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Random Fuzzy Delayed Renewal Processes

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Abstract

In renewal processes, fuzziness and randomness often coexist intrinsically. Based on the random fuzzy theory, a delayed renewal process with random fuzzy interarrival times is proposed in this paper. Relations between the renewal number and interarrival times in such a process are investigated, and useful theorems such as the elementary renewal theorem, Blackwell renewal theorem and Smith key renewal theorem in a conventional delayed renewal process are extended to their counterparts for random fuzzy delayed renewal processes.

Keywords: Fuzzy variable; random fuzzy variable; renewal process; delayed renewal process; random fuzzy delayed renewal process

1 Introduction

In solving many real world problems, the knowledge and information available is often imprecise and incomplete. Such uncertainty may be dealt with using the fuzzy set theory [32]. In particular, fuzzy sets have been successfully applied in modelling renewal processes. For example, Zhao and Liu [34] considered an ordinary renewal process in which the interarrival times and rewards are characterized as fuzzy variables, and established the fuzzy version of the elementary renewal theorem and reward renewal theorem. Li *et al* [16] discussed a delayed renewal process with fuzzy interarrival times and proposed some potentially useful properties on the average of the renewal number.

However, the fuzziness and the randomness often occur simultaneously in most practical situations. Under such circumstances, fuzzy random variables [13] [14] [26] appear to be more suitable to capture these two kinds of uncertainty in a unified manner. Hwang [12] investigated an ordinary renewal process in which the interarrival times were considered as fuzzy random variables and then created a theory of the fuzzy rate for this type of fuzzy random renewal process. Popova and Wu [27] considered a renewal reward process with random interarrival times and fuzzy rewards and investigated the long-run average fuzzy reward property per unit time. Dozzi *et al* [5] gave a limit theorem for counting renewal processes indexed by fuzzy sets. Li *et al* [17] researched a delayed renewal process with fuzzy random interarrival times, and gave the fuzzy random version of the elementary renewal theorem, Blackwell renewal theorem and Smith key renewal theorem. As a specific case of the delayed renewal processes as proposed in [17], Li *et al* [18] studied a homogeneous Poisson process in a fuzzy random environment.

Very recently, random fuzzy variables [21] have also been suitably employed to simultaneously capture these two types of uncertainty, and applied to model renewal processes. In contrast to fuzzy random renewal processes, there have been only a few articles on renewal processes in a random fuzzy environment. Zhao *et al* [35] introduced an ordinary renewal process with such a renewal reward process having random fuzzy interarrival times and rewards. Li *et al* [19] studied a Poisson process with fuzzy time-dependent intensity, where the rates of the process were deemed to be the fuzzy variables.

Delayed renewal process is an extension of ordinary renewal processes. The interarrival times of an ordinary renewal process are regarded to be independent and identically distributed (iid) random variables. However, a delayed renewal process permits the first interarrival time to have a different distribution from the remaining ones. This requires an extended theory to address both the renewal number and the interarrival times. Some important results such as the elementary renewal theorem, Blackwell renewal theorem and Smith key renewal theorem have been established for delayed renewal processes [28]. Based on these, this paper further incorporates the concept of random fuzzy variables into the modelling of delayed renewal processes, with a focus on the relations between the renewal number and interarrival times holding within such a process.

The paper is organized as follows. In Section 2, some concepts and propositions about fuzzy variables and random fuzzy variables are introduced. A brief overview of the theory of delayed renewal processes is described in Section 3. The concept of random fuzzy delayed renewal processes is then proposed in Section 4 with useful theorems such as the elementary renewal theorem, Blackwell renewal theorem and Smith key renewal theorem extended. Finally, Section 5 concludes the paper and points of further research.

2 Fuzzy Variables and Random Fuzzy Variables

Fundamental definitions and concepts of fuzzy variables and random fuzzy variables are briefly outlined in this section. Interested readers can refer to [20] for further details.

Possibility measure [33] (Pos) and necessity measure [6] [29] (Nec) have both been widely used. The main difference among these two measures is that they consider the same question from a different angle. The possibility measure assesses the possibility of a set A in terms of affirmation, and the necessity measure in terms of impossibility of the opposite set A^c. This may lead to situations where the former overrates the possibility of a set while the latter underrates the possibility of a set. To have a balanced approach, and based on the basic intuitions behind the measures of Pos and Nec, a self-dual measure Cr, called a credibility measure, has been introduced [22, 23], which is defined such that

$$\mathrm{Cr}\{\cdot\} = \frac{1}{2}\left(\mathrm{Pos}\{\cdot\} + \mathrm{Nec}\{\cdot\}\right).$$

Obviously, $\mathrm{Cr}\{A\} = 1 - \mathrm{Cr}\{A^c\}$.

A credibility space is represented by $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$, where Θ is a nonempty set, $\mathcal{P}(\Theta)$ the power set of Θ, and Cr the credibility measure. A fuzzy variable is defined as a function from a credibility space to the real number line. From a measure-theoretic point of view, the expected value of a fuzzy variable [23] can then be defined as

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geq r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leq r\}\mathrm{d}r.$$

Note that the definition of the scalar expected value of a fuzzy variable is more favorable in many applications than others (e.g. [7] and [11]).

Let ξ be a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then, its membership function $\mu_\xi(\cdot)$ can be derived from the credibility measure by

$$\mu_\xi(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1.$$

It is worth mentioning that the α-pessimistic value and the α-optimistic value (see [25]) of a fuzzy variable ξ, defined as

$$\xi_\alpha^L = \inf\left\{r \mid \mu_\xi(r) \geq \alpha\right\} \quad \text{and} \quad \xi_\alpha^U = \sup\left\{r \mid \mu_\xi(r) \geq \alpha\right\} \tag{1}$$

for any $\alpha \in (0, 1]$, are useful means of representing a fuzzy variable as well as its expected value.

Proposition 1 ([25]) *Let ξ be a fuzzy variable with the finite expected value $E[\xi]$. Then*

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\xi_\alpha^L + \xi_\alpha^U\right)\mathrm{d}\alpha.$$

Another two concepts to mention are fuzzy variables' independence [24] and identical distribution [20]. Fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if and only if for any sets $B_1, B_2, \cdots, B_n$ of $\Re$,

$$\mathrm{Cr}\{\xi_i \in B_i, i = 1, 2, \cdots, n\} = \min_{1 \leq i \leq n} \mathrm{Cr}\{\xi_i \in B_i\}.$$

Fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ are said to be identically distributed if and only if for any set B of $\Re$,

$$\mathrm{Cr}\{\xi_i \in B\} = \mathrm{Cr}\{\xi_j \in B\}, \quad i, j = 1, 2, \cdots, n.$$

Random fuzzy variables are extensions of both fuzzy variables and random variables. A random fuzzy variable can be seen as a function from a credibility space to a collection of random variables. For example, in the domain of crime prevention, when analyzing forensic data, it might be known that the concentration of aluminium in c-glass may be an exponentially distributed variable associated with an unknown parameter [1]. If this parameter is itself regarded as a fuzzy variable, then the aluminium concentration is a random fuzzy variable. A random fuzzy variable ξ defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ is probabilistically nonnegative if and only if $\Pr\{\xi(\theta) < 0\} = 0$ for each $\theta \in \Theta$ with $\mathrm{Cr}\{\theta\} > 0$.

Definition 1 ([25]) *The expected value of a random fuzzy variable ξ, also denoted by $E[\xi]$ without causing confusion, is defined by*

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\left\{\theta \in \Theta \;\middle|\; E[\xi(\theta)] \geq r\right\} \mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\left\{\theta \in \Theta \;\middle|\; E[\xi(\theta)] \leq r\right\} \mathrm{d}r$$

provided that at least one of the two integrals is finite.

Proposition 2 ([20]) *Let ξ be a random fuzzy variable defined on $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. Then, for any $\theta \in \Theta$, $E[\xi(\theta)]$ is a fuzzy variable provided that $E[\xi(\theta)]$ is finite for fixed $\theta \in \Theta$.*

Definition 2 (Random Fuzzy Arithmetic On Different Credibility Spaces) *Let $f : \Re^n \to \Re$ be a measurable function, and ξ_i random fuzzy variables on the credibility spaces $(\Theta_i, \mathcal{P}(\Theta_i), \mathrm{Cr}_i), i = 1, 2, \cdots, n$. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a random fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$, i.e.,*

$$\xi(\theta_1, \theta_2, \cdots, \theta_n) = f(\xi_1(\theta_1), \xi_2(\theta_2), \cdots, \xi_n(\theta_n))$$

for any $(\theta_1, \theta_2, \cdots, \theta_n) \in \Theta$, where $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$, and for any $A \in \mathcal{P}(\Theta)$,

$$\mathrm{Cr}\{A\} = \sup_{(\theta_1, \theta_2, \cdots, \theta_n) \in A} \min_{1 \leq i \leq n} \mathrm{Cr}_i\{\theta_i\}.$$

Definition 3 ([15]) *The random fuzzy variables $\xi_1, \xi_2, \cdots, \xi_n$ defined on $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ are independent if*

(1) $\xi_1(\theta)$, $\xi_2(\theta)$, $\cdots, \xi_n(\theta)$ are independent random variables for each $\theta \in \Theta$;

(2) $E[\xi_1(\cdot)]$, $E[\xi_2(\cdot)]$, $\cdots, E[\xi_n(\cdot)]$ are independent fuzzy variables.

Definition 4 ([15]) *The random fuzzy variables ξ and η are identically distributed if*

$$\sup_{\mathrm{Cr}\{A\}\geq\alpha}\inf_{\theta\in A}\{\Pr\{\xi(\theta)\in B\}\}=\sup_{\mathrm{Cr}\{A\}\geq\alpha}\inf_{\theta\in A}\{\Pr\{\eta(\theta)\in B\}\}$$

for any $\alpha\in(0,1]$ and Borel set B of real numbers.

Proposition 3 ([20]) *Let $\xi_1,\xi_2,\cdots,\xi_n$ be iid random fuzzy variables. Then $E[\xi_i(\theta)]$, $i=1,2,\cdots,n$ are iid fuzzy variables.*

3 Delayed Renewal Processes

Before discussing random fuzzy delayed renewal processes, it is helpful to review the basic definition and some useful theorems of a delayed renewal process.

Let ξ_n, $n=1,2,\cdots$, denote the elapsed times between the $(n-1)$th and the nth event, known as *interarrival times*. Suppose that such interarrival times are nonnegative mutually independent random variables, and that ξ_i have a common distribution for $i\geq 2$. Define

$$N(t)=\max_{n\geq 0}\left\{n \mid 0\leq \xi_1+\xi_2+\cdots+\xi_n\leq t\right\}. \tag{2}$$

$\{N(t),t\geq 0\}$ is called a *delayed renewal process* and $N(t)$ a renewal number.

The followings are several important theorems regarding delayed renewal processes (see [2, 3, 4, 8, 9, 10, 30] for more detail).

Theorem 1 (Elementary Renewal Theorem) *For a delayed renewal process $\{N(t),t\geq 0\}$ with interarrival times ξ_i, the following holds:*

$$\lim_{t\to+\infty}\frac{E[N(t)]}{t}=\frac{1}{E[\xi_2]}.$$

Theorem 2 *For a delayed renewal process $\{N(t),t\geq 0\}$ with interarrival times ξ_i, $E[N(t)]$ is a nondecreasing, finite, and right-continuous function of t.*

Definition 5 *A nonnegative random variable η is said to be lattice if and only if there exists $d>0$ such that $\sum_{n=0}^{\infty}\Pr\{\eta=nd\}=1$.*

Theorem 3 (Blackwell Renewal Theorem) *For a delayed renewal process $\{N(t),t\geq 0\}$, if random variables ξ_i are nonlattic, then, for any $a>0$,*

$$\lim_{t\to+\infty}\left(E[N(t+a)]-E[N(t)]\right)=\frac{a}{E[\xi_2]}. \tag{3}$$

Theorem 4 (Smith Key Renewal Theorem) *For a delayed renewal process* $\{N(t), t \geq 0\}$ *with interarrival times* ξ_i*, if random variables* ξ_i *are nonlattic, and* $\int_0^{+\infty} h(t)\mathrm{d}t < +\infty$*, then, for any* $a > 0$*,*

$$\lim_{t \to +\infty} \int_0^t h(t-x)\mathrm{d}E[N(x)] = \frac{1}{E[\xi_2]} \int_0^{+\infty} h(t)\mathrm{d}t.$$

4 Extension to Random Fuzzy Delayed Renewal Processes

Let $\xi_1, \xi_2, \cdots$, known as the *interarrival times*, be nonnegative mutually independent random fuzzy variables defined on the credibility spaces $(\Theta_i, \mathcal{P}(\Theta_i), \mathrm{Cr}_i)$. Suppose that $\xi_2, \xi_3, \cdots$ have a common distribution, and that

$$N(t) = \max_{n \geq 0} \left\{ n \mid 0 \leq \xi_1 + \xi_2 + \cdots + \xi_n \leq t \right\}. \tag{4}$$

The process $\{N(t), t \geq 0\}$ is called a *random fuzzy delayed renewal process* defined on the credibility space $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ (see Definition 2), and $N(t)$ is termed a renewal number.

According to Definition 2, the renewal number $N(t)$ is a random fuzzy variable. Furthermore, from Proposition 2, $E[N(t)(\theta)]$ is a fuzzy variable. For any $\alpha \in (0, 1]$, denote the α-pessimistic value and the α-optimistic value of $E[N(t)(\theta)]$ as $E[N(t)(\theta)]_\alpha^L$ and $E[N(t)(\theta)]_\alpha^U$, respectively. Similarly, $E[\xi_i(\theta)]$, $i = 1, 2, \cdots$, are fuzzy variables, and the α-pessimistic values and the α-optimistic values of $E[\xi_i(\theta)]$ can be written as $E\left[\xi_i(\theta)\right]_\alpha^L$ and $E\left[\xi_i(\theta)\right]_\alpha^U$.

Throughout the rest of this paper, the followings are assumed:

(a) $\Theta_2 = \Theta_3 = \cdots$;

(b) For each i and $\theta \in \Theta_i$, to avoid trivialities suppose that $\Pr\{\xi_i(\theta) = 0\} < 1$;

(c) For any $\theta \in \Theta_i$, $E[\xi_i(\theta)]$ are closed and bounded sets for $i \geq 1$;

(d) Let $\mathcal{F} = \{\xi_i(\theta), \theta \in \Theta_i, i = 1, 2, \cdots\}$. Suppose that the image set $\mathcal{F}$ of ξ_i is a totally ordered set with a *stochastic ordering*. That is, for any given $\xi_i(\theta_i), \xi_j(\theta_j) \in \mathcal{F}, i, j = 1, 2, \cdots$ and $r \in \Re$, either

$$\Pr\{\xi_i(\theta_i) \leq r\} \leq \Pr\{\xi_j(\theta_j) \leq r\} \quad (\text{denoted by } \xi_j(\theta_j) \leq_d \xi_i(\theta_i))$$

or

$$\Pr\{\xi_j(\theta_j) \leq r\} \leq \Pr\{\xi_j(\theta_j) \leq r\} \quad (\text{denoted by } \xi_i(\theta_i) \leq_d \xi_j(\theta_j))$$

(The symbol $\leq_d$ represents stochastic ordering; see [31] for proportion of stochastic ordering).

Lemma 1 *For a random fuzzy delayed renewal process* $\{N(t), t \geq 0\}$ *with random fuzzy interarrival times* ξ_i*, the following holds:*

$$\lim_{t \to +\infty} \frac{E\left[N(t)(\theta)\right]_\alpha^L}{t} = \frac{1}{E\left[\xi_2(\theta)\right]_\alpha^U},$$

$$\lim_{t \to +\infty} \frac{E\left[N(t)(\theta)\right]_\alpha^U}{t} = \frac{1}{E\left[\xi_2(\theta)\right]_\alpha^L}.$$

Proof. For any $\alpha \in (0,1]$, by Assumptions (a) and (c), there at least exist $\theta_i^{'}, \theta_i^{''} \in \Theta_i$ such that $\left\{\xi_i\left(\theta_i^{'}\right)\right\}$ and $\left\{\xi_i\left(\theta_i^{''}\right)\right\}$ are two iid random variable sequences for $i \geq 2$, and

$$E\left[\xi_i\left(\theta_i^{'}\right)\right] = E\left[\xi_i(\theta)\right]_\alpha^L, \tag{5}$$

$$E\left[\xi_i\left(\theta_i^{''}\right)\right] = E\left[\xi_i(\theta)\right]_\alpha^U, \tag{6}$$

$$\theta_2^{'} = \theta_3^{'} = \cdots, \quad \theta_2^{''} = \theta_3^{''} = \cdots$$

For any $\theta_i \in \Theta_i$ with membership degree $\mu(E\left[\xi_i(\theta_i)\right]) \geq \alpha$, using (1) it follows that

$$E\left[\xi_i\left(\theta_i^{'}\right)\right] \leq E\left[\xi_i(\theta_i)\right] \leq E\left[\xi_i\left(\theta_i^{''}\right)\right].$$

Furthermore, it follows by Assumption (d) that

$$\xi_i\left(\theta_i^{'}\right) \leq_d \xi_i\left(\theta_i\right) \leq_d \xi_i\left(\theta_i^{''}\right). \tag{7}$$

That is, for any real number $r \in \Re$,

$$\Pr\left\{\xi_i\left(\theta_i^{''}\right) \leq r\right\} \leq \Pr\left\{\xi_i(\theta_i) \leq r\right\} \leq \Pr\left\{\xi_i\left(\theta_i^{'}\right) \leq r\right\}.$$

Replacing ξ_i in (4) with $\xi_i(\theta_i)$, $\xi_i\left(\theta_i^{'}\right)$ and $\xi_i\left(\theta_i^{''}\right)$, respectively, and writing

$$N(t)(\theta) = \max\left\{n \mid 0 \leq \xi_1(\theta_1) + \xi_2(\theta_2) + \cdots + \xi_n(\theta_n) \leq t\right\},$$

$$N(t)\left(\theta^{'}\right) = \max\left\{n \mid 0 \leq \xi_1\left(\theta_1^{'}\right) + \xi_2\left(\theta_2^{'}\right) + \cdots + \xi_n\left(\theta_n^{'}\right) \leq t\right\}, \tag{8}$$

$$N(t)\left(\theta^{''}\right) = \max\left\{n \mid 0 \leq \xi_1\left(\theta_1^{''}\right) + \xi_2\left(\theta_2^{''}\right) + \cdots + \xi_n\left(\theta_n^{''}\right) \leq t\right\}, \tag{9}$$

then the following can be obtained: a counting process $\{N(t)(\theta), t \geq 0\}$ generated by $\xi_1(\theta_1), \xi_2(\theta_2), \cdots$, and two delayed renewal processes $\left\{N(t)\left(\theta^{'}\right), t \geq 0\right\}$ which is generated by $\xi_1\left(\theta_1^{'}\right), \xi_2\left(\theta_2^{'}\right), \cdots$, and $\left\{N(t)\left(\theta^{''}\right), t \geq 0\right\}$ which is generated by $\xi_1\left(\theta_1^{''}\right), \xi_2\left(\theta_2^{''}\right), \cdots$, where

$$\theta = (\theta_1, \theta_2, \cdots), \quad \theta^{'} = \left(\theta_1^{'}, \theta_2^{'}, \cdots\right), \quad \theta^{''} = \left(\theta_1^{''}, \theta_2^{''}, \cdots\right).$$

From (7), it follows that for any fixed positive integer k,

$$\sum_{i=1}^{k} \xi_i\left(\theta_i^{'}\right) \leq_d \sum_{i=1}^{k} \xi_i(\theta_i) \leq_d \sum_{i=1}^{k} \xi_i\left(\theta_i^{''}\right),$$

which is equivalent to

$$\Pr\left\{\sum_{i=1}^{k} \xi_i\left(\theta_i^{''}\right) \leq r\right\} \leq \Pr\left\{\sum_{i=1}^{k} \xi_i(\theta_i) \leq r\right\} \leq \Pr\left\{\sum_{i=1}^{k} \xi_i\left(\theta_i^{'}\right) \leq r\right\}$$

for any real number $r \in \Re$. Since

$$\Pr\left\{N(t)\left(\theta^{'}\right) \leq k\right\} \Longleftrightarrow \Pr\left\{\sum_{i=1}^{k} \xi_i\left(\theta_i^{'}\right) \geq t\right\},$$

$$\Pr\left\{N(t)(\theta) \leq k\right\} \Longleftrightarrow \Pr\left\{\sum_{i=1}^{k} \xi_i(\theta_i) \geq t\right\},$$

$$\Pr\left\{N(t)\left(\theta^{''}\right) \leq k\right\} \Longleftrightarrow \Pr\left\{\sum_{i=1}^{k} \xi_i\left(\theta_i^{''}\right) \geq t\right\},$$

then

$$\Pr\left\{N(t)\left(\theta^{'}\right) \leq k\right\} \leq \Pr\left\{N(t)(\theta) \leq k\right\} \leq \Pr\left\{N(t)\left(\theta^{''}\right) \leq k\right\},$$

which implies

$$N(t)\left(\theta^{''}\right) \leq_d N(t)(\theta) \leq_d N(t)\left(\theta^{'}\right). \tag{10}$$

Taking expectations over (10) yields

$$E\left[N(t)\left(\theta^{''}\right)\right] \leq E[N(t)(\theta)] \leq E\left[N(t)\left(\theta^{'}\right)\right]. \tag{11}$$

Owing to the arbitrariness of θ_i, using (1) leads to

$$E[N(t)(\theta)]_\alpha^L = E\left[N(t)\left(\theta^{''}\right)\right], \quad E[N(t)(\theta)]_\alpha^U = E\left[N(t)\left(\theta^{'}\right)\right]. \tag{12}$$

In addition, it follows from Assumption (b) and $\xi_i \geq 0$ that

$$0 < E[\xi_i(\theta)] \leq +\infty$$

for any $\theta \in \Theta_i$. For delayed renewal processes $\left\{N(t)\left(\theta^{''}\right), t \geq 0\right\}$ and $\left\{N(t)\left(\theta^{'}\right), t \geq 0\right\}$, as defined in (9) and (8) respectively, following the Elementary Renewal Theorem gives

$$\lim_{t\to+\infty} \frac{E\left[N(t)\left(\theta^{''}\right)\right]}{t} = \frac{1}{E\left[\xi_2\left(\theta_2^{''}\right)\right]},$$

$$\lim_{t\to+\infty} \frac{E\left[N(t)\left(\theta^{'}\right)\right]}{t} = \frac{1}{E\left[\xi_2\left(\theta_2^{'}\right)\right]}.$$

Substituting (12) into (5) and (6) completes the proof.

Theorem 5 (Random Fuzzy Delayed Elementary Renewal Theorem) *For a random fuzzy delayed renewal process $\{N(t), t \geq 0\}$ with random fuzzy interarrival times ξ_i, the following holds:*

$$\lim_{t \to +\infty} \frac{E[N(t)]}{t} = E\left[\frac{1}{\xi}\right],$$

where ξ is one of the fuzzy variables with the α-pessimistic value $E\left[\xi_2(\theta)\right]_\alpha^L$ and the α-optimistic value $E\left[\xi_2(\theta)\right]_\alpha^U$, and $E\left[\frac{1}{\xi}\right] < +\infty$.

Proof. For any given $\varepsilon > 0$, by Lemma 1, there exist two positive real numbers t_1 and t_2 such that

$$\frac{E\left[N(t)(\theta)\right]_\alpha^L}{t} < \frac{1}{E\left[\xi_2(\theta)\right]_\alpha^U} + \varepsilon$$

for $t > t_1$, and

$$\frac{E\left[N(t)(\theta)\right]_\alpha^U}{t} < \frac{1}{E\left[\xi_2(\theta)\right]_\alpha^L} + \varepsilon$$

for $t > t_2$. Consequently, for $t > \max(t_1, t_2)$,

$$\frac{E\left[N(t)(\theta)\right]_\alpha^L}{t} + \frac{E\left[N(t)(\theta)\right]_\alpha^U}{t} < \frac{1}{E\left[\xi_2(\theta)\right]_\alpha^L} + \frac{1}{E\left[\xi_2(\theta)\right]_\alpha^U} + 2\varepsilon.$$

Since

$$\begin{aligned}
E\left[\frac{1}{\xi}\right] &= \frac{1}{2}\int_0^1 \left(\left(\frac{1}{\xi}\right)_\alpha^L + \left(\frac{1}{\xi}\right)_\alpha^U\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{\xi_\alpha^L} + \frac{1}{\xi_\alpha^U}\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{E\left[\xi_2(\theta)\right]_\alpha^L} + \frac{1}{E\left[\xi_2(\theta)\right]_\alpha^U}\right) \mathrm{d}\alpha \\
&< +\infty,
\end{aligned}$$

by Definition 1, Proposition 1 and the dominated convergence theorem we get

$$\begin{aligned}
\lim_{t \to +\infty} \frac{E[N(t)]}{t} &= \lim_{t \to +\infty} \int_0^{+\infty} \mathrm{Cr}\left\{\theta \in \Theta \mid \frac{E[N(t)(\theta)]}{t} \geq r\right\} \mathrm{d}r \\
&= \lim_{t \to +\infty} \frac{1}{2}\int_0^1 \left(\frac{E[N(t)(\theta)]_\alpha^L}{t} + \frac{E[N(t)(\theta)]_\alpha^U}{t}\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(\lim_{t \to +\infty} \frac{E[N(t)(\theta)]_\alpha^L}{t} + \lim_{t \to +\infty} \frac{E[N(t)(\theta)]_\alpha^U}{t}\right) \mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(\frac{1}{E[\xi_2(\theta)]_\alpha^L} + \frac{1}{E[\xi_2(\theta)]_\alpha^U}\right) \mathrm{d}\alpha \\
&= E\left[\frac{1}{\xi}\right].
\end{aligned}$$

The proof is finished.

Example 1 *Assume that* $\xi_1 \sim \mu\left(\rho', \rho' + 1\right)$ *with* $\rho' = (0, 2, 4)$, *and* $\xi_i \sim \mu(\rho, \rho + 1)$ *with* $\rho = (0, 1, 2)$ *for* $i \geq 2$. *The result of random fuzzy simulation ([25]) with 10000 cycles is*

$$\lim_{t \to +\infty} \frac{E[N(t)]}{t} = E\left[\frac{1}{\xi}\right] = 0.5470.$$

Remark 1 *A renewal process* $\{N(t), t \geq 0\}$ *is discussed in [12] that involves fuzzy random interarrival times* ξ_i *whose expected value is* $E[\xi_i]$ *(as defined in [13, 14]). One of the main results obtained there is*

$$\frac{N(t)}{t} \to \frac{1}{E[\xi_1]}$$

with probability 1 *as* $t \to +\infty$. *However, the interarrival times between two events described in the present paper are characterized as random fuzzy variables and consequently, a delayed renewal process in a random fuzzy situation is defined. A random fuzzy variable and a fuzzy random variable are two distinct concepts [20].*

Remark 2 *A renewal process* $\{N(t), t \geq 0\}$ *is described in [35] that involves random fuzzy interarrival times* ξ_i. *One of the main results there is the random fuzzy elementary renewal theorem. That is,*

$$\lim_{t \to \infty} \frac{E[N(t)]}{t} = E\left[\frac{1}{\xi}\right]$$

where ξ *is a fuzzy variable with the* α*-pessimistic value* $E\left[\xi_1(\theta)\right]_\alpha^L$ *and the* α*-optimistic value* $E\left[\xi_1(\theta)\right]_\alpha^U$ *and* $E\left[\frac{1}{\xi}\right]$ *is the expected value of* $\frac{1}{\xi}$. *If* ξ_1 *is assumed to be of the same distribution as the remaining interarrival times* $\xi_i, i \geq 2$ *in Theorem 5, then* $E\left[\xi_1(\theta)\right]_\alpha^L = E\left[\xi_2(\theta)\right]_\alpha^L$, $E\left[\xi_1(\theta)\right]_\alpha^U = E\left[\xi_2(\theta)\right]_\alpha^U$ *and consequently, Theorem 5 degenerates into the random fuzzy elementary renewal theorem in [35]. Thus, the result obtained in [35] is a special case of Theorem 5.*

Remark 3 *A renewal process* $\{N(t), t \geq 0\}$ *is considered in [27] that involves random interarrival times* ξ_i *and fuzzy random rewards* η_i *whose* α*-pessimistic values and* α*-optimistic values are* η_i^L *and* η_i^U, *respectively. The main result is*

$$\frac{C(t)}{t} \to a$$

with probability 1 level-wise, where $C(t)$ *is the total reward and* a *is the fuzzy variable with the* α*-pessimistic value* $E\left[\eta_1^L\right]/E[\xi_1]$ *and the* α*-optimistic value* $E\left[\eta_1^U\right]/E[\xi_1]$. *This shows that the renewal process in [27] and the delayed renewal processes in this paper are two different processes.*

Theorem 6 *For a random fuzzy delayed renewal process* $\{N(t), t \geq 0\}$, $E[N(t)]$ *is a nondecreasing, finite, and right-continuous function of* t.

Proof. The proof involves three stages. First, prove $E[N(t)]$ to be a nondecreasing function of t. It follows from the stochastic renewal theorem in a random delayed renewal process (see [28]) that $E\left[N(t)\left(\theta^{''}\right)\right]$ and $E\left[N(t)\left(\theta^{'}\right)\right]$ are two nondecreasing functions of t. That is, for any $t_1 < t_2$,

$$E\left[N(t_1)\left(\theta^{''}\right)\right] \leq E\left[N(t_2)\left(\theta^{''}\right)\right] \quad \text{and} \quad E\left[N(t_1)\left(\theta^{'}\right)\right] \leq E\left[N(t_2)\left(\theta^{'}\right)\right],$$

which imply

$$E\left[N(t_1)(\theta)\right]_\alpha^L \leq E\left[N(t_2)(\theta)\right]_\alpha^L \quad \text{and} \quad E\left[N(t_1)(\theta)\right]_\alpha^U \leq E\left[N(t_2)(\theta)\right]_\alpha^U.$$

By Definition 1 and Proposition 1,

$$\begin{aligned} E[N(t_1)] &= \frac{1}{2}\int_0^1 \left(E[N(t_1)(\theta)]_\alpha^L + E[N(t_1)(\theta)]_\alpha^U\right) \mathrm{d}\alpha \\ &\leq \frac{1}{2}\int_0^1 \left(E[N(t_2)(\theta)]_\alpha^L + E[N(t_2)(\theta)]_\alpha^U\right) \mathrm{d}\alpha \\ &= E[N(t_2)]. \end{aligned}$$

Second, prove $E[N(t)]$ to be finite. It also follows from the stochastic renewal theorem in a random delayed renewal process (see [28]) that both $E\left[N(t)\left(\theta^{''}\right)\right]$ and $E\left[N(t)\left(\theta^{'}\right)\right]$ are finite. That is, there exist two positive real numbers M_1 and M_2 such that

$$E\left[N(t)\left(\theta^{''}\right)\right] \leq M_1 \quad \text{and} \quad E\left[N(t)\left(\theta^{''}\right)\right] \leq M_2$$

for any fixed t. Consequently, the following holds:

$$E\left[N(t)(\theta)\right]_\alpha^L + E\left[N(t)(\theta)\right]_\alpha^U = E\left[N(t)\left(\theta^{'}\right)\right] + E\left[N(t)\left(\theta^{''}\right)\right] \leq M_1 + M_2.$$

Hence,

$$\begin{aligned} E[N(t)] &= \frac{1}{2}\int_0^1 \left(E[N(t)(\theta)]_\alpha^L + E[N(t)(\theta)]_\alpha^U\right) \mathrm{d}\alpha \\ &\leq \frac{M_1 + M_2}{2}\int_0^1 \mathrm{d}\alpha \\ &= \frac{M_1 + M_2}{2}. \end{aligned}$$

Finally, prove $E[N(t)]$ to be a right-continuous function of t. For any fixed t, let $\{t_n, n \geq 0\}$ be a sequence of real numbers decreasing to t, i.e., $t_n \downarrow t$. Once again, it follows from the stochastic renewal theorem in random delayed renewal process (see [28]) that $E\left[N(t)\left(\theta^{''}\right)\right]$ and $E\left[N(t)\left(\theta^{'}\right)\right]$ are right-continuous functions of t, and consequently

$$\lim_{t_n \to t} E\left[N(t_n)\left(\theta^{''}\right)\right] = E\left[N(t)\left(\theta^{''}\right)\right] \quad \text{and} \quad \lim_{t_n \to t} E\left[N(t_n)\left(\theta^{'}\right)\right] = E\left[N(t)\left(\theta^{'}\right)\right].$$

Since $E\left[N(t_n)\left(\theta^{''}\right)\right]$ and $E\left[N(t_n)\left(\theta^{'}\right)\right]$ are two nondecreasing functions of t_n,

$$E\left[N(t_n)\left(\theta^{'}\right)\right] \le E\left[N(t_0)\left(\theta^{'}\right)\right] \quad \text{and} \quad E\left[N(t_n)\left(\theta^{''}\right)\right] \le E\left[N(t_0)\left(\theta^{''}\right)\right].$$

As $E\left[N(t_0)\left(\theta^{''}\right)\right]$ and $E\left[N(t_n)\left(\theta^{'}\right)\right]$ are finite, i.e.,

$$E\left[N(t_0)\left(\theta^{'}\right)\right] < +\infty \quad \text{and} \quad \le E\left[N(t_0)\left(\theta^{''}\right)\right] < +\infty,$$

and by Definition 1, Proposition 1 and the dominated convergence theorem, the following can be shown:

$$\begin{aligned}
\lim_{t_n \to t} E[N(t_n)] &= \lim_{t_n \to t} \frac{1}{2} \int_0^1 \left(E[N(t_n)(\theta)]_\alpha^L + E[N(t_n)(\theta)]_\alpha^U\right) \mathrm{d}\alpha \\
&= \frac{1}{2} \int_0^1 \left(\lim_{t_n \to t} E[N(t_n)(\theta)]_\alpha^L + \lim_{t_n \to t} E[N(t_n)(\theta)]_\alpha^U\right) \mathrm{d}\alpha \\
&= \frac{1}{2} \int_0^1 \left(E[N(t)(\theta)]_\alpha^L + E[N(t)(\theta)]_\alpha^U\right) \mathrm{d}\alpha \\
&= E[N(t)].
\end{aligned}$$

Thus, the theorem is proven.

Remark 4 *For a random fuzzy delayed renewal process* $\{N(t), t \ge 0\}$ *and* $r \ge 0$, $E\left[N^r(t)\right]$ *is a nondecreasing, finite, and right-continuous function of* t.

Theorem 7 (Random Fuzzy Delayed Blackwell Renewal Theorem) *For a random fuzzy delayed renewal process* $\{N(t), t \ge 0\}$, *if for any given* $\theta \in \Theta$, *random variables* $\xi_i(\theta)$ *are nonlattic, then, for any* $a > 0$,

$$\lim_{t \to +\infty} \left(E[N(t+a)] - E[N(t)]\right) = E\left[\frac{a}{\xi}\right]. \tag{13}$$

Proof. For delayed renewal processes $\left\{N(t)\left(\theta^{''}\right), t \ge 0\right\}$ and $\left\{N(t)\left(\theta^{'}\right), t \ge 0\right\}$, as defined in (9) and (8) respectively, using Blackwell Renewal Theorem leads to

$$\lim_{t \to +\infty} \left(E\left[N(t+a)\left(\theta^{''}\right)\right] - E\left[N(t)\left(\theta^{''}\right)\right]\right) = \frac{a}{E\left[\xi_2\left(\theta^{''}\right)\right]}$$

and

$$\lim_{t \to +\infty} \left(E\left[N(t+a)\left(\theta^{'}\right)\right] - E\left[N(t)\left(\theta^{'}\right)\right]\right) = \frac{a}{E\left[\xi_2\left(\theta^{'}\right)\right]},$$

which, by (5), (6) and (12), imply

$$\lim_{t \to +\infty} \left(E[N(t+a)(\theta)]_\alpha^L - E[N(t)(\theta)]_\alpha^L\right) = \frac{a}{E[\xi_2(\theta)]_\alpha^U} \tag{14}$$

and

$$\lim_{t \to +\infty} \left(E[N(t+a)(\theta)]_\alpha^U - E[N(t)(\theta)]_\alpha^U\right) = \frac{a}{E[\xi_2(\theta)]_\alpha^L}. \tag{15}$$

Consequently, for any $\varepsilon > 0$,

$$E[(t+a)(\theta)]^L_\alpha - E[N(t)(\theta)]^L_\alpha + E[N(t+a)(\theta)]^U_\alpha - E[N(t)(\theta)]^U_\alpha < \frac{a}{E[\xi_2(\theta)]^L_\alpha} + \frac{a}{E[\xi_2(\theta)]^U_\alpha} + \varepsilon$$

for any sufficiently large t. Finally, by Definition 1, Proposition 1, and the dominated convergence theorem, it follows that

$$\begin{aligned}
&\lim_{t\to+\infty}(E[N(t+a)] - E[N(t)]) \\
&= \lim_{t\to+\infty}\frac{1}{2}\int_0^1 \left(\left(E[N(t+a)(\theta)]^L_\alpha - E[N(t)(\theta)]^L_\alpha\right) + \left(E[N(t+a)(\theta)]^U_\alpha - E[N(t)(\theta)]^U_\alpha\right)\right)\mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(\lim_{t\to+\infty}\left(E[N(t+a)(\theta)]^L_\alpha - E[N(t)(\theta)]^L_\alpha\right) + \lim_{t\to+\infty}\left(E[N(t+a)(\theta)]^U_\alpha - E[N(t)(\theta)]^U_\alpha\right)\right)\mathrm{d}\alpha \\
&= \frac{1}{2}\int_0^1 \left(\frac{a}{E[\xi_2(\theta)]^L_\alpha} + \frac{a}{E[\xi_2(\theta)]^U_\alpha}\right)\mathrm{d}\alpha \\
&= E\left[\frac{a}{\xi}\right].
\end{aligned}$$

The proof is completed.

Remark 5 *A renewal process $\{N(t), t \geq 0\}$ is described in [35] that involves random fuzzy interarrival times ξ_i. One of the main results there is the random fuzzy Blackwell renewal theorem. That is,*

$$E[N(t+a)] - E[N(t)] \to E\left[\frac{a}{\xi}\right],$$

where ξ is a fuzzy variable with the α-pessimistic values $E\left[\xi_i(\theta)\right]^L_\alpha$ and the α-optimistic values $E\left[\xi_i(\theta)\right]^U_\alpha$, and $E\left[\frac{1}{\xi}\right]$ is the expected value of $\frac{1}{\xi}$. This result is a special case of Theorem 7.

Theorem 8 (Random Fuzzy Delayed Smith Key Renewal Theorem) *Let $\{N(t), t \geq 0\}$ be a random fuzzy delayed renewal process. If for any given $\theta \in \Theta$, random variables $\xi_i(\theta)$ are nonlattic, and $\int_0^{+\infty} h(t)\mathrm{d}t < +\infty$, then*

$$\lim_{t\to+\infty}\int_0^t h(t-x)\mathrm{d}E[N(x)] = E\left[\frac{1}{\xi}\right]\int_0^{+\infty} h(t)\mathrm{d}t.$$

Before showing the proof of Theorem 8, it is helpful to present two lemmas first. Without loss of generality, assume that $h(t)$ is a positive decreasing function of t such that $h(0) < +\infty$ and $h(t) = 0$ for $t < 0$.

Lemma 2 *If $\int_0^{+\infty} h(t)\mathrm{d}t < +\infty$, then*

$$\lim_{t\to+\infty} t \cdot h(t) = 0.$$

Proof. By $\int_0^{+\infty} h(t)\mathrm{d}t < +\infty$, for any given $\varepsilon > 0$, there exists a positive real number t_1 with $t \geq 2t_1$ such that

$$\int_{\frac{t}{2}}^{t} h(u)\mathrm{d}u < \frac{\varepsilon}{2}.$$

Furthermore, since

$$\int_{\frac{t}{2}}^{t} h(u)\mathrm{d}u \geq h(t)\int_{\frac{t}{2}}^{t} \mathrm{d}u = \frac{h(t)\cdot t}{2},$$

it follows that

$$\frac{t\cdot h(t)}{2} < \frac{\varepsilon}{2}, \tag{16}$$

which shows

$$\lim_{t\to+\infty} t\cdot h(t) = 0.$$

The proof is finished.

Lemma 3 *If $\varphi(\cdot)$ is a positive decreasing function of t with $\varphi(t) = 0$ for $t \in (-\infty, 0)$ or $t \in (T_0, \infty)$ (here T_0 is a positive real number), and $\int_0^{+\infty} \varphi(t)\mathrm{d}t < +\infty$, and if it is integrable with respect to the Lebesgue-Stieltjes measure $\mu(a, b] = E[N(b)] - E[N(a)]$ on any finite interval of $\Re$, then*

$$\lim_{t\to+\infty} \int_0^t \varphi(t-x)\mathrm{d}E[N(x)] = E\left[\frac{1}{\xi}\right]\int_0^{+\infty} \varphi(t)\mathrm{d}t.$$

Proof. First, it is known that

$$\int_0^t \varphi(t-x)\mathrm{d}E[N(x)] = \int_0^t \varphi(u)\mathrm{d}(-E[N(t-u)]) = \int_0^{T_0} \varphi(u)\mathrm{d}(-E[N(t-u)]).$$

Since $\varphi(\cdot)$ is integrable with respect to the Lebesgue-Stieltjes measure $\mu(a, b] = E[N(b)] - E[N(a)]$ on any finite interval of $\Re$,

$$\int_0^t \varphi(t-x)\mathrm{d}E[N(x)] = \lim_{n\to\infty}\sum_{k=0}^{n} \varphi(u_k)\left(E\left[N\left(t - \frac{k}{n}T_0 + \frac{T_0}{n}\right)\right] - E\left[N\left(t - \frac{k}{n}T_0\right)\right]\right),$$

where

$$x_0 = 0, x_k = \frac{kT_0}{n},\ \text{and } u_k \in (x_{k-1}, x_k] \text{ for } k = 1, 2, \cdots, n.$$

By Theorem 7,

$$\lim_{t\to+\infty}\left(E\left[N\left(t - \frac{k}{n}T_0 + \frac{T_0}{n}\right)\right] - E\left[N\left(t - \frac{k}{n}T_0\right)\right]\right) = \frac{T_0}{n}E\left[\frac{1}{\xi}\right].$$

Furthermore, there exist two positive real numbers t_1 and M (e.g. $M = E\left[\frac{1}{\xi}\right] + 1$) such that, for $t > t_1$,

$$E\left[N\left(t - \frac{k}{n}T_0 + \frac{T_0}{n}\right)\right] - E\left[N\left(t - \frac{k}{n}T_0\right)\right] < \frac{MT_0}{n},$$

and consequently

$$\varphi(u_k)\left(E\left[N\left(t-\frac{k}{n}T_0+\frac{T_0}{n}\right)\right]-E\left[N\left(t-\frac{k}{n}T_0\right)\right]\right)<\varphi(u_k)\frac{MT_0}{n}.$$

In addition, since

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\varphi(u_k)\frac{T_0}{n}=\int_0^{T_0}\varphi(u)\mathrm{d}u<+\infty,$$

then, the series of functions

$$\sum_{k=1}^{n}\varphi(u_k)\left(E\left[N\left(t-\frac{k}{n}T_0+\frac{T_0}{n}\right)\right]-E\left[N\left(t-\frac{k}{n}T_0\right)\right]\right)$$

convergences uniformly for $t \geq t_1$. Thus,

$$\begin{aligned}
&\lim_{t\to+\infty}\lim_{n\to+\infty}\sum_{k=1}^{n}\varphi(u_k)\left(E\left[N\left(t-\frac{k}{n}T_0+\frac{T_0}{n}\right)\right]-E\left[N\left(t-\frac{k}{n}T_0\right)\right]\right)\\
&=\lim_{n\to+\infty}\sum_{k=1}^{n}\varphi(u_k)\lim_{t\to+\infty}\left(E\left[N\left(t-\frac{k}{n}T_0+\frac{T_0}{n}\right)\right]-E\left[N\left(t-\frac{k}{n}T_0\right)\right]\right)\\
&=\lim_{n\to+\infty}\sum_{k=1}^{n}\varphi(u_k)\left(\frac{T_0}{n}E\left[\frac{1}{\xi}\right]\right)\\
&=E\left[\frac{1}{\xi}\right]\lim_{n\to+\infty}\sum_{k=1}^{n}\varphi(u_k)\frac{T_0}{n}\\
&=E\left[\frac{1}{\xi}\right]\int_0^{+\infty}\varphi(t)\mathrm{d}t.
\end{aligned}$$

The proof is completed.

Proof of Theorem 8. Let

$$\varphi_n(x)=\begin{cases} h(x), & 0\leq x\leq n\\ 0, & x>n.\end{cases}$$

On the one hand, when $0 \leq x \leq n$, $\varphi_n(x) = h(x)$ and consequently $|\varphi_n(x) - h(x)| = 0$. On the other hand, when $x > n$,

$$|\varphi_n(x)-h(x)|=h(x).$$

By Lemma 2, for any given $\varepsilon > 0$, there exists a positive real number t_1 with $t > t_1$ such that

$$h(x)<\frac{1}{x}\text{ (by taking }\varepsilon=1\text{ in (16)).}$$

Taking $N = [\frac{1}{\varepsilon}] + 1$ with $n > \max(N, t_1)$ results in

$$|\varphi_n(x)-h(x)|=h(x)<\frac{1}{x}<\frac{1}{n}<\frac{1}{N}<\varepsilon.$$

Hence, the following holds:

$$\lim_{n\to+\infty}\varphi_n(x)=h(x) \tag{17}$$

uniformly of x. Therefore,

$$\lim_{n\to+\infty}\int_0^{+\infty}\varphi_n(u)\mathrm{d}u=\int_0^{+\infty}\lim_{n\to+\infty}\varphi_n(u)\mathrm{d}u=\int_0^{+\infty}h(u)\mathrm{d}u. \tag{18}$$

In addition, using lemma 3,

$$\lim_{t\to+\infty}\int_0^{+\infty}\varphi_n(t-u)\mathrm{d}E[N(x)]=E\left[\frac{1}{\xi}\right]\int_0^{+\infty}\varphi_n(u)\mathrm{d}u. \tag{19}$$

Moreover, by Theorem 7, there exist two positive real numbers t_2 and M (e.g. $M=E[\frac{1}{\xi}]+1$) such that

$$|E[N(t-x)]-E[N(t-y)]|<M|x-y| \tag{20}$$

for any x,y and $t>t_2$. Since h is Lebesgue integrable on $\Re$, then h is integral with respect to Lebesgue-Stieltjes measure $\mu(a,b]=E[N(b)]-E[N(a)]$ on $\Re$. Thus, for any given $\varepsilon>0$, there exists a sufficiently large number t_3 with $t>t_3$ such that

$$\left|\int_{t_3}^{t}h(t-x)\mathrm{d}E[N(x)]\right|<\varepsilon.$$

Therefore, the integer n can be taken, with $t_3\le n\le t$, such that

$$\left|\int_{n}^{t}h(t-x)\mathrm{d}E[N(x)]\right|<\varepsilon.$$

Hence,

$$\left|\int_0^t\varphi_n(t-x)\mathrm{d}E[N(x)]-\int_0^t h(t-x)\mathrm{d}E[N(x)]\right|=\int_n^t h(t-x)\mathrm{d}E[N(x)]<\varepsilon,$$

then

$$\lim_{n\to+\infty}\int_0^t\varphi_n(t-x)\mathrm{d}E[N(x)]=\int_0^t h(t-x)\mathrm{d}E[N(x)], \tag{21}$$

uniformly of t.

Finally, from above,

$$\begin{aligned}
&\lim_{t\to+\infty}\int_0^t h(t-x)\mathrm{d}E[N(x)]\\
&=\lim_{t\to+\infty}\lim_{n\to+\infty}\int_0^t\varphi_n(t-x)\mathrm{d}E[N(x)]\quad\text{(by (21))}\\
&=\lim_{n\to+\infty}\lim_{t\to+\infty}\int_0^t\varphi_n(t-x)\mathrm{d}E[N(x)]\quad\text{(by (21))}\\
&=\lim_{n\to+\infty}\left(E\left[\frac{1}{\xi}\right]\int_0^{+\infty}\varphi_n(u)\mathrm{d}u\right)\quad\text{(by (19))}\\
&=E\left[\frac{1}{\xi}\right]\lim_{n\to+\infty}\int_0^{+\infty}\varphi_n(u)\mathrm{d}u\\
&=E\left[\frac{1}{\xi}\right]\int_0^{+\infty}h(t)\mathrm{d}t.\quad\text{(by (18))}
\end{aligned}$$

The theorem is proved.

Remark 6 *Generally, the following holds:*

$$\lim_{t\to+\infty}(E[N(t+a)]-E[N(t)])=E\left[\frac{a}{\xi}\right]\Longleftrightarrow\lim_{t\to+\infty}\int_0^t h(t-x)\mathrm{d}E[N(x)]=E\left[\frac{1}{\xi}\right]\int_0^{+\infty}h(t)\mathrm{d}t,$$

i.e., random fuzzy delayed Smith key renewal theorem is equivalent to random fuzzy delayed Blackwell renewal theorem.

5 Conclusion

In this paper, by employing random fuzzy variables to describe the interarrival times, random fuzzy delayed renewal processes were proposed, forming an extension of conventional delayed renewal processes. Furthermore, some useful theorems such as random fuzzy delayed elementary renewal theorem, random fuzzy delayed Blackwell renewal theorem, random fuzzy delayed Smith key renewal theorem were developed. All these theorems are proven to be generalized versions of the corresponding theorems that can be found in random delayed renewal processes. These theorems provide a good way to deal with fuzzy delayed renewal processes and similar types of random fuzzy processes. This may include applications to addressing problems of crime investigation and prevention where say, forensic data may be gathered and represented by exponential distributions with associated unknown parameters [1]. A possible and interesting direction for future research is the computation of the rate of convergence for the limit theorems presented in this paper. Such an investigation can be based on the work on rates of convergence in classical renewal processes that can be found in [2].

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