Symplectic Noise and the Classical Analog of the Lindblad Generator
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Does the Regression Hypothesis also fail in Classical Physics?

John E. Gough

Abstract We introduce the concepts of Poisson Brackets for classical noise and, in particular, of pairs of canonically conjugate Wiener processes (symplectic noise). Phase space diffusions driven by these processes are considered and the general form of a stochastic process preserving the full (system and noise) Poisson structure is obtained. We show that, once the classical stochastic model is required to preserve the joint system and noise Poisson Bracket, it has much in common with quantum markovian models.
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1 Introduction

In 1931 Onsager proposed a markovian dynamical model for fluctuations of a classical dissipative system close to equilibrium to derive the celebrated reciprocal relations [1]. A more general theory was given by Casimir [2] who included the time-derivatives of the extensive variables of Onsager's theory as new Markov state variables. Subsequently, Onsager and Machlup added the further assumption of Gaussianity [3] to derive the regression theorem. We mention briefly that this work is in the time-domain, as opposed to the fluctuation-dissipation theory of Nyquist [4] which is frequency domain. In the quantum version, there are known to be problems reconciling the Onsager and Nyquist theories, often interpreted as a failure of the quantum regression theorem [5].

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It is well known that the generator of a quantum dynamical semigroup (QDS) of superoperators takes the GKS-Lindblad form [6,7]

$$\mathcal{L}X = \frac{1}{i\hbar}[X,H] + \frac{1}{2}\sum_k \left\{ L_k^\dagger[X,L_k] + [L_k^\dagger,X]L_k \right\},$$

with H self-adjoint. Lindblad [7] characterized the generators by their dissipation which he defined as

$$\mathcal{D}_\mathcal{L}(X,Y) \triangleq \mathcal{L}(X^\dagger Y) - \mathcal{L}(X^\dagger)Y - X^\dagger \mathcal{L}Y,$$

and showed that complete positivity implied dissipativity, that is $\mathcal{D}_\mathcal{L}(X,X) = \sum_k [L_k,X]^\dagger[L_k,X] \geq 0$ (in fact, that it implied the stronger condition of complete dissipativity). We may obtain the generator from a quantum stochastic model [8,9] with quantum stochastic differential equation (QSDE) defining at unitary process

$$dU_t = \left\{ \sum_k L_k dB_k^\dagger(t) - \sum_k L_k^\dagger dB_k(t) - (\frac{1}{2}\sum_k L_k^\dagger L_k + iH)dt \right\} U_t$$

where $B_k(t)$ are quantum annihilator processes living on a Fock space describing the environment, and we have the nontrivial quantum Itō relation $dB_j(t)\,dB_k(t)^\dagger = \delta_{jk}\,dt$. In particular, for an observable X of the system, we set $j_t(X) = U_t^\dagger(X \otimes I_{\mathrm{Fock}})U_t$ and find

$$dj_t(X) = j_t(\mathcal{L}X)\,dt + \sum_k j_t([X,L_k])\,dB_k^\dagger(t) + \sum_k j_t([L_k^\dagger,X])\,dB_k(t).$$

Taking the partial trace of $j_t(X)$ over the Fock vacuum leads to the QDS generated by $\mathcal{L}$. We note that the noise can be written in terms of the two *quadratures*

$$Q_k(t) = B_k(t) + B_k(t)^\dagger, \quad P_k(t) = \frac{1}{i}\left(B_k(t) - B_k(t)^\dagger\right).$$

Separately, these are Wiener processes for the Fock vacuum state, but they in fact are non-commuting operator-valued processes with $[Q_j(t),P_k(s)] = \delta_{jk}\min(t,s)$, and

$$(dQ_k)^2 = (dP_k)^2 = dt, \quad dQ_k\,dP_k = -dP_k\,dQ_k = i\,dt.$$

The quantum stochastic model essentially achieves a form of fluctuation-dissipation balance at the dynamical level, which in quantum theory is algebraic in nature. The problem of including thermal noise, properties of detailed balance, whether there is a regression theorem, the relationship with the Nyquist theory, etc., has been addressed extensively [10,11]. The quantum noise however becomes essentially classical if only at most one of the process in each quadrature noise (Q_k, P_k) appears: more generally, we could add in classical jump processes along with the Wiener noise, and the class of generator for quantum systems with essentially classical noise is known [12] and

is smaller that the GKS-Lindblad class. It is therefore the case in quantum Markovian evolutions that both the system and the noise are quantum!

It is at this point we return to the classical theory mechanics, and to the programme initiated by Onsager. Classical mechanics has a geometric structure characterized by the Poisson Bracket. We argue that the general "de-quantized" version of quantum Markov stochastic models should capture the commutation relations of the noise as well as the system: that is, there is a Poisson Brackets for both the system and noise processes!

2 Classical Noise

We now look at the classical the analog where we fix a phase space with global canonical coordinates (q, p) and standard Poisson Brackets

$$\{f, g\} = \frac{\partial f(q, p)}{\partial q} \frac{\partial g(q, p)}{\partial p} - \frac{\partial g(q, p)}{\partial q} \frac{\partial f(q, p)}{\partial p}.$$

The restriction to one mechanical degree of freedom is for convenience only.

2.1 Deterministic Flows on Phase Space

Let us consider a deterministic dynamical flow on phase space given by

$$\dot{q}_t = v^q(q_t, p_t), \; \dot{p}_t = v^p(q_t, p_t) \tag{1}$$

and we have locally the solutions $q_t = q_t(q, p)$ and $p_t = (q, p)$ integral to the velocity vector field $v = (v^q, v^p)$ and with initial phase point (q, p) at $t = 0$. Then for any function f on phase space we have

$$\frac{d}{dt} f(q_t, p_t) = v.\nabla(f)|_{(q_t, p_t)}. \tag{2}$$

From a geometric viewpoint, $v.\nabla$ is of course a tangent vector field on the phase space. In what follows, we could take the phase space to be a general cotangent bundle with the canonical symplectic structure, or more generally a Poisson manifold [14], however we restrict to the standard case for transparency.

The Poisson Bracket of the evolved functions is

$$\{f(q_t, p_t), g(q_t, p_t)\} = \frac{\partial f(q_t(q, p), p_t(q, p))}{\partial q} \frac{\partial g(q_t(q, p), p_t(q, p))}{\partial p}$$

$$- \frac{\partial g(q_t(q, p), p_t(q, p))}{\partial q} \frac{\partial f(q_t(q, p), p_t(q, p))}{\partial p}.$$

In what follows we wish to consider flow semi-groups generated by differential operators $\mathcal{L}$ other than vector fields. In particular, we have in mind the generators of Markov diffusions, which will be second order differential operators. As the classical analog of Lindblad's dissipation, we make the following definition.

Definition: We say that the flow is *canonical* if we have

$$\{f(q_t, p_t), g(q_t, p_t)\} = \{f, g\}|_{(q_t, p_t)}, \tag{3}$$

for ever pair of smooth functions f, g.

For example, the linearly damped harmonic oscillator is

$$v_{\text{DHO}}^q = \frac{p}{m}, \quad v_{\text{DHO}}^p = -m\omega^2 q - \gamma p \tag{4}$$

and we find $\nabla.v = -\gamma$, constant. For this case we see that the Poisson Bracket is $\{f(q_t, p_t), g(q_t, p_t)\} = e^{-\gamma t}\{f, g\}$.

Definition: Let $\mathcal{L}$ be a differential operator on phase space. Its *dissipation* is the object $\mathscr{D}_{\mathcal{L}}$ defined by

$$\mathscr{D}_{\mathcal{L}}(f, g) \triangleq \mathcal{L}\{f, g\} - \{\mathcal{L}f, g\} - \{f, \mathcal{L}g\} \tag{5}$$

for arbitrary twice-differentiable functions f, g.

Some commentary is appropriate here. The classical definition of dissipation involves the Poisson Brackets, and so differs from the obvious analog of Lindblad's definition which would be $\Gamma_{\mathcal{L}}(f, g) \triangleq \mathcal{L}(fg) - \mathcal{L}(f), g - f\mathcal{L}(g)$. The $\Gamma_{\mathcal{L}}$ operator is the well-known *squared-field* operator from the geometric analysis of Markov generators [13]. Classically we have $\Gamma_{\mathcal{L}} \equiv 0$ if and only if $\mathcal{L}$ is a tangent vector field (that is, a first order differential operator). In this sense $\Gamma_{\mathcal{L}}$ is a natural analog of the Lindblad dissipation in the sense that the both measure how far $\mathcal{L}$ is away from being a derivation on the corresponding algebras, namely the algebra of smooth functions classically and the C*-algebra of operators on the Hilbert space in the quantum case. However, the product of operators encodes kinematic information, for instance the Heisenberg canonical commutation relations $qp - pq = i\hbar$, and our definition gives the appropriate physical definition of dissipation in the classical setting. The next two propositions single out Hamiltonian systems as those having no dissipation.

Proposition 1 *The dissipation associated with a vector field v on phase space is*

$$\mathscr{D}_{v.\nabla}(f, g) = -(\nabla.v)\{f, g\}.$$

Definition: The dynamics is said to be *Hamiltonian* if $v^q(q, p) = \frac{\partial H}{\partial p}$, $v^p(q, p) = -\frac{\partial H}{\partial q}$, for some function H on phase space. (Equivalently, $v.\nabla \equiv \{\cdot, H\}$.)

Proposition 2 *The dissipation associated with a vector field v on phase space with differentiable components v^q and v^p will vanish if and only if it is Hamiltonian.*

Proof: By Proposition 1, we have $\mathscr{D}_v \equiv 0$ only if $\nabla.v = \frac{\partial}{\partial q}(v^q) + \frac{\partial}{\partial p}(v^p) = 0$ at every point of phase space. It is well-known from potential theory that this will then imply that $v^q = \frac{\partial H}{\partial p}$, $v^p = -\frac{\partial H}{\partial q}$ for some function H. Conversely, if v is Hamiltonian then $\nabla.v = \frac{\partial}{\partial q}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0$, which is of course Liouville's theorem. $\square$

Note that the converse also follows from the Jacobi identity for Poisson Brackets: $\mathscr{D}_{\{\cdot,H\}}(f,g) = \{\{f,g\},H\} - \{\{f,H\},g\} - \{f,\{g,H\}\} = 0$.

Proposition 3 *The flow generated by a velocity field v on phase space is canonical if and only if v is Hamiltonian.*

Proof: Using (2), we see that the infinitesimal form for the canonical criterion is $v.\nabla\{f,g\} = \{v.\nabla f,g\} + \{f,v.\nabla g\}$ for all f and g, and this requires $\mathscr{D}_{v.\nabla} \equiv 0$. However, being dissipation-free is equivalent from Proposition 2 to being Hamiltonian. $\square$

2.2 Stochastic Flows on Phase space

We now consider stochastic diffusion on phase space driven by independent Wiener processes $\{Q_k(t) : k = 1, \cdots, n, t \geq 0\}$ and satisfying the stochastic differential equation (SDE)

$$dq_t = v^q(q_t, p_t)\, dt + \sum_k \sigma_k^q(q_t, p_t)\, dQ_k(t),$$

$$dp_t = v^p(q_t, p_t)\, dt + \sum_k \sigma_k^p(q_t, p_t)\, dQ_k(t). \tag{6}$$

The SDE is understood in the Itō sense, so all increments are future pointing. The vector fields $\sigma_k = (\sigma_k^q(q,p), \sigma_k^p(q,p))$, for $k = 1, \cdots, n$, fix the strength of the fluctuations. For a function f on phase space, we find

$$df(q_t, p_t) = (\mathcal{L}f)|_{(q_t,p_t)}\, dt + \sum_k (\sigma_k.\nabla f)|_{(q_t,p_t)}\, dQ_k(t)$$

where the generator $\mathcal{L}$ of the diffusion is the second order differential operator

$$\mathcal{L} = v^q \frac{\partial}{\partial q} + v^p \frac{\partial}{\partial p} + \frac{1}{2}g^{qq}\frac{\partial^2}{\partial q^2} + g^{qp}\frac{\partial^2}{\partial q \partial p} + \frac{1}{2}g^{pp}\frac{\partial^2}{\partial p^2}$$

and the diffusion coefficients are $g^{qq} = \sum_k \sigma_k^q \sigma_k^q$, $g^{qp} = \sum_k \sigma_k^q \sigma_k^p$, $g^{pp} = \sum_k \sigma_k^p \sigma_k^p$. We note that the corresponding Fokker-Planck equation [22] is

$$\frac{\partial \varrho}{\partial t} = -\frac{\partial}{\partial q}(v^q \varrho) - \frac{\partial}{\partial p}(v^p \varrho) + \frac{1}{2}\frac{\partial^2}{\partial q^2}(g^{qq}\varrho) + \frac{\partial^2}{\partial q \partial p}(g^{qp}\varrho) + \frac{1}{2}\frac{\partial^2}{\partial p^2}(g^{pp}\varrho).$$

The definition of canonical flows (3) makes sense in the stochastic setting. From the Itō calculus [17], we now obtain the infinitesimal form

$$d\{f_t, g_t\} = \{df_t, g_t\} + \{f_t, dg_t\} + \{df_t, dg_t\}, \tag{7}$$

which implies that

$$\left(\mathscr{D}_{\mathcal{L}}(f,g) - \sum_k \{\sigma_k(f), \sigma_k(g)\} \right) dt + \sum_k \mathscr{D}_{\sigma_k.\nabla}(f,g) dQ_k(t) = 0.$$

For this to hold, we must have $\mathscr{D}_{\sigma_k.\nabla} \equiv 0$ for each k requiring that $\sigma_k.\nabla \equiv \{\cdot, F_k\}$, for some functions F_k, and so

$$\mathscr{D}_{\mathcal{L}}(f,g) = \sum_k \{\{f, F_k\}, \{g, F_k\}\}.$$

It is easy to see [15] that the generator must then have the form

$$\mathcal{L} = \{\cdot, H\} + \frac{1}{2} \sum_k \{\{\cdot, F_k\}, F_k\}. \tag{8}$$

We then have

$$v^q = \mathcal{L}(q) = \frac{\partial H}{\partial p} + \frac{1}{2} \sum_k \{\frac{\partial F_k}{\partial p}, F_k\},$$

$$v^p = \mathcal{L}(p) = -\frac{\partial H}{\partial q} - \frac{1}{2} \sum_k \{\frac{\partial F_k}{\partial q}, F_k\},$$

so that the vector field v (Itō drift) for canonical fields is then dissipative with

$$\nabla.v = -\sum_k \left(\frac{\partial^2 F_k}{\partial q^2} \frac{\partial^2 F_k}{\partial p^2} - \left(\frac{\partial^2 F_k}{\partial q \partial p} \right)^2 \right).$$

We summarize as follows.

Theorem 1 *The stochastic process on phase space defined by (6) is canonical with respect to the standard Poisson Brackets if and only if the σ_k are Hamiltonian vector fields with Hamiltonian F_k and $\mathcal{L}$ takes the form (8).*

Example 1: We may try a linear function of q and p for the fluctuation functions F_k :

$$F_k(q,p) = \alpha_k p + \beta_k q$$

for constants α_k and β_k. Unfortunately, this leads to no dissipation since the Itō drift is Hamiltonian: $v^q = \frac{\partial H}{\partial p}, v^p = -\frac{\partial H}{\partial q}$. This model does nothing more than add Wiener noise onto an already Hamiltonian model. The fluctuations have no dissipation to balance here.

Example 2: If we wish to obtain nonzero dissipation in the Itō drift, then we need the F_k to be nonlinear. The damped harmonic oscillator velocity field v_{DHO} (4) can be obtained from the single Wiener noise model with the choices

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 + \frac{1}{2}\gamma qp, \quad F = \sqrt{\gamma} \left(\frac{p^2}{2z} + \frac{1}{2}zq^2 \right).$$

3 Symplectic Noise Models

We start from the observation that in the quantum case the noises come in canonically conjugate pairs $(Q_k(t), P_k(t))$, taken as dimensionless. We similarly postulate pairs of classical Wiener processes satisfying canonical Poisson Bracket relations of the form

$$\{Q_j(t), P_k(s)\} = \frac{1}{\mathsf{s}} \delta_{jk} \min(t, s), \tag{9}$$

where $\mathsf{s} > 0$ has units of action, and then apply the same dynamical principles to the classical case.

   This is achieved by introducing the following definition of the full Poisson Brackets for a classical system with classical noise.

   Definition: Let $(Q_k(t), P_k(t))$, $k = 1, \cdots, n$ be a family of $2n$ independent Wiener processes, and on the space of functions of the system variables q, p and the coordinate processes $(Q_k(t), P_k(t))$ define the Poisson Bracket

$$\{F, G\} \triangleq \frac{\partial F}{\partial q} \frac{\partial G}{\partial p} - \frac{\partial G}{\partial q} \frac{\partial F}{\partial p} + \frac{1}{\mathsf{s}} \sum_k \int dt \left(\frac{\delta F}{\delta Q_k(t)} \frac{\delta G}{\delta P_k(t)} - \frac{\delta G}{\delta Q_k(t)} \frac{\delta F}{\delta P_k(t)} \right),$$
$$\tag{10}$$

for every pair of functionals F, G, with $\frac{\delta F}{\delta Q_k(t)}$ denoting functional derivative.

   A more concrete construction is given by taking $\mathcal{Q}(\cdot), \mathcal{P}(\cdot)$ to be a pair of Gaussian random fields with moment generating functions

$$\mathbb{E}\left[e^{i\mathcal{Q}(f) + i\mathcal{P}(g)} \right] = \exp\left\{ -\frac{1}{2} \int |f(t)|^2 dt - \frac{1}{2} \int |g(t)|^2 dt \right\}$$

and we define the noise Poisson Brackets as the antisymmetric bi-derivation with basic relations

$$\{\mathcal{Q}(f), \mathcal{Q}(g)\} = 0 = \{\mathcal{P}(f), \mathcal{P}(g)\}, \quad \{\mathcal{Q}(f), \mathcal{P}(g)\} = \int f(t)g(t)\, dt,$$

as well as satisfying a Jacobi identity. We then set $Q(t) = \mathcal{Q}(1_{[0,t]})$ and $P(t) = \mathcal{P}(1_{[0,t]})$, where $1_{[0,t]})$ is the indicator function of interval $[0, t]$. Let $(X_t), (Y_t)$ be stochastic processes adapted to the filtration generated by the Weiner processes Q, P, then we have the infinitesimal relations

$$\{X_t\, dQ(t), Y_t\} = \{X_t, Y_t\}\, dQ(t), \ \{X_t\, dP(t), Y_t\} = \{X_t, Y_t\}\, dP(t),$$

and

$$\{X_t dQ(t), Y_t dP(t)\} = \{X_t, Y_t\}\, dQ(t)dP(t) + X_tY_t\{dQ(t), dP(t)\} \equiv X_tY_t\, dt.$$

3.1 Canonical Evolutions Under Symplectic Noise

For a function f of q and p we shall write f_t for $f(q_t, p_t)$. Let us consider a classical stochastic process of the form

$$df_t = (\mathcal{L}f)_t \, dt + (\sigma.\nabla f)_t \, dQ(t) + (\varsigma.\nabla f)_t \, dP(t) \tag{11}$$

and we now require the stochastic flow to be canonical with respect to the full Poisson Brackets (10). (For simplicity we consider a single pair of canonically conjugate Wiener processes.)

The equation (7) is required to hold with the full Poisson Brackets (10) now understood. Here we have now

$$
\begin{aligned}
\{df, dg\} = &\{(\sigma.\nabla f)_t \, dQ + (\varsigma.\nabla f)_t \, dP, \\
&(\sigma.\nabla g)_t \, dQ + (\varsigma.\nabla g)_t \, dP\} = \{\sigma.\nabla f, \sigma.\nabla g\}_t \, dt + \{\varsigma.\nabla f, \varsigma.\nabla g\}_t \, dt \\
&+ \mathsf{s}^{-1} \left(\sigma.\nabla f)(\varsigma.\nabla g)_t dt - (\varsigma.\nabla f)(\sigma.\nabla g)\right)_t \, dt.
\end{aligned}
$$

The equation (7) implies the following identities

$$
\begin{aligned}
\mathscr{D}_{\mathcal{L}}(f, g) &= \{(\sigma.\nabla f), (\sigma.\nabla g)\} + \{(\varsigma.\nabla f), (\varsigma.\nabla g)\} \\
&\quad + \mathsf{s}^{-1} \left(\sigma.\nabla f)(\varsigma.\nabla g\right) - \mathsf{s}^{-1}(\varsigma.\nabla f)(\sigma.\nabla g), \\
\mathscr{D}_{\sigma.\nabla}(f, g) &= 0, \\
\mathscr{D}_{\varsigma.\nabla}(f, g) &= 0.
\end{aligned}
$$

Evidently, this requires that $\sigma.\nabla \equiv \{\cdot, F\}$ and $\varsigma.\nabla \equiv \{\cdot, G\}$, for some functions F, G of (q, p). From arguments similar to before, we see that the generator takes the form $\mathcal{L} = \{\cdot, H\} + \frac{1}{2}\{\{\cdot, F\}, F\} + \frac{1}{2}\{\{\cdot, G\}, G\} + \mathcal{K}$, where $\mathscr{D}_{\mathcal{K}}(f, g) = \mathsf{s}^{-1} \left(\sigma.\nabla f)(\varsigma.\nabla g\right) - \mathsf{s}^{-1}(\varsigma.\nabla f)(\sigma.\nabla g)$, and remarkably $\mathscr{D}_{\mathcal{K}}(f, g)$ then equals

$$
\begin{aligned}
&\{f, F\}\{g, G\} - \{f, G\}\{g, F\} \\
&= \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\left(\frac{\partial g}{\partial q}\frac{\partial G}{\partial p} - \frac{\partial G}{\partial q}\frac{\partial g}{\partial p}\right) - (F \leftrightarrow G) \\
&\equiv \{f, g\}\{F, G\}.
\end{aligned}
$$

Comparison with Proposition 1 shows that it is enough to take $\mathcal{K}$ to be a first order differential operator $u.\nabla$ with divergence $-\mathsf{s}^{-1}\{F, G\}$. This determines u uniquely up to a Hamiltonian vector field which can always be absorbed. We now synthesis the main result.

Theorem 2 *A diffusion on phase space driven by canonically conjugate pairs of Wiener processes will be canonical for the full Poisson Brackets (10) if it takes the form*

$$df_t = (\mathcal{L}f)_t \, dt + \sum_k \{f, F_k\}_t \, dQ_k(t) + \{f, G_k\}_t \, dP_k(t)$$

with generator

$$\mathcal{L} = \{\cdot, H\} + \frac{1}{2} \sum_k \left(\{\{\cdot, F_k\}, F_k\} + \{\{\cdot, G_k\}, G_k\} \right) + u.\nabla,$$

where u is a vector field with $\nabla.u = -\mathsf{s}^{-1} \sum_k \{F_k, G_k\}$, for functions H, F_k, G_k on phase space.

Note that the dissipation $\mathscr{D}_{\mathcal{L}}(f, g)$ takes the general form

$$\sum_k \{\{f, F_k\}, \{g, F_k\}\} + \sum_k \{\{f, G_k\}, \{g, G_k\}\}$$

$$+ \mathsf{s}^{-1} \sum_k \{F_k, G_k\} \{f, g\}. \tag{12}$$

The Fokker-Planck equation for the diffusion may be written in the form

$$\frac{\partial \varrho}{\partial t} = \{H, \varrho\} + \frac{1}{2} \sum_k \{\{\varrho, F_k\}, F_k\} + \frac{1}{2} \sum_k \{\{\varrho, G_k\}, G_k\} - \nabla.(\varrho\, u).$$

3.2 Linear Symplectic Stochastic Models

We now shall describe general examples leading to a Itō drift velocity v linear in the canonical coordinates, with bounded below Hamiltonian, and constant diffusion matrix. For simplicity we restrict to a single canonical pair, and set

$$H = H_0 + zqp,$$
$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2,$$
$$F = aq + bp,$$
$$G = cq + dp,$$

and we set $\gamma = \mathsf{s}\{F, G\} = \mathsf{s}(ad - bc)$ which we assume greater than zero.

A particular solution a vector field u satisfying $\nabla.u = -\gamma$ is

$$u.\nabla = -\gamma p \frac{\partial}{\partial p}.$$

The general solution is the particular solution plus some Hamiltonian vector field which we may absorb into H, but which we ignore in the present context.

This leads to phase space velocity field

$$v^q = zq + \frac{1}{m}p, \quad v^p = -m\omega^2 q - zp - \gamma p. \tag{13}$$

The second-order part of the generator has the diffusion tensor

$$g = \begin{bmatrix} b^2 + d^2 & ab + cd \\ ab + cd & a^2 + c^2 \end{bmatrix}$$

which is easily seen to be positive definite. Without loss of generality, we shall fix

$$F = -\sqrt{\gamma \mathsf{s} \epsilon}\, p,\, G = \sqrt{\gamma / \mathsf{s} \epsilon}\, q,$$

where $\epsilon > 0$ has units $\mathtt{mN^{-1}s^{-1}}$ (the general case follows from a linear canonical transformation of the conjugate Wiener noises).

If we set $z = 0$ then the system becomes invariant under position translation and stochastic differential equations for the coordinate processes are

$$dq_t = \frac{p_t}{m}\, dt - \sqrt{\gamma \mathsf{s} \epsilon}\, dQ(t),$$
$$dp_t = -(m\omega^2 q_t + \gamma p_t)\, dt - \sqrt{\gamma \mathsf{s}/\epsilon}\, dP(t), \tag{14}$$

which is canonical with respect to the full Poisson Brackets (Theorem 2), though not if we considered the system Poisson Brackets only (Theorem 1). The Fokker-Planck equation for the diffusion is

$$\frac{\partial \varrho}{\partial t} = -\frac{\partial}{\partial q}\left(v_{\mathrm{DHO}}^q\, \varrho\right) - \frac{\partial}{\partial p}\left(v_{\mathrm{DHO}}^p\, \varrho\right) + \frac{1}{2}\gamma \mathsf{s} \epsilon \frac{\partial^2 \varrho}{\partial q^2} + \frac{1}{2}\frac{\gamma \mathsf{s}}{\epsilon}\frac{\partial^2 \varrho}{\partial p^2}$$

and a steady state solution is given by a Gaussian density [22],

$$\varrho = Z^{-1} \exp\left\{ -\frac{1}{2}\, [q, p]\, \sigma^{-1} \begin{bmatrix} q \\ p \end{bmatrix} \right\}$$

with covariance matrix σ satisfying $A\sigma + \sigma A = -D$ where $A = \begin{bmatrix} 0 & 1/m \\ -m\omega & -\gamma \end{bmatrix}$. One finds that the Gaussian steady state is mean zero and (positive definite!) covariance matrix σ given by

$$\begin{bmatrix} \langle q^2 \rangle_{\mathrm{ss}} & \langle qp \rangle_{\mathrm{ss}} \\ \langle pq \rangle_{\mathrm{ss}} & \langle p^2 \rangle_{\mathrm{ss}} \end{bmatrix} = \frac{1}{2}\mathsf{s} \begin{bmatrix} \frac{1 + \epsilon^2 m^2 (\omega^2 + \gamma^2)}{\epsilon^2 m^2 \omega^2} & -\epsilon m \gamma \\ -\epsilon m \gamma & \frac{1 + \epsilon^2 m^2 \omega^2}{\epsilon} \end{bmatrix}.$$

It is customary to identify the quadratic function $\frac{1}{2}\, [q, p]\, \sigma^{-1} \begin{bmatrix} q \\ p \end{bmatrix}$ with $\beta H_{\mathrm{eff}}(q, p)$ where β is the inverse temperature and H_{eff} is the effective Hamiltonian. However, it is clear that H_{eff} is not the oscillator Hamiltonian H_0. Indeed, we have cross terms in H_{eff} due to the nonzero covariances $\langle qp \rangle_{\mathrm{ss}} = -\varepsilon m \gamma$, and so $H_{\mathrm{eff}}(q, -p) \neq H_{\mathrm{eff}}(q, p)$.

One may try to fix this by changing the Hamiltonian of the stochastic model to $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 + zqp$ for some real scalar z, in which case one computes that

$$\langle q^2 \rangle_{\mathrm{ss}} = \frac{1}{Y(z)}\left(2\varepsilon^2 m^2 \omega^2 + 2\varepsilon^2 m^2 z^2 + 4\varepsilon^2 m^2 \gamma z + 2\varepsilon^2 m^2 \gamma^2 + 2\right)$$

$$\langle p^2 \rangle_{\mathrm{ss}} = \frac{1}{Y(z)}\left(2m^4 \omega^4 \varepsilon^2 - m^2 z^2 - m^2 \gamma z + 2m^2 \omega^2\right)$$

$$\langle qp \rangle_{\mathrm{ss}} = -\frac{1}{Y(z)}\left(2m^3 \varepsilon^2 \omega^2 z + 2m^3 \varepsilon^2 \omega^2 \gamma + mz\right)$$

where $Y(z) = \frac{2m^2}{\gamma s}\left(z\omega^2 - z^3 - 2z^2\gamma - z\gamma^2 + 2\omega^2\gamma\right)$. The specific choice $z = -2\varepsilon^2 m^2\omega^2\frac{\gamma}{2\varepsilon^2 m^2\omega^2+1}$ leads to steady state $\langle qp\rangle_{\mathrm{ss}} = 0$, that is

$$\sigma = \frac{1}{2}s\frac{2\varepsilon^2 m^2\omega^2 + 1}{\varepsilon m^2\omega^2}\begin{bmatrix} 1 & 0 \\ 0 & m^2\omega^2 \end{bmatrix}.$$

This is the Gibbs state giving the canonical ensemble for the energy H_0 with inverse temperature $\beta = 1/k_B T$ given by

$$k_B T = \frac{1}{2}s\frac{2\epsilon^2 m^2\omega^2 + 1}{\epsilon m},$$

and we have a proper equipartition of energy. However, the underlying Hamiltonian is now have $H(q,p) = H_0(q,p) + zqp$ and the stochastic differential equations no longer equals (14) as the drift velocity is now given by (13) the non-zero value of z above.

4 Comparison With Quantum Models

It is instructive to consider the quantum Markov of a linearly damped harmonic oscillator. For linear damped motion for an open system with canonical observables operators q, p satisfying the Heisenberg commutation relations $[q,p] = i\hbar$, the most general form can be obtained by taking $H = H_0 + \frac{\mu}{2}(qp + pq)$, $L_k = \alpha_k p + \beta_k q$, where $H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$ is a harmonic oscillator Hamiltonian, $z\mu$ is a real constant, the α_k, β_k are complex constants, and it suffices to take just two coupling terms $k = 1, 2$ [19, 20, 21]. We follow the presentation given in [18].

For instance, let us, introduce the annihilator $a = \sqrt{m\omega/2\hbar}(q + ip/m\omega)$ so that $H = \hbar\omega(a^\dagger a + \frac{1}{2})$, and take a pair of independent quantum Wiener processes $B_k(t)$ $(k = 1, 2)$ with coupling operators $L_1 = \sqrt{\gamma(n+1)}a$ and $L_2 = -\sqrt{\gamma n}a^\dagger$, leads to

$$dq_t = \left(\frac{p_t}{m} + (\frac{1}{2}\gamma - \mu)q_t\right)dt - \sqrt{\frac{\gamma(n+1)\hbar}{2m\omega}}\,dQ_1(t) - \sqrt{\frac{\gamma n\hbar}{2m\omega}}\,dQ_2(t),$$

$$dp_t = \left(-m\omega^2 q_t - (\frac{1}{2}\gamma + \mu)p_t\right)dt$$
$$- \sqrt{\frac{\gamma(n+1)\hbar m\omega}{2}}\,dP_1(t) + \sqrt{\frac{\gamma n\hbar m\omega}{2}}\,dP_2(t),$$

where $q_t = j_t(q)$, $p_t = j_t(p)$ and the Q_k, P_k are the quadrature processes.

We may take $\mu = \gamma$ and set $n = 0$ (vacuum noise) then the Langevin equations are identical in form to the classical symplectic SDEs (14) above under the identification $s = \hbar/2$ and $\varepsilon = 1/m\omega$. However, once again we do not have convergence a Gibbs state of the form $e^{-\beta H_0}/Z$, with H_0 the oscillator Hamiltonian.

If, however, we set $\mu = 0$ and $n > 0$ then we find convergence to the Gibbs state $e^{-\beta H_0}/Z$ with temperature determined by $n = 1/(e^{\beta \hbar \omega} - 1)$. (The same applies to the vacuum case $n = 0$ where the oscillator decays to its ground state as equilibrium state.) However, the drift velocity in phase space is now $v^q = \frac{p}{m} + \frac{1}{2}\gamma q$, $v^p = -m\omega^2 q - \frac{1}{2}\gamma p$, and this is derived from the Hamiltonian $H = H_0 + \frac{\gamma}{2}(qp + pq)$, as opposed to the vector field v_{DHO} of linearly damped harmonic oscillator.

5 Conclusion

The Poisson Brackets are the algebraic ghosts of quantum mechanics in the classical world. The Hamiltonian and Poisson bracket formulation of classical mechanics dates from the 1830's, and it was Dirac who realised that the commutator $\frac{1}{i\hbar}[\cdot, \cdot]$ is the analog, in spirit, of the Poisson Brackets. It has become fashionable to view quantum probability as the natural extension of classical theory that formulates the classical theory axiomatically, and then drops the assumption of commutativity of random variables. While this has indeed been productive, it ignores the fact that specific form of the non-commutativity - the Heisenberg commutation relations - have a physical importance that transcends the abstract formalism.

The new ingredient introduced in this paper is the idea of canonically conjugate noises, and more generally of a Poisson Brackets for the noise in stochastic classical mechanical models. The idea is not unreasonable as it suggests that a classical environment is ultimately a mechanical system, and that an idealized model of the noise should somehow retain the symplectic structure. There is a central program in Mathematical and Statistical Physics to realise quantum models describing dissipation and relaxation to thermal equilibrium. Here there is two-way transfer from quantising classical stochastic models and dequantising quantum open systems back to the classical world. We in effect provide a missing link - a classical noise which its own inherent symplectic structure and where the physical noises do not Poisson commute!

If we take the noise to have no symplectic nature, but require the diffusion on phase space to be canonical for the system Poisson Bracket, then we arrive at the general situation described in Theorem 1, which was previously derived in [15], see also [16]. Here we have the surprising fact that to dilate the equations of motion of the linearly damped Harmonic oscillator (4) that we take a quadratic Hamiltonian H but must have a coupling F to the noise that is *quadratic*. In the quantum case it suffices to take the coupling operators L_k to be *linear* in the canonical observables. Indeed, we see in Example 1 that linear F_k lead to zero dissipation.

If we now allow the noise to have a symplectic structure, and ask for the overall system plus noise Poisson bracket to be respected, then the general situation is as given in Theorem 2. In the quantum model, the system interact with the noise, so that in the Heisenberg picture observables of the system evolves as stochastic process on the tensor product system plus noise space.

When we use the commutator in the quantum stochastic setting we really do be the commutator of operators on the Hilbert space of the system tensored with the Fock space of the noise. In that sense, our use of the full Poisson Brackets for system plus noise capture the spirit of what goes on in the quantum models.

The q and p play the roles of the α and β variables in the Onsager-Casimir-Machlup theory. A standing assumption of Onsager and Machlup [3] in the case of Gaussian fluctuations for the $\alpha = q$ and $\beta = p$ variables is that there are vanishing covariances between the q's and p's, consistent with the intuition that the entropy cannot change under time reversal. Interpreting the state as a Gibbs state means that we have a quadratic Hamiltonian satisfying $H(q, -p) = H(q, p)$, so there can be no q and p cross terms. We propose that the stochastic flows of the form described in Theorem 2 give the appropriate noise model on which to base the theory of dissipation and spontaneous fluctuations about equilibrium.

Inevitably, whenever we consider Wiener noise we are making an approximation to the physical spectrum of frequencies by the flat spectrum of white noise. However, we should note that the approximation is an excellent and widely accepted one in quantum optics, and what we gain is computationally tractable model that allows us to use the Itō calculus. Noting that the model noise is just a mathematical proxy for the physical environment, what we have tried to do is to retain the canonical nature of the environment in the noise model [10,23]. This is automatic in the quantum theory, but classically we have to define a Poisson structure for the noise along with the notion of canonically conjugate Wiener process.

Within the framework of Theorem 2, we now see that it is possible to have a linear noise coupling leading to a stochastic diffusion on phase space with the linearly damped Harmonic oscillator (4) as drift. This much agrees with the quantum situation. Moreover, if we wish to have a linear model with constant damping, then we cannot have both the drift given by (4) and a steady state that is not a thermal equilibrium state for the oscillator. This is well-known in the quantum setting, where it has been much touted as the failure of the quantum regression theorem. This is frequently blamed on the structure of GKS-Lindblad generator, which is in turn attributed to the quantum Markov assumption or complete positivity, however, we see here that virtually the same situation occurs classically and that the real transgressor is the requirement of a symplectic model for both system and noise.

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