda'}{\Lambda} + 2 \frac{\Lambda''}{\Lambda} p''^2 \right) + 2 \left( \frac{\Lambda' p''}{\Lambda} \right)^2 \right] \langle \hat{p}^2 \rangle \\ &\left. + 6\mu\nu \frac{\chi'}{\chi} \frac{\Lambda'}{\Lambda} q' p'' \langle \hat{q}\hat{p} + \hat{p}\hat{q} \rangle + \dots \right\} \end{aligned}$$

We therefore obtain the approximate form

$$\frac{\hat{\mathcal{V}}(q', p'')}{\sqrt{\rho(q', p'')}} = 1 - 2\mu \frac{\chi'}{\chi} q' (\hat{q} - \langle \hat{q} \rangle) - 2\nu \frac{\Lambda'}{\Lambda} p'' (\hat{p} - \langle \hat{p} \rangle)$$

$$\begin{aligned}
& +\mu^2 \left[ \left( \frac{\chi'}{\chi} + 2\frac{\chi''}{\chi} q'^2 \right) \hat{q}^2 - \left( \frac{\chi'}{\chi} + 2\frac{\chi''}{\chi} q'^2 + 2 \left( \frac{\chi' q'}{\chi} \right)^2 \right) \langle \hat{q}^2 \rangle \right. \\
& \left. + 6 \left( \frac{\chi' q'}{\chi} \right)^2 \langle \hat{q} \rangle^2 - 4 \left( \frac{\chi' q'}{\chi} \right)^2 \hat{q} \langle \hat{q} \rangle \right] \\
& +\nu^2 \left[ \left( \frac{\Lambda'}{\Lambda} + 2\frac{\Lambda''}{\Lambda} p''^2 \right) \hat{p}^2 - \left( \frac{\Lambda'}{\Lambda} + 2\frac{\Lambda''}{\Lambda} p''^2 + 2 \left( \frac{\Lambda' p''}{\Lambda} \right)^2 \right) \langle \hat{p}^2 \rangle \right. \\
& \left. + 6 \left( \frac{\Lambda' p''}{\Lambda} \right)^2 \langle \hat{p} \rangle^2 - 4 \left( \frac{\Lambda' p''}{\Lambda} \right)^2 \hat{p} \langle \hat{p} \rangle \right] \\
& + 6\mu\nu \frac{\chi'}{\chi} \frac{\Lambda'}{\Lambda} q' p'' (2 \langle \hat{q} \hat{p} + \hat{p} \hat{q} \rangle - 3 \langle \hat{q} \hat{p} + \hat{p} \hat{q} \rangle + 6 \langle \hat{q} \rangle \langle \hat{p} \rangle - 4 \hat{q} \langle \hat{p} \rangle - 4 \hat{p} \langle \hat{q} \rangle) + \dots .
\end{aligned}$$

Our objective is to study the random propagator  $\frac{\hat{v}(Q', P'')}{\sqrt{\rho(Q', P'')}}$  where now the classical random variables  $Q'$  and  $P''$  have  $\rho$  as their joint probability density function. As we have seen, the exact distributions of  $Q'$  and  $P''$  are non-trivially related to the original states of the system and apparatus. However, we are only interested in the lowest order dependence in terms of parameters  $\mu$  and  $\nu$ . The decompositions (20) may be rewritten as

$$Q' = Q'_0 + \mu \langle \hat{q} \rangle + \mu F, \quad P'' = P''_0 + \nu \langle \hat{p} \rangle + \nu G \quad (21)$$

where  $Q'_0$  and  $P''_0$  are independent variables with the pre-interaction densities  $\chi (|q'_0|^2)^2$  and  $\Lambda (|p''_0|^2)^2$  respectively, and  $F$  and  $G$  are further independent variables with means of order  $\nu$  and  $\mu$ , respectively. To lowest order, we obtain

$$\begin{aligned}
\frac{\chi' (|Q'|^2)}{\chi (|Q'|^2)} Q' &= \frac{\chi'_0}{\chi_0} Q'_0 + \mu \left[ \frac{\chi'_0}{\chi_0} + 2 \left( \frac{\chi''_0}{\chi_0} - \left( \frac{\chi'_0}{\chi_0} \right)^2 \right) Q'^2_0 \right] (\langle \hat{q} \rangle + F) + \dots \\
\frac{\Lambda' (|P''|^2)}{\Lambda (|P''|^2)} P'' &= \frac{\Lambda'_0}{\Lambda_0} P''_0 + \nu \left[ \frac{\Lambda'_0}{\Lambda_0} + 2 \left( \frac{\Lambda''_0}{\Lambda_0} - \left( \frac{\Lambda'_0}{\Lambda_0} \right)^2 \right) P''^2_0 \right] (\langle \hat{p} \rangle + G) + \dots
\end{aligned}$$

where  $\chi_0 = \chi (|Q'_0|^2)$ ,  $\Lambda_0 = \Lambda (|P''_0|^2)$ , etc. Making these replacements leads to

$$\frac{\hat{v}(Q', P'')}{\sqrt{\rho(Q', P'')}} = 1 - 2\mu \frac{\chi'_0}{\chi_0} Q'_0 (\hat{q} - \langle \hat{q} \rangle) - 2\nu \frac{\Lambda'_0}{\Lambda_0} P''_0 (\hat{p} - \langle \hat{p} \rangle)$$

$$\begin{aligned}
& +\mu^2 \left[ \left( \frac{\chi'_0}{\chi_0} + 2\frac{\chi''_0}{\chi_0} Q_0'^2 \right) \hat{q}^2 - \left( \frac{\chi'_0}{\chi_0} + 2\frac{\chi''_0}{\chi_0} Q_0'^2 + 2 \left( \frac{\chi'_0 Q'_0}{\chi_0} \right)^2 \right) \langle \hat{q}^2 \rangle + 6 \left( \frac{\chi'_0 Q'_0}{\chi_0} \right)^2 \langle \hat{q} \rangle^2 \right. \\
& \left. - 4 \left( \frac{\chi'_0 Q'_0}{\chi_0} \right)^2 \hat{q} \langle \hat{q} \rangle - 2 \left[ \frac{\chi'_0}{\chi_0} + 2 \left( \frac{\chi''_0}{\chi_0} - \left( \frac{\chi'_0}{\chi_0} \right)^2 \right) Q_0'^2 \right] (\langle \hat{q} \rangle + F) (\hat{q} - \langle \hat{q} \rangle) \right] \\
& +\nu^2 \left[ \left( \frac{\Lambda'_0}{\Lambda_0} + 2\frac{\Lambda''_0}{\Lambda_0} P_0''^2 \right) \hat{p}^2 - \left( \frac{\Lambda'_0}{\Lambda_0} + 2\frac{\Lambda''_0}{\Lambda_0} P_0''^2 + 2 \left( \frac{\Lambda'_0 P_0''}{\Lambda_0} \right)^2 \right) \langle \hat{p}^2 \rangle + 6 \left( \frac{\Lambda'_0 P_0''}{\Lambda_0} \right)^2 \langle \hat{p} \rangle^2 \right. \\
& \left. - 4 \left( \frac{\Lambda'_0 P_0''}{\Lambda_0} \right)^2 \hat{p} \langle \hat{p} \rangle - 2 \left[ \frac{\Lambda'_0}{\Lambda_0} + 2 \left( \frac{\Lambda''_0}{\Lambda_0} - \left( \frac{\Lambda'_0}{\Lambda_0} \right)^2 \right) P_0'' \right] (\langle \hat{p} \rangle + G) (\hat{p} - \langle \hat{p} \rangle) \right] \\
& + 6\mu\nu \frac{\chi'_0 \Lambda'_0}{\chi_0 \Lambda_0} Q_0' P_0'' (2 \langle \hat{q} \hat{p} \rangle + \hat{p} \hat{q}) - 3 \langle \hat{q} \hat{p} \rangle + \hat{p} \hat{q} + 6 \langle \hat{q} \rangle \langle \hat{p} \rangle - 4 \hat{q} \langle \hat{p} \rangle - 4 \hat{p} \langle \hat{q} \rangle + \dots \quad (22)
\end{aligned}$$

## 4 Repeated Measurements

Next suppose that simultaneous measurements of position and momentum are made at regular intervals of time  $\tau$ . Again, we assume that the pre-interaction states  $\phi_{A(j)}$  are all copies of the fixed state  $\phi_A$ .

If we denote the conditioned wave-function immediately after the  $n$ -th measurement by  $\psi_n = \psi_n(x|Q'_1, P''_1; \dots, Q'_n, P''_n)$ , then we have the relation

$$\psi_n = e^{\tau H/i\hbar} \frac{\hat{Y}(Q'_n, P''_n)}{\sqrt{\rho(Q'_n, P''_n)}} \psi_{n-1}.$$

Setting  $\mu = \nu = \sqrt{\tau}$ , we can use (22) to obtain

$$\begin{aligned}
\frac{1}{\tau} [\psi_n - \psi_{n-1}] &= \left\{ \frac{1}{i\hbar} \hat{H} - 2\sqrt{\frac{1}{\tau}} \left( \frac{\chi'_0}{\chi_0} Q'_0 \right) (q - \langle \hat{q} \rangle) - 2\sqrt{\frac{1}{\tau}} \left( \frac{\Lambda'_0}{\Lambda_0} P_0'' \right) (\hat{p} - \langle \hat{p} \rangle) \right. \\
& \left. - 2\frac{1}{\tau} \left( \frac{\chi'_0}{\chi_0} Q'_0 \right)_n^2 (q - \langle \hat{q} \rangle)^2 - 2\frac{1}{\tau} \left( \frac{\Lambda'_0}{\Lambda_0} P_0'' \right)_n^2 (\hat{p} - \langle \hat{p} \rangle)^2 \right\} \psi_{n-1} + O(\tau^{1/2}). \quad (23)
\end{aligned}$$

To obtain (23) from (22) we made the replacements  $\frac{\chi'_0}{\chi_0} + 2\frac{\chi''_0}{\chi_0} Q_0'^2$  with  $-2 \left( \frac{\chi'_0}{\chi_0} Q'_0 \right)^2$  and  $\frac{\Lambda'_0}{\Lambda_0} + 2\frac{\Lambda''_0}{\Lambda_0} P_0''^2$  with  $-2 \left( \frac{\Lambda'_0}{\Lambda_0} P_0'' \right)^2$  which is consistent with (13). In particular, the coefficients of  $\langle \hat{q}^2 \rangle$  and  $\langle \hat{p}^2 \rangle$  disappear. The terms involving  $F$  and  $G$  are dropped since they are  $O(\tau^{1/2})$  and so will give negligible contribution in a law of large numbers limit. Finally the cross terms vanish since

$$\lim_{\tau \rightarrow 0} \frac{1}{\tau} \sum_{j=1}^{\lfloor t/\tau \rfloor} \left( \frac{\chi'_0}{\chi_0} Q'_0 \right)_j \left( \frac{\Lambda'_0}{\Lambda_0} P_0'' \right)_j = 0$$

on account of the fact that  $\int \chi(y^2) \chi'(y^2) y dy = 0 = \int \Lambda(x^2) \Lambda'(x^2) x dx$ .

Comparison with position-only situation shows that we are lead to the stochastic differential equation

$$|d\Psi\rangle = \left\{ \frac{1}{i\hbar} \hat{H} - \kappa_q (\hat{q} - \langle \hat{q} \rangle)^2 - \kappa_p (\hat{p} - \langle \hat{p} \rangle)^2 \right\} |\Psi\rangle dt - \sqrt{2\kappa_q} (\hat{q} - \langle \hat{q} \rangle) |\Psi\rangle dB_t^{(q)} - \sqrt{2\kappa_p} (\hat{p} - \langle \hat{p} \rangle) |\Psi\rangle dB_t^{(p)}$$

where

$$\kappa_q = 2 \int (\chi'(y^2) y)^2 dy, \quad \kappa_p = 2 \int (\Lambda'(x^2) x)^2 dx$$

and we have the following limits in mean-square sense to independent Wiener processes

$$B_t^{(q)} = \lim_{\tau \rightarrow 0} \sqrt{\frac{2}{\kappa_q \tau}} \sum_{j=1}^{\lfloor t/\tau \rfloor} \left( \frac{\chi'_0}{\chi_0} Q'_0 \right)_j, \quad B_t^{(p)} = \lim_{\tau \rightarrow 0} \sqrt{\frac{2}{\kappa_p \tau}} \sum_{j=1}^{\lfloor t/\tau \rfloor} \left( \frac{\Lambda'_0}{\Lambda_0} P''_0 \right)_j.$$

## 4.1 Stochastic Dynamics

The associated Lindblad generator for this monitored dynamics is

$$\mathcal{L}(\hat{Y}) = \frac{1}{i\hbar} [\hat{Y}, \hat{H}] + \kappa_q \left\{ [\hat{q}, \hat{Y}] \hat{q} + \hat{q} [\hat{Y}, \hat{q}] \right\} + \kappa_p \left\{ [\hat{p}, \hat{Y}] \hat{p} + \hat{p} [\hat{Y}, \hat{p}] \right\}.$$

We note that  $\mathcal{L}(\hat{Y}) = \frac{1}{i\hbar} [\hat{Y}, \hat{H}]$  occurs for the special case of observables of the type  $\hat{Y} = \alpha \hat{q} + \beta \hat{p} + \gamma \hat{q} \hat{p}$ . This means that when  $\hat{H} = \frac{1}{2m} \hat{p}^2 + \Phi(\hat{q})$ , the averages of the canonical observables evolve in a non-random way according to the Ehrenfest theorem for closed systems:

$$\frac{d}{dt} \langle \hat{q} \rangle = \frac{1}{m} \langle \hat{p} \rangle, \quad \frac{d}{dt} \langle \hat{p} \rangle = -\langle \Phi'(\hat{q}) \rangle.$$

However, we do not have the derivational property  $\mathcal{L}(\hat{X}\hat{Y}) = \hat{X}\mathcal{L}(\hat{Y}) + \mathcal{L}(\hat{X})\hat{Y}$  and so the dynamics is dissipative. In particular,

$$\mathcal{L}(\hat{H}) = \kappa_q \frac{\hbar^2}{m} + \kappa_p \hbar^2 \Phi''(\hat{q})$$

and so energy is not conserved.

## 4.2 Conclusion

We have shown that the central limit effect allows us to derive a stochastic Schrödinger equation in a very general setting. Essentially we need only be in the domain of attraction for Gaussian statistics. (It is possible that more

general results hold for stable laws - leading to stochastic Schrödinger equations driven by Levy processes - however, this clearly would be indicative of an imperfection in the measurement apparatus only.) The striking result is that the non-commuting observables can be measured simultaneously with negligible interference. This backs up the general phenomenological approach used in areas of quantum control and filtering. In their article, Scott & Milburn [12] investigate several models, including classically chaotic ones, and show that the system is localized with the monitored trajectory in phase space corresponding to a quantum average plus noise. Whereas there are still many interesting open questions regarding, for instance, the semi-classical limit, the present result does show claim that the analysis of Scott & Milburn approach is generic for continuously monitored phase variables.

## 5 Appendix

Let  $(q, p) \in \Gamma$  where  $\Gamma = \mathbb{R}^2$  is phase space. For a given function  $f = f(q, p)$  on phase space, the association of an operator  $f(\hat{q}, \hat{p})$  is ambiguous due to the problem of operator ordering. We shall adopt the Weyl quantization convention.

The Weyl operator at phase point  $(q, p)$  is defined to be  $\hat{W}(q, p) = \exp(q\hat{p} - p\hat{q})/i\hbar$ . From the CCR and an application of the Baker-Hausdorff-Campbell theorem, we find that

$$\hat{W}(q, p) \hat{W}(q', p') = e^{(qp' - p'q')/2i\hbar} \hat{W}(q + q', p + p'). \quad (24)$$

Let  $f = f(q, p)$  be absolutely integrable on  $\Gamma$  and define its phase space Fourier (Weyl) transform to be

$$\tilde{f}(q, p) := \int_{\Gamma} \frac{d\bar{q}d\bar{p}}{2\pi\hbar} e^{(p\bar{q} - q\bar{p})/i\hbar} f(\bar{q}, \bar{p}). \quad (25)$$

The Weyl quantization of  $f$  is then defined to be the operator

$$\begin{aligned} [f(\hat{q}, \hat{p})]_{\text{Weyl}} & : = \int_{\Gamma} \frac{dqdp}{2\pi\hbar} \hat{W}(q, p) \tilde{f}(q, p) \\ & = \int_{\Gamma \times \Gamma} \frac{dqdp}{2\pi\hbar} \frac{d\bar{q}d\bar{p}}{2\pi\hbar} e^{(p\bar{q} - q\bar{p})/i\hbar} f(\bar{q}, \bar{p}) \hat{W}(q, p) \end{aligned} \quad (26)$$

and we refer to the map  $f(q, p) \mapsto [f(\hat{q}, \hat{p})]_{\text{Weyl}}$  as Weyl quantization.

We have for instance  $[\exp(t(q\hat{p} - p\hat{q})/i\hbar)]_{\text{Weyl}} = \hat{W}(tq, tp)$  and expanding in powers of  $t$ , we obtain  $[(\hat{q}\alpha + \hat{p}\beta)^n]_{\text{Weyl}} = (\hat{q}\alpha + \hat{p}\beta)^n$ . Further expansion in terms of  $\alpha, \beta$  show that polynomials will be mapped to the symmetrically (Weyl) ordered form.. For instance,  $[\hat{q}^2\hat{p}]_{\text{Weyl}} = \frac{1}{3}(\hat{q}^2\hat{p} + \hat{q}\hat{p}\hat{q} + \hat{p}\hat{q}^2)$ , etc.

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