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Conditions for the spectrum associated with an asymptotically straight leaky wire to contain an interval $[-\alpha^2/4, \infty)$

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1. Introduction

Let Γ be a continuous and piecewise smooth curve extending to infinity in the plane $\mathbf{R}^2$, and consider the spectrum σ associated with the perturbed Laplacian

$$H := -\Delta - \alpha(\mathbf{x})\delta(\mathbf{x} - \Gamma) \quad (1.1)$$

in $\mathbf{R}^2$ where δ is the Dirac delta function and $\alpha(\mathbf{x}) \geq 0$ is a given continuous and bounded function. In this spectral context, Γ is called a leaky wire and the Hamiltonian (1.1) represents the motion of a particle under the influence of a singular attraction (since $\alpha(\mathbf{x}) \geq 0$) along Γ . We refer to the recent extensive survey [5] for the physical motivation of studying this model and details of the influence that the geometry of Γ has on the nature of the spectrum. In the simplest case where α is constant and Γ is a straight line, we have $\sigma = [-\alpha^2/4, \infty)$ [6, (5.1)] but, for much more general curves Γ , it was shown in [6, section 5] that the essential spectrum σ_{ess} is

$$\sigma_{ess} = [-\alpha^2/4, \infty) \quad (1.2)$$

under certain global conditions of Γ which include the idea of asymptotic straightness [6, (3.1) and (3.2)]. An example of this idea [6, Remark 5.6] is that, in terms of the arc length s , the curvature $k(s)$ of Γ satisfies $|k(s)| \leq (\text{const.}) |s|^{-\beta}$ for some $\beta > 5/4$.

There are two aspects to the proof of (1.2) in [6]. One is that

$$\sigma_{ess} \supset [-\alpha^2/4, \infty) \quad (1.3)$$

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and the other is that $(-\infty, -\alpha^2/4)$ is not in σ_{ess} . In this paper we are concerned with (1.3) and, in view of the considerable technicalities in [6], we give a much simpler proof of (1.3) using the singular (or Weyl) sequence method. Further, our approach covers the case of $\alpha(\mathbf{x})$, a bounded and continuous function, tending to a finite limit at infinity. This approach is in the spirit of the early work on the Schrödinger operator in [2], [3] and [8, section 49] and requires conditions imposed only on long disjoint sections of Γ , rather than globally. Our main result is given in Theorem 3.1.

2. Operator realisation

The formal definition (1.1) can be made precise by the procedure indicated in [1, section 4] and [5, section 2] (see also [6] and [7]). Thus we assume that Γ is a piecewise C^1 curve without cusps and that for each compact subset K of $\mathbf{R}^2$ we have $\int_K \alpha(\mathbf{x}) \delta(\mathbf{x} - \Gamma) d\mathbf{x} < \infty$. (For simplicity we only consider the case when Γ divides the plane into two regions R_1 and R_2 .) In addition we assume that $\alpha(\mathbf{x})$ is non-negative, bounded and continuous on Γ . In this case we have that

$$\int_{\mathbf{R}^2} (1 + \alpha(\mathbf{x})) |\psi(\mathbf{x})|^2 \delta(\mathbf{x} - \Gamma) d\mathbf{x} \leq c \int_{\mathbf{R}^2} (|\nabla \psi(\mathbf{x})|^2 + |\psi(\mathbf{x})|^2) d\mathbf{x}$$

for $\psi(\cdot) \in C_0^\infty(\mathbf{R}^2)$. Therefore we can define the quadratic form

$$q(f, g) := \int_{\mathbf{R}^2} \nabla f(\mathbf{x}) \nabla \bar{g}(\mathbf{x}) d\mathbf{x} - \int_{\Gamma} \alpha(\mathbf{x}) f(\mathbf{x}) \bar{g}(\mathbf{x}) ds$$

with the domain $W^{1,2}(\mathbf{R}^2)$ which gives rise to the selfadjoint operator H from (1.1) by the same construction as described in [5, section 2]. The same operator can be constructed from the essentially self-adjoint operator $\tilde{H}$ defined by

$$\tilde{H}\psi(\mathbf{x}) = -\Delta\psi(\mathbf{x}) \quad (\mathbf{x} \in \mathbf{R}^2 \setminus \Gamma) \quad (2.1)$$

with domain $D(\tilde{H})$ consisting of functions $\psi \in W^{2,2}(\mathbf{R}^2 \setminus \Gamma)$ which are continuous at Γ and with the normal derivatives having a jump in the sense that

$$\frac{\partial \psi}{\partial n_1}(\mathbf{x}) + \frac{\partial \psi}{\partial n_2}(\mathbf{x}) = -\alpha(\mathbf{x})\psi(\mathbf{x}) \quad (\mathbf{x} \in \Gamma). \quad (2.2)$$

Here n_1 and n_2 denote the normals directed away from Γ on the two sides of Γ . This operator reproduces the form q on the core $C_0^\infty(\mathbf{R}^2)$ of q (see [5, Section 2] and [1, Remark 4.1]). Thus H is the closure of $\tilde{H}$ and, in particular, $D(\tilde{H}) \subset D(H)$.

A real number λ is in σ_{ess} if and only if there is a sequence f_m in $D(H)$ such that

$$\|f_m\| = 1, \quad f_m \rightharpoonup 0 \text{ (weak convergence)}$$

and

$$\|(H - \lambda I)f_m\| \rightarrow 0 \quad (2.3)$$

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as $m \rightarrow \infty$ [4, p.415]. Such a sequence is called a singular (or Weyl) sequence. In section 3, our choice of f_m will lie in $D(\tilde{H})$ so that (2.3) becomes simply

$$\int_{R_1} |(\Delta + \lambda I)f_m|^2 d\mathbf{x} + \int_{R_2} |(\Delta + \lambda I)f_m|^2 d\mathbf{x} \rightarrow 0 \quad (2.4)$$

by (2.1), subject to f_m satisfying the normal derivative condition (2.2).

3. The singular sequence

Our idea of asymptotic straightness is simply that Γ should lie close to arbitrarily long disjoint line segments as Γ recedes to infinity. The segments can be located without restriction in $\mathbf{R}^2$ but, purely for convenience in the proof which follows, we take them to lie along the x -axis. Thus we assume that there are disjoint intervals $I_m = (c_m - a_m, c_m + a_m)$ on the x -axis with $c_m \rightarrow \infty$ and $a_m \rightarrow \infty$ and, for x in each I_m , Γ has the equation $y = \gamma(x)$ with

$$\gamma(x) \rightarrow 0 \quad (3.1)$$

as $x \rightarrow \infty$ through the I_m . As in section 2, we take it that $\alpha(\mathbf{x})$ is non-negative, bounded and continuous on Γ .

Theorem 3.1. *Let $\gamma(x)$ have continuous derivatives up to order 3 in each I_m and, in addition to (3.1), let*

$$\gamma^{(r)}(x) \rightarrow 0 \quad (r = 1, 2, 3) \quad (3.2)$$

as $x \rightarrow \infty$ through the I_m . For $x \in I_m$, we write $\alpha(x) := \alpha(x, \gamma(x))$ and assume that $\alpha(x)$ has a continuous second derivative. As $x \rightarrow \infty$ through the I_m , let $\alpha(x)$ tend to a finite limit α_0 (> 0) with $\alpha^{(r)}(x) \rightarrow 0$ ($r = 1, 2$). Then $\sigma_{ess} \supset [-\alpha_0^2/4, \infty)$.

Proof. In the square $S_m = (c_m - a_m, c_m + a_m) \times (-a_m, a_m)$ we define

$$f_m(\mathbf{x}) = b_m h_m(x - c_m) h_m(y) \exp\{-\beta(x) |y - \gamma(x)| + i\nu x\} \quad (3.3)$$

where b_m is the normalisation factor making $\|f_m\| = 1$, $\nu \geq 0$ and h_m is as usual a $\mathbf{C}^{(2)}(-\infty, \infty)$ function such that

$$h_m(t) = 1 \quad (|t| \leq a_m - 1) = 0 \quad (|t| \geq a_m)$$

and with derivatives independent of m . Finally, $\beta(x)$ (≥ 0) is chosen so that f_m satisfies (2.2), and we deal with this choice now.

When f_m is substituted into the left-hand side of (2.2), the net result comes only from the modulus term in (3.3). Let $\theta(x) = \tan^{-1} \gamma(x)$ ($|\theta(x)| \leq \pi/2$). Then a simple calculation shows that (2.2) holds if

$$2\beta(\cos \theta + \gamma' \sin \theta) = \alpha,$$

giving

$$\beta = \frac{1}{2} \alpha \cos \theta = \frac{1}{2} \alpha (1 + \gamma'^2)^{-1/2}. \quad (3.4)$$

Then, as $x \rightarrow \infty$ through the I_m , we have from (3.2)

$$\beta(x) \rightarrow \frac{1}{2}\alpha_0, \quad \beta^{(r)}(x) \rightarrow 0 \quad (r = 1, 2). \quad (3.5)$$

It follows now from (3.1) and (3.3) that

$$1 = b_m^2 \{1 + o(1)\} \int_{-a_m}^{a_m} dt \int_{-a_m}^{a_m} \exp\{-2\beta(t + c_m) | y | \} dy$$

and hence, by (3.2) and (3.5),

$$b_m \sim (4a_m/\alpha_0)^{-1/2} \quad (m \rightarrow \infty). \quad (3.6)$$

The weak convergence condition $f_m \rightharpoonup 0$ is easily verified from (3.3) and (3.6). Then, to apply (2.4), we consider Δf_m for $\mathbf{x}$ in S_m and $\mathbf{x} \notin \Gamma$. The situation is similar on the two sides of Γ , and we concentrate on $y > \gamma(x)$. Then, by (3.3),

$$\begin{aligned} \Delta f_m = & \{\beta^2 + (\beta\gamma' + \beta'\gamma - y\beta' + i\nu)^2 \\ & + \beta\gamma'' + 2\beta'\gamma' + \beta''\gamma - y\beta''\} f_m + E_m, \end{aligned} \quad (3.7)$$

where E_m denotes terms containing derivatives of h_m . Now $h'_m(t)$ and $h''_m(t)$ are only non-zero when $|a_m| - 1 < t < |a_m|$, and it follows from (3.6) that $\|E_m\| = o(1)$ ($m \rightarrow \infty$).

Finally, by (3.1), (3.2) and (3.5), we have from (3.7)

$$\int_{R_1} |(\Delta + \lambda I)f_m|^2 d\mathbf{x} + \int_{R_2} |(\Delta + \lambda I)f_m|^2 d\mathbf{x} = (\alpha_0^2/4 - \nu^2 + \lambda)^2 \|f_m\|^2 + o(1),$$

and hence (2.4) is satisfied with $\lambda = -\alpha_0^2/4 + \nu^2$. Since $\nu \geq 0$ is arbitrary, this proves that $\sigma_{ess} \supset [-\alpha_0^2/4, \infty)$ as required. $\square$

The proof includes the case of constant $\alpha(\mathbf{x}) (= \alpha > 0)$ in the I_m , and then $\sigma_{ess} \supset [-\alpha^2/4, \infty)$. We also note that the theorem remains true when $\alpha_0 = 0$. We simply choose an f_m like (3.3) but with $\beta = 0$ and supported on a large square which does not intersect Γ . We omit the familiar details which are as in [2] for example.

References

- [1] J. F. Brasche, P. Exner, Y. A. Kuperin and P. Seba. Schrödinger operators with singular interactions. *J. Math. Anal. and Appl.*, 184:112–139, 1994.
- [2] M. S. P. Eastham. A condition for the spectrum in eigenfunction theory to contain $[0, \infty)$. *Quart. J. Math. Oxford Ser. (2)*, 17:146–150, 1966.
- [3] M. S. P. Eastham. Conditions for the spectrum in eigenfunction theory to consist of $(-\infty, \infty)$. *Quart. J. Math. Oxford Ser. (2)*, 18:147–153, 1967.
- [4] D. E. Edmunds and W. D. Evans. *Spectral theory and differential operators*. Oxford Mathematical Monographs. The Clarendon Press, New York, 1987. Oxford Science Publications.
- [5] P. Exner. Leaky quantum graphs: a review. *Proc INI programme "Analysis on graphs and applications"*.

- [6] P. Exner and T. Ichinose. Geometrically induced spectrum in curved leaky wires. *J. Phys. A*, 34(7):1439–1450, 2001.
- [7] P. Exner and K. Němcová. Leaky quantum graphs: approximations by point-interaction Hamiltonians. *J. Phys. A*, 36(40):10173–10193, 2003.
- [8] I. M. Glazman. *Direct methods of qualitative spectral analysis of singular differential operators*. Israel Program for Scientific Translations, Jerusalem, 1965.

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