Fuzzy qualitative trigonometry

Liu, Honghai; Coghill, George; Barnes, Dave

Published in:
International Journal of Approximate Reasoning

DOI:
10.1016/j.ijar.2009.07.003

Publication date:
2009

Citation for published version (APA):
Liu, H., Coghill, G., & Barnes, D. (2009). Fuzzy qualitative trigonometry. *International Journal of Approximate Reasoning*, *51*(1), 71-88. https://doi.org/10.1016/j.ijar.2009.07.003

International Journal of Approximate Reasoning 51 (2009) 71–88

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Fuzzy qualitative trigonometry

Honghai Liu [a], George M. Coghill [b,*], Dave P. Barnes [c]

[a] Intelligent Systems & Robotics Group, School of Creative Technologies, University of Portsmouth, Portsmouth PO1 2DJ, United Kingdom
[b] Department of Computing Science, University of Aberdeen, King's College, Meston Walk, Aberdeen AB24 3UE, United Kingdom
[c] Department of Computer Science, University of Wales, Aberystwyth SY23 3DB, United Kingdom

ARTICLE INFO

Article history:
Received 28 August 2008
Received in revised form 30 July 2009
Accepted 31 July 2009
Available online 6 September 2009

Keyword:
Fuzzy qualitative reasoning

ABSTRACT

This paper presents a fuzzy qualitative representation of conventional trigonometry with the goal of bridging the gap between symbolic cognitive functions and numerical sensing & control tasks in the domain of physical systems, especially in intelligent robotics. Fuzzy qualitative coordinates are defined by replacing a unit circle with a fuzzy qualitative circle; a Cartesian translation and orientation are defined by their normalized fuzzy partitions. Conventional trigonometric functions, rules and the extensions to triangles in Euclidean space are converted into their counterparts in fuzzy qualitative coordinates using fuzzy logic and qualitative reasoning techniques. This approach provides a promising representation transformation interface to analyze general trigonometry-related physical systems from an artificial intelligence perspective.

Fuzzy qualitative trigonometry has been implemented as a MATLAB toolbox named XTRIG in terms of 4-tuple fuzzy numbers. Examples are given throughout the paper to demonstrate the characteristics of fuzzy qualitative trigonometry. One of the examples focuses on robot kinematics and also explains how contributions could be made by fuzzy qualitative trigonometry to the intelligent connection of low-level sensing & control tasks to high-level cognitive tasks.

1. Introduction

Trigonometry is a branch of mathematics that deals with the relationships between the sides and angles of triangles and with the properties and application of trigonometric functions of angles. It began as the computational component of geometry in the second century BC and plays a crucial role in domains such as mathematics & engineering. In order to bridge the gap between qualitative and quantitative descriptions of physical systems, we propose a fuzzy qualitative representation of trigonometry (FQT), which provides theoretical foundations for the representation of trigonometric properties.

It is often desirable and sometimes necessary to reason about the behavior of a system on the basis of incomplete or sparse information. The methods of model-based technology provide a means of doing this [1,2]. The initial approaches to model-based reasoning were seminal but focused on symbolic qualitative reasoning (QR) only, providing a means whereby the global picture of how a system might behave could be generated using only the sign of the magnitude and direction of change of the system variables. This makes qualitative reasoning complementary to quantitative simulation. However, quantitative and qualitative simulation form the two ends of a spectrum; and semi-quantitative methods were developed to fill the gap. For the most part these were interval reasoners bolted on to existing qualitative reasoning systems (e.g. [3]), Blackwell did pioneering work on spatial reasoning on robots [4]; however, one exception to this was fuzzy qualitative reasoning

* Corresponding author. Tel.: +44 (0)1224 273829.
E-mail addresses: honghai.liu@port.ac.uk (H. Liu), g.coghill@abdn.ac.uk (G.M. Coghill), dpb@aber.ac.uk (D.P. Barnes).

0888-613X/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.ijar.2009.07.003

which integrated the strengths of approximate reasoning with those of qualitative reasoning to form a more coherent semi-quantitative approach than their predecessors [5,6]. Model-based technology methods have been successfully applied to a number of tasks in the process domain. However, while some effort has been expended on developing qualitative kinematic models, the results have been limited [7–10]. The basic requirement for progressing in this domain is the development of a qualitative version of the trigonometric rules. Buckley and Eslami [11] proposed the definition of fuzzy trigonometry from the fuzzy perspective without consideration of the geometric meaning of trigonometry. Some progress has been made in this direction by Liu [12], but as with other applications of qualitative reasoning, the flexibility gained in variable precision by integrating fuzzy and qualitative approaches is no less important in the kinematics domain. In this paper we present an extension of the rules of trigonometry to the fuzzy qualitative case, which will serve as the basis for fuzzy qualitative reasoning about the behavior and possible diagnosis of kinematic robot devices.

Fuzzy qualitative reasoning combines the advantages of fuzzy reasoning and qualitative reasoning techniques. Research into the integration of fuzzy reasoning and qualitative reasoning has been carried out in both theory and application in the past two decades [5,6,13–16]. The use of fuzzy reasoning methods are becoming more and more popular in intelligent systems[17,18], especially hybrid methods and their applications integrating with evolutionary computing [19–22], decision trees [23,24], neural networks [25–27], data mining [28], and so on [29–33]. Qualitative reasoning is reviewed in [34–37]. The integration of fuzzy reasoning and qualitative reasoning (i.e., fuzzy qualitative reasoning) provides an opportunity to explore research (e.g., spatial reasoning) with both the advantages of fuzzy reasoning and qualitative reasoning. Some of fuzzy qualitative reasoning contributions can be found in [5,6,14–16]. Shen and Leitch [5] use a fuzzy quantity space (i.e., normalized fuzzy partition), which allows for a more detailed description of the values of the variables. Such an approach relies on the extension principle and approximation principle in order to express the results of calculations in terms of the fuzzy sets of the fuzzy quantity space.

Fuzzy reasoning has been significantly developed and has attracted much attention and exploitation from industry and research communities in the past four decades. Fuzzy reasoning is good at communicating with sensing and control level subsystems by means of fuzzification and defuzzification methods. It has powerful reasoning strategies utilizing compiled knowledge through conditional statements so as to easily handle mathematical and engineering systems in model free manner. Fuzzy reasoning also provides a means of handling uncertainty in a natural way making it robust in significantly noisy environments. However, the fact that its knowledge is primarily shallow, and the questions over the computational overhead associated with handling grades of membership of discrete fuzzy sets must be taken into account if multi-step reasoning is to be carried out. On the other hand, Qualitative and Model-based Reasoning has been successfully deployed in many applications such are autonomous spacecraft support [38], Systems Biology [39] and qualitative systems identification [40]. It has the advantage of operating at the conceptual modeling level, reasoning symbolically with models which retain the mathematical structure of the problem rather than the input/output representation of rule bases (fuzzy or otherwise). These models are incomplete in the sense that, being symbolic, they do not contain, or require, exact parameter information in order to operate. Qualitative reasoning can make use of multiple ontologies, can explicitly represent causality, enable the construction of more sophisticated models from simpler constituents by means of compositional modeling, and infer the global behavior of a system from a description of its structure [34,37]. These features can, when combined with fuzzy values and operators, compensate for the lack of ability in fuzzy reasoning alone to deal with that kind of inference about complex systems. The computational cause-effect relations contained in qualitative models facilitates analyzing and explaining the behavior of a structural model. Based on a scenario generated from fuzzy reasoning's fuzzification process, fuzzy qualitative reasoning may be able to build a behavioral model automatically, and use this model to generate a behavior description, acceptable by symbolic systems, either by abstraction and qualitative simulation or as a comprehensive representation of all possible behaviors utilizing linguistic fuzzy values. Liu, Coghill and Brown had attempted two completely different approaches [41–43] based on fuzzy qualitative trigonometry [44]. Research reported in [41] proposed a normalized based qualitative representation from cognition perspective, it converts both numeric and subsymbolic data into a normalization reference where transfer of different types of data is carried out, the method was not implemented into spatial robots due to its costly computational cost and complex spatial relation though it was applied to planar robots. On the other hand, conventional robotics had been adapted with the fuzzy qualitative trigonometry, not only did it implement feasible for spatial robots but also it shows the promising potential for intelligent robotics [42]. Though fuzzy qualitative trigonometry was briefly reviewed in the both papers, a full account of fuzzy qualitative trigonometry is presented in this paper with the goal of solving the intelligent connection problem (also known as symbol grounding problem) for physical systems, in particular robotic systems. This problem is one of the key issues in AI robotics [45] and relates to a wide range of research areas such as computer vision [46]. This paper is organized as follows. Section 2 reviews the technical background of fuzzy qualitative reasoning. Section 3 presents fuzzy qualitative Cartesian coordinates. Section 4 derives fuzzy qualitative trigonometric functions. It converts trigonometric functions into those in terms of fuzzy qualitative descriptions. Section 5 addresses fuzzy qualitative trigonometric rules. Section 6 presents fuzzy qualitative triangle theorems. Section 7 addresses discussions and conclusions in the end.

2. Fuzzy qualitative reasoning

Fundamentals of fuzzy qualitative reasoning are provided in this section.

2.1. Fuzzy numbers

The membership distribution of a normal convex fuzzy number can be approximated by a 4-tuple fuzzy number representation (i.e., $[a, b, \tau, \beta]$) with the condition $a \leqslant b$ and $a \times b \geqslant 0$. The representation of the 4-tuple fuzzy numbers is a better qualitative representation for trigonometry, because this representation has high resolution and good compositionality. The degree of resolution can be adjusted by the choice of fuzzy numbers. Such a representation provides the flexibility to carry out computation based on real numbers, intervals, triangular numbers and trapezoidal fuzzy intervals, which comprise nearly all of the computing elements in fuzzy qualitative reasoning. This representation has the ability to combine representations for different aspects of a phenomenon or system to create a representation of the phenomenon or systems as a whole. The computation of fuzzy numbers is based on its arithmetic operations. The arithmetic operations on the 4-tuple parametric representation of fuzzy numbers are shown in Table 1, where $<_0$ is the partial order $<_\alpha$ when $\alpha = 0$, as defined in [5]. The partial order has been extended to a general form, i.e., $\{\mathbb{O}_\alpha | (\mathbb{O} \in \{>, <, =, \leqslant, \geqslant, \in, \subset, \supset, \subseteq, \supseteq\})$. For instance, partial order $<_\alpha$ is defined such that for $m,\ n\ (m \neq n)$, we say m is α-less than n, $m <_\alpha n$, iff $a < b,\ a \in m_\alpha,\ b \in n_\alpha$ with m_α and n_α being the α-cuts of m and n, respectively.

It should be noted that the extension principle and normalized fuzzy partitions generation allow the extension of classical mathematical operators on crisp sets to the fuzzy domain. However, though the result of an arithmetic operation is a fuzzy number, it may not map exactly onto any of the members of the fuzzy partition of the constrained variable. In order to have better approximation, fuzzy qualitative constraints are proposed, for instance, fuzzy qualitative trigonometric identity in Eq. (9c). In addition, fuzzy qualitative states are mapped back into their fuzzy partitions and the results used for fuzzy qualitative calculations. Further the fuzzy qualitative states can be grounded into symbols, which allows describing a same object in numerical and symbolic terms.

2.2. Quantity representation in fuzzy qualitative reasoning

Qualitative reasoning has explored tradeoffs in representations for continuous parameters ranging in resolution from sign algebras to the hyperreals [2]. Intervals are a well-known variable-resolution representation for numerical values, and have been heavily used in qualitative reasoning [47]. A fuzzy partition is utilized to represent continuous values via sets of ordinal relations, it can be thought of as partial information about a set of intervals [14]. The natural mapping between fuzzy partitions and intervals has been exploited by a variety of systems that use intervals, whose endpoints are known numerical values, to refine predictions produced by purely qualitative reasoning [2]. Fuzzy intervals have also been used in fuzzy reasoning about mechatronics systems [5]. Fuzzy qualitative trigonometry (FQT) has chosen the concept of a fuzzy partition due to the fact that it is a form of semi-qualitative QR and the success of fuzzy qualitative simulation. In this case the fuzzy partition is a set of overlapping fuzzy numbers, an example is shown in Fig. 1.

The fuzzy partition for every variable in the system is a finite and convex discretisation of the real number line. The fuzzy partition for each variable individually is a subset of the real number line, which still covers the real number line. The fuzzy partition in Fig. 1 would be suitable for a variable whose domain has been normalized.

3. Fuzzy qualitative coordinates

Fuzzy qualitative coordinates are proposed in order to clearly describe the geometric meaning of FQT.

3.1. Angle and distance measurement

Angle and distance measurements play an important role in FQT due to the fact that trigonometry is centered on angle measurement and quantities determined by the measure of an angle. A fuzzy qualitative fuzzy partition, Q, is introduced to

Table 1
Arithmetic operations with 4-tuple fuzzy numbers.

Operation	Result	Conditions
$-n$	$[-d, -c, \delta, \gamma]$	All n
$\sqrt{n}$	$\left[\sqrt{c}, \sqrt{d}, \sqrt{c} - \sqrt{c-\gamma}, -\sqrt{d} + \sqrt{d+\delta}\right]$	$n >_0 0$
$1/n$	$[1/d, 1/c, \delta/(d(d+\delta)), \gamma/(c(c-\gamma))]$	$n >_0 0,\ n <_0 0$
$m+n$	$[a+c, b+d, \tau+\gamma, \beta+\delta]$	All $m,\ n$
$m-n$	$[a-d, b-c, \tau+\delta, \beta+\gamma]$	All $m,\ n$
$m \times n$	$[ac, bd, a\gamma + c\tau - \tau\gamma, b\delta + d\beta + \beta\delta]$	$m >_0 0,\ n >_0 0$
	$[ad, bc, d\tau - a\delta + \tau\delta, c\beta - b\gamma + \beta\gamma]$	$m <_0 0,\ n >_0 0$
	$[bc, ad, b\gamma - c\beta + \beta\gamma, a\delta - d\tau - \tau\delta]$	$m >_0 0,\ n <_0 0$
	$[bd, ac, -b\delta - d\beta - \beta\delta, \tau\gamma - a\gamma - c\tau]$	$m <_0 0,\ n <_0 0$
	$m = [a, b, \tau, \beta],\ n = [c, d, \gamma, \delta]$	

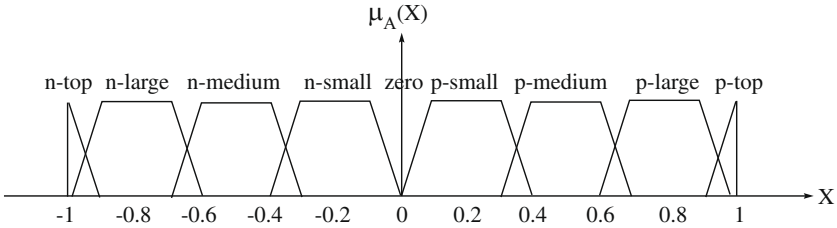

Fig. 1. A normalized fuzzy partition, i.e., quantity space.

represent qualitative states of an orientation component Q^a and a translation component Q^d. It means that angle and distance measurements in fuzzy qualitative coordinates depend on the numbers and fuzzy characteristic of the elements of a fuzzy qualitative fuzzy partition. For example, given an orientation range $[0\ \Theta]$ and a translation range $[0\ L]$, its fuzzy partition can be described as,

$$Q = \{Q^a, Q^d\} \tag{1}$$

where

$$Q^a = [QS_a(\theta_1), \ldots, QS_a(\theta_i), \ldots QS_a(\theta_m)]$$
$$Q^d = [QS_d(l_1), \ldots, QS_d(l_j), \ldots QS_d(l_n)]$$

$QS_a(\theta_i)$ denotes the state of an angle θ_i, $QS_d(l_j)$ denotes the state of a distance l_j, m and n are the number of the elements of the two components. The state is defined by the area covered by fuzzy numbers. The position measurement of $P(QS_a(\theta_i), QS_d(l_j))$ determined by both the characteristics of the fuzzy membership functions of $QS_a(\theta_i)$ and $QS_d(l_j)$.

The geometric meaning of FQT is demonstrated in a proposed fuzzy qualitative unit circle, in which the motion of an orientation component and a translation component are constructed. A fuzzy qualitative unit circle is nothing but a conventional trigonometric circle whose axes are replaced by unit (normalized) fuzzy partitions, for instance, Eq. (1) is a representation for a fuzzy qualitative unit circle if $\Theta = 2\pi$ and $L = 1$. Further, the fuzzy qualitative Cartesian coordinates can be constructed by the combination of the fuzzy partitions of the two motion components in real numbers $\mathbb{R}$. The position state of a fuzzy qualitative point is defined by the projections of the point onto fuzzy qualitative axes. For instance, let the X axis projection of a fuzzy qualitative point P be X_p, and the Y axis projection as Y_p, its fuzzy qualitative position can be described as $P = (X_p, Y_p)$. It means that a point position in fuzzy qualitative coordinates is described by fuzzy sets, hence, fuzzy arithmetic and its characteristics can be applied to those in fuzzy qualitative coordinates. For simplicity, we use membership functions of 4-tuple fuzzy numbers, based on which a fuzzy qualitative unit circle is constructed, please see Fig. 2. Compared with the quantitative trigonometric circle, its translation is replaced by the fuzzy partitions constructed by a set of 4-tuple fuzzy numbers (i.e., μ_x, μ_y) on the distance range $[-1\ 1]$; its orientation is replaced by the fuzzy partition constructed by a

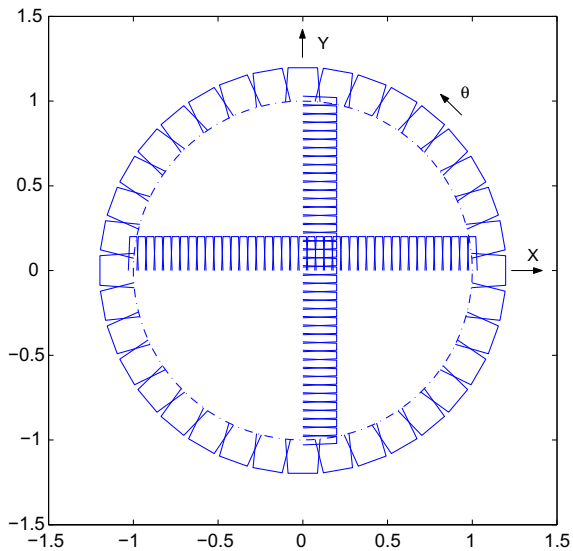

Fig. 2. A fuzzy qualitative trigonometric circle with $p = 36$, $q_x = 41$ and $q_y = 41$.

set of 4-tuple fuzzy numbers (i.e., μ_θ) on the circle range [0 2π]. Additionally, in accordance with Cartesian coordinates convention, the position $P_o(X_o, Y_o) = ([0,0,0,0],[0,0,0,0])$ is the fuzzy qualitative origin, where the axes intersect. Counterclockwise is the positive orientation, the number of qualitative orientation states in a full circle starting from the X axis is denoted by p, the number of qualitative translation states is denoted by q_{axis} (i.e., $q_x,\ q_y$). Though different methods can be employed to generate the fuzzy numbers in a fuzzy partition, two conditions must hold. First, the fuzzy numbers in each component must be origin symmetric. Second, special real numbers (e.g., 0, $\pi/2,\ \pi,\ 3\pi/2$ in an orientation, and -1, 0, -1 in a translation) are the centers of some certain fuzzy numbers in their corresponding fuzzy partition. For instance, a right angle in conventional trigonometry corresponds to the fuzzy number $QS_a(p/4+1)$, whose center is equal to $\pi/2$. Please note that, for a 4-tuple fuzzy number (e.g., $[a,b,\tau,\beta]$), its center is defined by $(a+b)/2$. It should be noted that the fuzzy numbers in the orientation fuzzy partition need to be multiplied by 2π before they can be utilized in the fuzzy arithmetic operations of Table 1.

3.2. Fuzzy qualitative Cartesian position

The fuzzy qualitative Cartesian coordinates are described by the combination of translation fuzzy partitions. Hence, fuzzy qualitative position in terms of 4-tuple fuzzy number in such coordinates can be derived; its representation is given by Eq. (2). The position is described in a compact row vector, in which the first four entries denote the fuzzy qualitative variable in the X axis, the rest denote the fuzzy qualitative variable in the Y axis. The graphic description of the position is shown in Fig. 3:

$$P_C = [[X]_{1\times4}[Y]_{1\times4}] = [[x_1 \quad x_2 \quad x_3 \quad x_4][y_1 \quad y_2 \quad y_3 \quad y_4]] \tag{2}$$

The fuzzy qualitative Cartesian position is divided into nine partitions by the matrix in Eq. (2). The partition elements can be described by a matrix PE, in which each entry corresponds to a partition. Please see Fig. 3:

$$PE = \begin{bmatrix} pe_{11} & pe_{12} & pe_{13} \\ pe_{21} & pe_{22} & pe_{23} \\ pe_{31} & pe_{32} & pe_{33} \end{bmatrix} \tag{3}$$

where element pe_{22} denotes the fuzzy qualitative partition where both the fuzzy values of the X and Y axes are equal to one; elements $pe_{12},\ pe_{21},\ pe_{32}$ and pe_{23} denote the partitions in which either the fuzzy value of the X axis or that of the Y axis is equal to one; elements $pe_{11},\ pe_{13},\ pe_{31}$ and pe_{33} denote the partitions in which neither of the fuzzy values of the X or Y axis is equal to zero. The matrix PE represents an uncertainty distribution to some extent. The position element in Eq. (2) can be converted to that in fuzzy qualitative polar coordinates. Its description is given in the following:

$$P_O = [[R]_{4\times4},[\Theta]_{4\times4}] = \left[\begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{bmatrix}\begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \theta_{14} \\ \theta_{21} & \theta_{22} & \theta_{23} & \theta_{24} \\ \theta_{31} & \theta_{32} & \theta_{33} & \theta_{34} \\ \theta_{41} & \theta_{42} & \theta_{43} & \theta_{44} \end{bmatrix}\right] \tag{4}$$

where,

$$r_{ij} = \sqrt{x_i'^2 + y_j'^2}$$
$$\theta_{ij} = arctg\frac{y_j'}{x_i'}$$

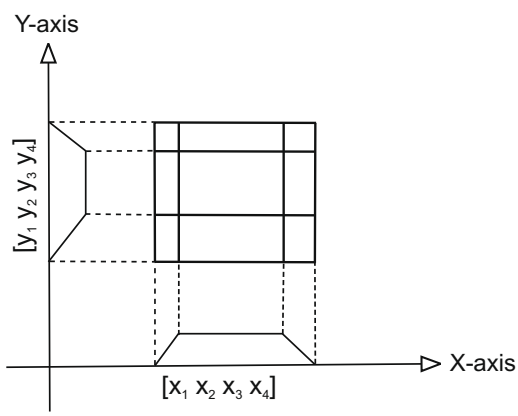

Fig. 3. A point position in fuzzy qualitative Cartesian coordinates.

and

$$[x'_1 \quad x'_2 \quad x'_3 \quad x'_4] = [x_1 - x_3 \quad x_1 \quad x_2 \quad x_2 + x_4]$$
$$[y'_1 \quad y'_2 \quad y'_3 \quad y'_4] = [y_1 - y_3 \quad y_1 \quad y_2 \quad y_2 + y_4]$$

Each row vector in $[R]_{4\times4}$ and $[\Theta]_{4\times4}$ is the element of the fuzzy partition of the radius and angle in the fuzzy qualitative polar coordinates. Besides, the above clearly indicates that the partition equation *PE* also applies to the position description in the polar coordinates, except that each entry is described by a radius and a angle rather than two elements in the *X* and *Y* axes.

3.3. Fuzzy qualitative polar position

It has shown in the description in Section 3.1 that the fuzzy qualitative polar coordinates are described by the combination of a translation and an orientation fuzzy partition. Hence, the fuzzy qualitative position in polar coordinates can be represented; its mathematical description is given in Eq. (2). The position in the polar coordinates has the same notation as its counterpart in Cartesian coordinates. It is also described in a compact row vector, in which the first four entries denote the fuzzy qualitative translation variable, the rest denote the fuzzy qualitative orientation variable. The graphic description of the position is shown in Fig. 4:

$$P_O = [[R]_{1\times4}[\Theta]_{1\times4}] = [[r_1 \quad r_2 \quad r_3 \quad r_4][\theta_1 \quad \theta_1 \quad \theta_1 \quad \theta_1]] \tag{5}$$

Eq. (5) also can be converted into fuzzy qualitative Cartesian coordinates (i.e., Eq. (2)). The conversion equation is given in the following:

$$P_C = [[X]_{4\times4}[Y]_{4\times4}] \tag{6}$$

where

$$[X]_{4\times4} = [R']^T_{1\times4} \cdot \cos([\Theta']_{1\times4})$$
$$[Y]_{4\times4} = [R']^T_{1\times4} \cdot \sin([\Theta']_{1\times4})$$

and

$$[R'] = [r'_1 \quad r'_2 \quad r'_3 \quad r'_4] = [r_1 - r_3 \quad r_1 \quad r_2 \quad r_2 + r_4]$$
$$[\Theta'] = [\theta'_1 \quad \theta'_2 \quad \theta'_3 \quad \theta'_4] = [\theta_1 - \theta_3 \quad \theta_1 \quad \theta_2 \quad \theta_2 + \theta_4]$$

3.4. XTRIG implementation

A MATLAB toolbox named XTRIG has been developed to implement the proposed fuzzy qualitative trigonometry. XTRIG is developed in terms of the 4-tuple fuzzy numbers, the characteristics of XTRIG are demonstrated by several examples throughout this paper. Examples have been provided in this section to demonstrate the above-mentioned definitions. They have been used to calculate trigonometric functions and triangle theorems by converting real numbers in Cartesian coordinates into fuzzy numbers in the fuzzy qualitative coordinates using fuzzy logic and qualitative reasoning techniques. In addition, the examples are based on a fuzzy number, $[a, b, \tau, \beta]$, the function is $b - a = \kappa_0\tau$ with the condition $\tau = \beta$. κ_0 is a threshold parameter to define the shape of fuzzy numbers. For instance, let the number of the fuzzy partition of an orientation p be 16 for a fuzzy qualitative trigonometric circle, those of a translation, q_x and q_y, be 21 each, and set κ_0 as 5,

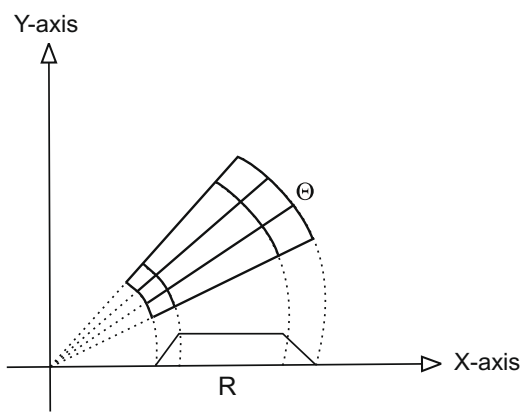

Fig. 4. A point position in fuzzy qualitative polar coordinates.

the corresponding fuzzy partitions of the orientation Q^a and translation Q^d can be generated. A version of the fuzzy partitions produced by XTRIG are shown in Figs. 5 and 6. It shows that the fuzzy-number generation function distributes the 16 fuzzy numbers in the translation component in the range [0 1], and two sets of the 21 fuzzy numbers in the range [−1 1]. The fuzzy qualitative right angle in this example is the fifth orientation angle (i.e., $[0.2240, 0.2760, 0.0104, 0.0104]$), whose center value is,

$$\frac{a+b}{2} \times 2\pi = \frac{0.224 + 0.276}{2} \times 2\pi = \frac{\pi}{2}$$

Though any translation quantitative range and orientation range can be described by 4-tuple fuzzy numbers through the adjustment of a fuzzy partition and its fuzzy membership functions, the relation between them needs to be defined. Fig. 5 also shows the conversion from a fuzzy qualitative angle to its fuzzy qualitative position, where the qualitative angle is the third orientation state in the orientation fuzzy partition Q^a (i.e., $[0.099 \quad 0.151 \quad 0.0104 \quad 0.0104]$). The conversion is carried out as follows. Firstly, we calculate the positions of the crossing points, A & B between the fuzzy number (i.e., the third ori-

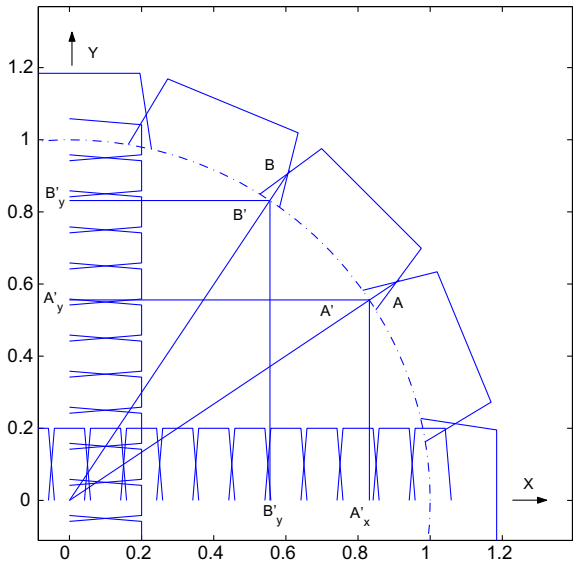

Fig. 5. The relation between qualitative translation and orientation.

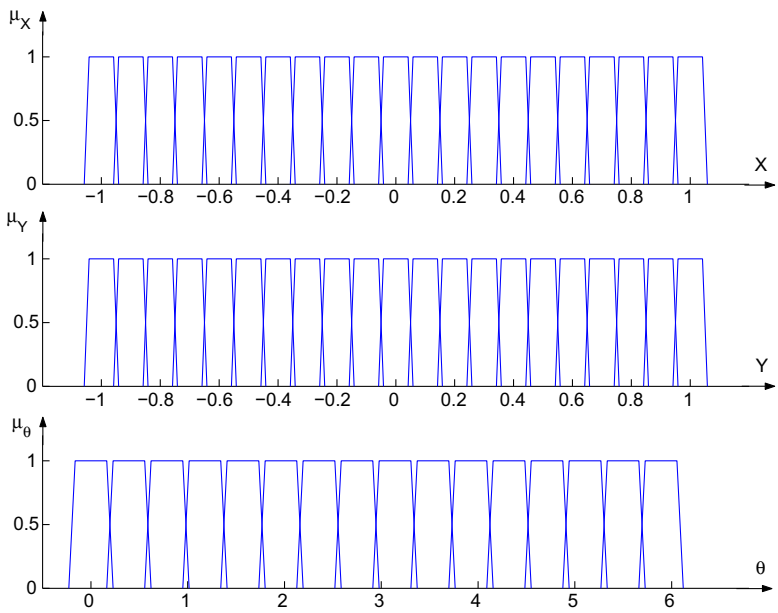

Fig. 6. Fuzzy partitions with $p = 17$, $q_x = 23$ and $q_y = 23$.

entation angle) and its adjacent fuzzy numbers; Secondly, we calculate crossing point positions A', B' between the unit circle curve and the two line segments OA and OB; Thirdly, we obtain two line segments $A'_xB'_x$ and $A'_yB'_y$ by projecting points A', B' into X and Y coordinates, respectively. Finally we generate two sets of fuzzy numbers which totally or partially are in the range $A'_xB'_x$ and $A'_yB'_y$. The set of fuzzy numbers that are within $A'_xB'_x$ is the fuzzy qualitative position on the X axis of the orientation angle (i.e., $QS_a(3)_X$); the other set within $A'_yB'_y$ is for Y axis projection (i.e., $QS_a(3)_Y$). The fuzzy qualitative position is given below, α stands for α-cut, as is zero by default,

$$P_{QS_a(3)}{=}_0(P_{QS_a(3)_X}, P_{QS_a(3)_Y}) \tag{7a}$$

where

$$P_{QS_a(3)_X}{=}_0\begin{bmatrix} 0.5583 & 0.6417 & 0.0167 & 0.0167 \\ 0.6583 & 0.7417 & 0.0167 & 0.0167 \\ 0.7583 & 0.8417 & 0.0167 & 0.0167 \end{bmatrix} \tag{7b}$$

$$P_{QS_a(3)_Y}{=}_0\begin{bmatrix} 0.5583 & 0.6417 & 0.0167 & 0.0167 \\ 0.6583 & 0.7417 & 0.0167 & 0.0167 \\ 0.7583 & 0.8417 & 0.0167 & 0.0167 \end{bmatrix} \tag{7c}$$

It shows that the fuzzy qualitative values of the third orientation angle in the X and Y axes are denoted by two sets of three 4-tuple fuzzy numbers, which are equal in this case, from its translation fuzzy partition. The conversion methodology can also be applied to the conversion from fuzzy numbers in its translation fuzzy partition to those in its orientation fuzzy partition. A closer look at line segment $B'B'_y$ shows that the line segment intersects with two fuzzy numbers with the X axis. The distances between the projection line segment and the centers of the two adjacent fuzzy numbers in the X axis are used to select whether a fuzzy number belongs to its corresponding fuzzy qualitative position or not. In addition, the relation of the non-selected fuzzy number to a qualitative position is needed to clarify in that it has potential to be part of the position. The relation is defined by a proposed relevance index in Section 4.2. The conversion provides an opportunity to analyze and calculate the relationship between quantitative ranges of an orientation and translation or both using fuzzy arithmetic and qualitative reasoning.

4. Fuzzy qualitative trigonometric functions

The fuzzy qualitative trigonometric functions are presented in this section. The extension principle allows the extension of classical mathematical operators (e.g., conventional trigonometric functions) to the fuzzy domain [48,49]. It means that an arithmetic operation performed between n fuzzy sets will yield a fuzzy set of the same form. Fuzzy qualitative trigonometric functions are derived from the geometric perspective; each of them is actually a set of its quantitative counterparts within a corresponding angular region. That is, fuzzy qualitative trigonometric functions provide an overview or a fuzzy qualitative description of a set of their quantitative counterparts. Each trigonometric function is derived and illustrated using FQT using the fuzzy partitions of the fuzzy qualitative coordinates. For simplicity, the symbols of quantitative trigonometric functions are used to describe their counterparts in FQT but with qualitative variables instead (e.g., $QS_i(j)$, where $i \in \{a, d\}$ and $j \in \{p, q\}$).

4.1. Trigonometric functions

In order to give a clear explanation of fuzzy qualitative trigonometric functions, let us consider the specific example for the third fuzzy qualitative angle in Figs. 5 and 6 as a general case of the ith orientation angle (i.e., $QS_a(i)$), please refer to Fig. 2 for the distribution of fuzzy partitions. $P_{QS_a(i)}$ stands for the fuzzy qualitative position of the intersection of $QS_a(i)$ to a unit circle curve, and its projected fuzzy qualitative positions on the axes of X and Y are $QS_d(|A'_xB'_x|)$ and $QS_d\left(\left|A'_yB'_y\right|\right)$ as shown in Fig. 5. We are now ready to define fuzzy qualitative trigonometric functions, which are given in the following,

$$\sin(QS_a(i)){=}_\alpha \frac{QS_d\left(\left|A'_yB'_y\right|\right)}{[1\quad 1\quad 0\quad 0]}{=}_\alpha QS_d\left(\left|A'_yB'_y\right|\right) \tag{8a}$$

$$\cos(QS_a(i)){=}_\alpha \frac{QS_d(|A'_xB'_x|)}{[1\quad 1\quad 0\quad 0]}{=}_\alpha QS_d(|A'_xB'_x|) \tag{8b}$$

$$\sec(QS_a(i)){=}_\alpha \frac{[1\quad 1\quad 0\quad 0]}{QS_d(|A'_xB'_x|)}{=}_\alpha \frac{[1\quad 1\quad 0\quad 0]}{\cos(QS_a(i))} \tag{8c}$$

$$\csc(QS_a(i)){=}_\alpha \frac{[1\quad 1\quad 0\quad 0]}{QS_d\left(\left|A'_yB'_y\right|\right)}{=}_\alpha \frac{[1\quad 1\quad 0\quad 0]}{\sin(QS_a(i))} \tag{8d}$$

$$\arcsin(QS_d\left(\left|A'_yB'_y\right|\right)){=}_\alpha QS_a(i) \tag{8e}$$

$$\arccos\left(QS_d(|A'_xB'_x|)\right){=}_\alpha QS_a(i) \tag{8f}$$

where $\sin(QS_a(i))$, $\cos(QS_a(i))$, $\sec(QS_a(i))$ and $\csc(QS_a(i))$ fuzzy qualitative functions are straightforward as shown in Fig. 5. Please note that a 4-tuple fuzzy number is an atomic element in the proposed trigonometric functions. For instance, say, $\sin(QS_a(i))$ is a fuzzy qualitative value including three 4-tuple fuzzy numbers, $\csc(QS_a(i))$ can be obtained by the operation that fuzzy qualitative division in Table 1 applies [1 1 0 0] to the three fuzzy numbers, respectively. In addition, the ith fuzzy qualitative position's arcsine and arccosine functions shown in Eqs. (8e) and (8f), i.e., $\arcsin(QS_d(i))$ and $\arccos(QS_d(i))$, are the inverse of its sine and cosine functions given in Eqs. (8a) and (8b). Refer to Fig. 5, if points B'_y and A'_y are the crossing points of the ith fuzzy number and its adjacent fuzzy numbers in Y coordinate, the fuzzy numbers in the orientation fuzzy partition within arc $A'B'$ are its corresponding qualitative angle. it should be noted that there are limitations for fuzzy qualitative arcsine and arccosine functions as their quantitative counterparts have, the limitations are $\arcsin(QS_d(i)) \subseteq_\alpha [-\frac{p}{4}-1, \frac{p}{4}+1]$, and $\arccos(QS_d(i)) \subseteq_\alpha [1, \frac{p}{2}+2]$.

Further, the tangent of $QS_a(i)$, written $\tan(QS_a(i))$, can be defined as the ratio of the opposite to the adjacent sides, that is $QS_d\left(\left|A'_yB'_y\right|\right)/QS_d(|A'_xB'_x|)$. It can clearly see that the tangent of $QS_a(i)$ is equal to the sine of $QS_a(i)$ divided by the cosine of $QS_a(i)$. The same can apply to cotangent function, then we have the two functions,

$$\tan(QS_a(i)) =_\alpha \frac{QS_d\left(\left|A'_yB'_y\right|\right)}{QS_d(|A'_xB'_x|)} =_\alpha \frac{\sin(QS_a(i))}{\cos(QS_a(i))} \tag{9a}$$

$$\cot(QS_a(i)) =_\alpha \frac{QS_d(|A'_xB'_x|)}{QS_d\left(\left|A'_yB'_y\right|\right)} =_\alpha \frac{\cos(QS_a(i))}{\sin(QS_a(i))} \tag{9b}$$

where

$$\cos^2(QS_a(i)) + \sin^2(QS_a(i)) =_\alpha QS_d(p) \tag{9c}$$

A fuzzy qualitative version of a fundamental trigonometric identity (i.e., Pythagorean identity) must hold for Eqs. (9a) and (9b). It filters out those fuzzy numbers in a fuzzy qualitative angle, which do not meet their geometric definition. For instance, $QS_d(p)$ is $[0.9583, 1.0417, 0.0167, 0.0167]$ in the example in Section 3.4, when $\alpha = 0$, the fundamental trigonometric identity can be checked by the fact whether the square root result of $\cos^2(QS_a(i)) + \sin^2(QS_a(i))$ is within $[0.9583, 1.0417, 0.0167, 0.0167]$. In fuzzy set theory, the explanation also can be given as follows. The extension principle allows the extension of classical mathematical operators to the fuzzy domain. That is to say that an arithmetic operation (e.g., those given in Table 1) performed between n 4-tuple fuzzy numbers will yield a fuzzy number of the same form. Although the result of an arithmetic operation is a fuzzy number, it may not map exactly onto any of the members of the fuzzy partition of the constrained variable. The fact is also addressed as linguistic approximation problem in relevant literature [50,48]. Therefore, a constraint related to a mapping from the calculated value onto the closed fuzzy partition is required. In this case the constraint is the fuzzy qualitative trigonometric identity given in Eq. (9c).

4.2. The relevance index

A fuzzy qualitative state is described by a set of overlapping fuzzy numbers, see Figs. 2–6. The ambiguity raised by the relation of projection line segments of a fuzzy qualitative state (i.e., $QS_r(i)$) to its adjacent fuzzy numbers needs to be clarified. A relevance index $QS_r(i)$ is introduced to present the relation between the ith fuzzy qualitative state and its adjacent fuzzy numbers, where the adjacent fuzzy numbers are not part of the fuzzy qualitative state. Its formula is given,

$$QS_r(i) = [\,\lambda_1 \quad X \quad X \quad \lambda_2\,] \quad (r \in \{d, a\}) \tag{10}$$

where λ_1 is the forward relevance index that denotes the relationship between the first fuzzy number of $QS_r(i)$ and its forward boundary fuzzy number, and λ_2 is backward relevance index that denotes the relationship between the last fuzzy numbers of $QS_r(i)$ and its backward boundary fuzzy number. Hence the relationship of two adjacent fuzzy qualitative states is solved by the description of the relation of their boundary fuzzy numbers. For instance, in Fig. 5, the X-axis projection of $P_{QS_a(3)_X}$ contains three fuzzy numbers (i.e., $QS_d(17)$, $QS_d(18)$ and $QS_d(19)$), its forward boundary fuzzy number is $QS_d(16)$ and its backward boundary fuzzy number is $QS_d(20)$. Note that the elements X in Eq. (10) could be any number in order to keep a 4-tuple fuzzy number form. X is replaced by NaN in the XTRIG MATLAB toolbox, where NaN means not a number.

New definitions have to be introduced before defining the elements of the relevance index, it is very complex to analyze the relationship of any two fuzzy numbers, it is also an open problem [51]. For simplicity, the relation of fuzzy numbers is analyzed for 4-tuple fuzzy numbers only. Consider two adjacent 4-tuple fuzzy numbers m and n, where $m = [a, b, \tau, \beta]$, $n = [c, d, \gamma, \delta]$. Besides the projection line l of FQT functions is shown in Fig. 7. The crossing point of the projection line l and fuzzy numbers m and n are $p_m(u, v_m)$ and $p_n(u, v_n)$, where u is the x-axis projection of the crossing points between line l and fuzzy numbers m and n, besides v_m and v_n are the degree of membership of $\mu(X)$ of the crossing points. A boundary point $p_b(u_b, v_b)$ is defined by the crossing point of the adjacent fuzzy numbers. The relation of a fuzzy number and its corresponding FQT functional projection line is determined by the following relevance index,

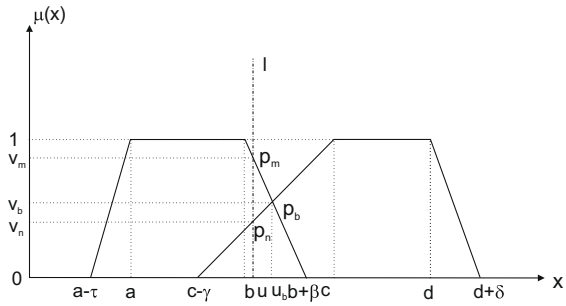

Fig. 7. Relationship between adjacent 4-tuple fuzzy numbers.

$$\lambda_1 = \begin{cases} \text{Strong-approximation} & u \leqslant b \\ \text{Weak-approximation} & b < u \leqslant u_b \\ \text{Bare-approximation} & (b+\beta) \geqslant u > u_b \end{cases}$$

$$\lambda_2 = \begin{cases} \text{Strong-approximation} & u \geqslant c \\ \text{Weak-approximation} & c > u \geqslant u_b \\ \text{Bare-approximation} & (c-\gamma) \leqslant u < u_b \end{cases}$$

where

$$u = f(QS_a(i))$$
$$u_b = \frac{1}{\gamma+\beta}(c \times \beta + b \times \gamma)$$
$$v_b = -\frac{1}{\gamma+\beta}(c - \beta - b - \gamma)$$

where $f(QS_a(i))$ is the projection numerical positions on X and Y coordinates of fuzzy qualitative state $QS_a(i)$. For instance, in Fig. 5, the X-coordinate positions (i.e., $f_X(QS_a(3))$) are A'_x and B'_x. The relevance index is defined in terms of membership distribution. Strong-approximation, weak-approximation and bare-approximation are employed to represent the degree of fuzzy qualitative approximation. Three real numbers 0, 1 and 2 are employed in the MATLAB XTRIG toolbox to denote them, respectively. For instance, the relevant index λ_2 of fuzzy number m is bare-approximation in Fig. 7, due to $b < u < u_b$, where $v_b = 0.5$ in XTRIG toolbox. And the relevant index λ_1 of fuzzy number n is strong-approximation.

The above definition of approximations is the fuzzy qualitative description of algebraic equality. It should be noted that fuzzy qualitative sine and secant functions of a FQT angle have the same relevant index in that a fuzzy convex and normal fuzzy number with bounded support can be mapped by a continuous real-valued function into a fuzzy number [52], it indicates that the relationship of the fuzzy numbers is not changed after a continuous real-valued mapping. As we know that the approximation principle merely states that any member of the fuzzy partition for the constrained variable, which intersects the calculated fuzzy number, is an approximation to it. The relevance index allows one to tell which members of a fuzzy partition are better approximations than others.

4.3. XTRIG implementation

Examples of fuzzy qualitative trigonometric functions using the fuzzy partitions generated in Section 3.4 are given in this section. First consider the fuzzy qualitative sine function, the result of the sine function of the 3rd fuzzy qualitative orientation angle is as follows,

$$\sin(QS_a(3)) = {}_0\begin{bmatrix} 0.5583 & 0.6417 & 0.0167 & 0.0167 \\ 0.6583 & 0.7417 & 0.0167 & 0.0167 \\ 0.7583 & 0.8417 & 0.0167 & 0.0167 \\ 1 & NaN & NaN & 2 \end{bmatrix}$$

In this case there are three fuzzy numbers within the fuzzy qualitative range (i.e., $|A'_yB'_y|$) and it shows that its forward relevance relation is weak-approximation and its backward relevant relation is bare-approximation. The results of the rest fuzzy qualitative trigonometric values of the 3rd orientation angle are,

$$\cos(QS_a(114))=_0\begin{bmatrix}0.7583 & 0.8417 & 0.0167 & 0.0167\\ 0.8583 & 0.9417 & 0.0167 & 0.0167\\ 0.9583 & 1.0417 & 0.0167 & 0.0167\\ 2 & NaN & NaN & 2\end{bmatrix}$$

$$\csc(QS_a(3))=_0\begin{bmatrix}1.5584 & 1.7910 & 0.0395 & 0.0551\\ 1.3483 & 1.5190 & 0.0296 & 0.0395\\ 1.1881 & 1.3187 & 0.0231 & 0.0296\\ 1 & NaN & NaN & 2\end{bmatrix}$$

$$\tan(QS_a(3))=_0\begin{bmatrix}0.6634 & 0.8462 & 0.0323 & 0.0415\\ 0.8876 & 1.1266 & 0.0415 & 0.0552\\ 1.1818 & 1.5075 & 0.0552 & 0.0772\end{bmatrix}$$

$$\arccos(QS_d(20))=_0\begin{bmatrix}0.0365 & 0.0885 & 0.0104 & 0.0104\\ 1 & NaN & NaN & 2\end{bmatrix}$$

There are two points, which should be noted, in the above examples. First, fuzzy qualitative trigonometric functions have periodic characteristics, e.g., $\cos(QS_a(114)) = \cos(QS_a(2+16\times 7))$. Secondly, fuzzy qualitative tangent and cotangent functions are calculated based on fuzzy qualitative sine, cosine functions and the fundamental trigonometric identity shown in Eqs. (9a)-(9c). The use of the latter ensures the elimination of those fuzzy numbers that are the results of fuzzy qualitative sine and cosine functions but do not have geometric meaning. For instance, before applying Eqs. (9a) and (9b), the result of function $tan(QS_a(3))$ are 9 fuzzy numbers, however, 6 of them do not meet their fundamental trigonometry identity.

5. Fuzzy qualitative trigonometric compound angles

The characteristics of FQT are presented in this section. This includes related value rules and compound angle formulae.

5.1. FQT characteristic rules

Fuzzy qualitative characteristic rules of trigonometric compound angle formulae and related value rules can be achieved by the extension principle and origin symmetry of fuzzy partitions [48,49,53]. Zadeh's extension principle is an important theoretical foundation and tool in fuzzy set theory and applications. It is demonstrated that each function Let $f: X \to Y$ $(X \in \mathbb{R}^n,\ Y \in \mathbb{R}^n)$ induces a corresponding function $\hat{f}: F(X) \to F(Y)$. It is proved that if f is continuous, then $\hat{f}: F(X,D) \to F(Y,D)$ is also continuous, D is the supremum over Hausdorff distances between their corresponding level sets. It leads to the following lemma for general fuzzy qualitative characteristics,

FQT Lemma. *If $y = RV(x)$ is the set of trigonometric related values, including supplementary, complementary, opposite and anti-supplementary characteristic, which map a fuzzy partition X to a fuzzy partition Y $(x \in \bigcup FQT(Q_X),\ y \in \bigcup FQT(Q_Y))$, where $FQT(\,)$ is one of fuzzy qualitative trigonometric functions in Section 4.1, and $Q_x,\ Q_y \in Q$ in Eq. (1), then with A being a fuzzy set in X, function $RV(\,)$ maps from A to a fuzzy set B in Y such that,*

$$\mu_B(y)=_\alpha \sup_{x\in RV^{-1}(y),\alpha\in[0,1]} \mu_A(x) \tag{11}$$

With the definition for the support of a fuzzy set, we obtain,

$$RV(supp(A))=_0 supp(B) \quad (\alpha = 0) \tag{12}$$

Fuzzy numbers herein are fuzzy sets of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. $supp(A)$ stands for the support of a fuzzy set A. Let $QS_a(i),\ QS_a(j)$ be the ith and jth qualitative state of two orientation angles, their relationship can be derived in terms of geometric analysis, fuzzy qualitative rules can be derived as follows,

$$\begin{array}{ll}\text{FQT supplementary} & \text{if } i+j=\frac{p}{2}+2\\ \text{FQT complementary} & \text{if } i+j=\frac{p}{4}+2\\ \text{FQT opposite} & \text{if } i+j=p+2\\ \text{FQT anti-supplementary} & \text{if } i-j=\frac{p}{2}\end{array} \tag{13}$$

For instance, consider two fuzzy qualitative states in Fig. 8, which are fuzzy qualitative supplementary. This clearly represents the fuzzy qualitative relation,

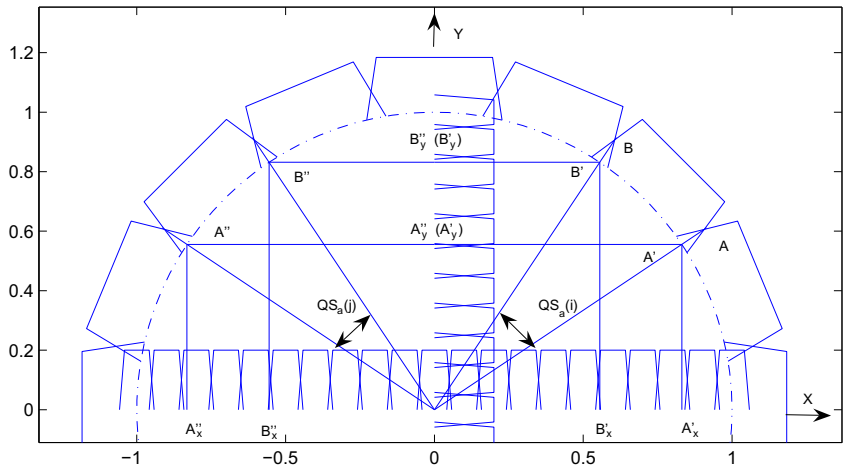

Fig. 8. Fuzzy qualitative supplementary.

$$\sin(QS_a(i)) =_\alpha \frac{QS_d\left(\left|A'_yB'_y\right|\right)}{[1\quad 1\quad 0\quad 0]} =_\alpha \frac{QS_d\left(\left|A''_yB''_y\right|\right)}{[1\quad 1\quad 0\quad 0]} =_\alpha \sin(QS_a(j)) \tag{14a}$$

$$\cos(QS_a(i)) =_\alpha \frac{QS_d\left(\left|A'_xB'_x\right|\right)}{[1\quad 1\quad 0\quad 0]} =_\alpha -\frac{QS_d\left(\left|A''_xB''_x\right|\right)}{[1\quad 1\quad 0\quad 0]} =_\alpha -\cos(QS_a(j)) \tag{14b}$$

where $QS_a(i) + QS_a(j) =_\alpha \frac{p}{2} + 1$. Substituting Eqs. (14a) and (14b) into Eqs. (9a) and (9b), we obtain,

$$\tan(QS_a(i)) =_\alpha -\tan(QS_a(j)), \quad \cot(QS_a(i)) =_\alpha -\cot(QS_a(j)) \tag{15}$$

Furthermore, the other relations of the fuzzy qualitative functions can be derived, similarly. When the relation of $QS_a(i) + QS_a(j)$ is fuzzy qualitative opposite, they are,

$$\begin{aligned} &\sin(QS_a(i)) =_\alpha -\sin(QS_a(j)), \quad \cos(QS_a(i)) =_\alpha \cos(QS_a(j)) \\ &\tan(QS_a(i)) =_\alpha -\tan(QS_a(j)), \quad \cot(QS_a(i)) =_\alpha -\cot(QS_a(j)) \end{aligned} \tag{16}$$

When the relation of $QS_a(i) + QS_a(j)$ is fuzzy qualitative complementary, they are:

$$\begin{aligned} &\sin(QS_a(i)) =_\alpha \cos(QS_a(j)), \quad \cos(QS_a(i)) =_\alpha \sin(QS_a(j)) \\ &\tan(QS_a(i)) =_\alpha \cot(QS_a(j)), \quad \cot(QS_a(i)) =_\alpha \tan(QS_a(j)) \end{aligned} \tag{17}$$

When the relation of $QS_a(i) + QS_a(j)$ is fuzzy qualitative anti-supplementary, they are:

$$\begin{aligned} &\sin(QS_a(i)) =_\alpha -\sin(QS_a(j)), \quad \cos(QS_a(i)) =_\alpha -\cos(QS_a(j)) \\ &\tan(QS_a(i)) =_\alpha \tan(QS_a(j)), \quad \cot(QS_a(i)) =_\alpha \cot(QS_a(j)) \end{aligned} \tag{18}$$

Further applying the extension principle and Eqs. (11) and (12) to arithmetic operations in Table 1, fuzzy qualitative compound angle formulae can be derived in terms of α-cut values. The compound angle formulae for sines and cosines of the sum and difference of two fuzzy qualitative angles $QS_a(i)$ and $QS_a(j)$ are,

$$\sin(QS_a(i) + QS_a(j)) =_\alpha \sin(QS_a(i))\cos(QS_a(j)) + \cos(QS_a(i))\sin(QS_a(j)) \tag{19a}$$

$$\sin(QS_a(i) - QS_a(j)) =_\alpha \sin(QS_a(i))\cos(QS_a(j)) - \cos(QS_a(i))\sin(QS_a(j)) \tag{19b}$$

$$\cos(QS_a(i) + QS_a(j)) =_\alpha \cos(QS_a(i))\cos(QS_a(j)) - \sin(QS_a(i))\sin(QS_a(j)) \tag{19c}$$

$$\cos(QS_a(i) - QS_a(j)) =_\alpha \cos(QS_a(i))\cos(QS_a(j)) + \sin(QS_a(i))\sin(QS_a(j)) \tag{19d}$$

$$\tan(QS_a(i) + QS_a(j)) =_\alpha \frac{\tan(QS_a(i)) + \tan(QS_a(j))}{1 - \tan(QS_a(i))\tan(QS_a(j))} \tag{19e}$$

$$\tan(QS_a(i) - QS_a(j)) =_\alpha \frac{\tan(QS_a(i)) - \tan(QS_a(j))}{1 + \tan(QS_a(i))\tan(QS_a(j))} \tag{19f}$$

The fuzzy qualitative counterpart of the compound angle formulae of the sum and difference of sines and cosines are derived as provided in Eqs. (19a)–(19f). For instance, as shown in Eq. (19b), fuzzy qualitative operation and aggregation are applied to $\sin(QS_a(i))\cos(QS_a(j))$ and $\cos(QS_a(i))\sin(QS_a(j))$, and their fuzzy qualitative subtraction. Similarly, the double angle formulae for two fuzzy qualitative angles $QS_a(i)$ and $QS_a(j)$ are derived as follows,

$$\sin 2(QS_a(i)) =_\alpha 2\sin(QS_a(i))\cos(QS_a(j)) \tag{20a}$$

$$\cos 2(QS_a(i)) =_\alpha 1 - 2\sin^2(QS_a(i)) \tag{20b}$$

$$\tan 2(QS_a(i)) =_\alpha \frac{2\tan(QS_a(i))}{1-\tan^2(QS_a(i))} \tag{20c}$$

$$\sin 3(QS_a(i)) =_\alpha 3\sin(QS_a(i)) - 4\sin^3(QS_a(j)) \tag{20d}$$

It is given below that products of sines and cosines are changed into sums or difference for two fuzzy qualitative angles $QS_a(i)$ and $QS_a(j)$,

$$\sin(QS_a(i))\cos(QS_a(j)) =_\alpha \frac{1}{2}[\sin(QS_a(i)+QS_a(j)) + \sin(QS_a(i)-QS_a(j))] \tag{21a}$$

$$\cos(QS_a(i))\sin(QS_a(j)) =_\alpha \frac{1}{2}[\sin(QS_a(i)+QS_a(j)) - \sin(QS_a(i)-QS_a(j))] \tag{21b}$$

$$\cos(QS_a(i))\cos(QS_a(j)) =_\alpha \frac{1}{2}[\cos(QS_a(i)+QS_a(j)) + \cos(QS_a(i)-QS_a(j))] \tag{21c}$$

$$\sin(QS_a(i))\sin(QS_a(j)) =_\alpha -\frac{1}{2}[\sin(QS_a(i)+QS_a(j)) - \sin(QS_a(i)-QS_a(j))] \tag{21d}$$

On the other hand, change sums or differences of sines and cosines into products for two fuzzy qualitative angles $QS_a(i)$ and $QS_a(j)$ are provided as below,

$$\sin(QS_a(i)) + \sin(QS_a(j)) =_\alpha 2\left[\sin\left(g\left(\frac{QS_a(i)+QS_a(j)}{2}\right)\right)\cos\left(g\left(\frac{QS_a(i)-QS_a(j)}{2}\right)\right)\right] \tag{22a}$$

$$\sin(QS_a(i)) - \sin(QS_a(j)) =_\alpha 2\left[\cos\left(g\left(\frac{QS_a(i)+QS_a(j)}{2}\right)\right)\sin\left(g\left(\frac{QS_a(i)-QS_a(j)}{2}\right)\right)\right] \tag{22b}$$

$$\cos(QS_a(i)) + \cos(QS_a(j)) =_\alpha 2\left[\cos\left(g\left(\frac{QS_a(i)+QS_a(j)}{2}\right)\right)\cos\left(g\left(\frac{QS_a(i)-QS_a(j)}{2}\right)\right)\right] \tag{22c}$$

$$\cos(QS_a(i)) - \cos(QS_a(j)) =_\alpha -2\left[\sin\left(g\left(\frac{QS_a(i)+QS_a(j)}{2}\right)\right)\sin\left(g\left(\frac{QS_a(i)-QS_a(j)}{2}\right)\right)\right] \tag{22d}$$

where $g(QS_a(X))$ is a function which rounds $QS_a(X)$ to the nearest fuzzy qualitative angle. It is demonstrated that the two versions have similar equation forms though the calculation elements are in different forms, i.e., numerical and qualitative representations, respectively. It is evident that the extension principle extends classical mathematical operators to the fuzzy qualitative domain. The extension mapping is computationally costly due to the individual fuzzy qualitative operations as provided in Table 1 and aggregation [42,43].

5.2. XTRIG implementation

The examples of the related values of fuzzy qualitative trigonometric functions are given in this section based on the content in Sections 3.4 and 4.3, the supplementary, complementary, opposite and anti-supplementary values of the 3rd orientation angle are calculated to demonstrate the correctness of the derived rules,

(1) *FQT supplementary:*

$$\sin\left(QS_a\left(\frac{p}{2}+2-3\right)\right) =_0 \sin(QS_a(7)) =_0 \begin{bmatrix} 0.5583 & 0.6417 & 0.0167 & 0.0167 \\ 0.6583 & 0.7417 & 0.0167 & 0.0167 \\ 0.7583 & 0.8417 & 0.0167 & 0.0167 \\ 1.000 & NaN & NaN & 2.0000 \end{bmatrix}$$

(2) *FQT complementary:*

$$\cos\left(QS_a\left(\frac{p}{4}+2-3\right)\right) =_0 \cos(QS_a(3)) =_0 \begin{bmatrix} 0.5583 & 0.6417 & 0.0167 & 0.0167 \\ 0.6583 & 0.7417 & 0.0167 & 0.0167 \\ 0.7583 & 0.8417 & 0.0167 & 0.0167 \\ 1.0000 & NaN & NaN & 2.0000 \end{bmatrix}$$

(3) *FQT opposite:*

$$\sin(QS_a(p+2-3)) =_0 \sin(QS_a(15)) =_0 \begin{bmatrix} -0.8417 & -0.7583 & 0.0167 & 0.0167 \\ -0.7417 & -0.6583 & 0.0167 & 0.0167 \\ -0.6417 & -0.5583 & 0.0167 & 0.0167 \\ 2.0000 & NaN & NaN & 1.0000 \end{bmatrix}$$

(4) *FQT anti-supplementary:*

$$\sin\left(QS_a\left(\frac{p}{2}+3\right)\right)=_0 \sin(QS_a(11))=_0 \begin{bmatrix} -0.8417 & -0.7583 & 0.0167 & 0.0167 \\ -0.7417 & -0.6583 & 0.0167 & 0.0167 \\ -0.6417 & -0.5583 & 0.0167 & 0.0167 \\ 2.0000 & NaN & NaN & 1.0000 \end{bmatrix}$$

6. Fuzzy qualitative triangle theorems

Fuzzy qualitative triangle theorems are presented based on the proposed fuzzy qualitative trigonometric functions. The role that the counterparts of the fuzzy qualitative triangle theorems play in the quantitative geometry indicates its contribution to fuzzy qualitative calculation and analysis. First, a fuzzy qualitative triangle is defined as,

Definition I. Three fuzzy qualitative angles are denoted as $QS_a(A)$, $QS_a(B)$ and $QS_a(C)$, and three fuzzy qualitative sides are denoted as $QS_d(a)$, $QS_d(b)$ and $QS_d(c)$, also, each side is opposite to its corresponding fuzzy qualitative angle, (e.g., side $QS_d(a)$ is opposite to angle $QS_a(A)$). The constructed shape is a fuzzy qualitative triangle (i.e., $QS_\Delta(ABC)$), iff the following holds,

$$QS_d(a)+QS_d(b)>_\alpha QS_d(c) \tag{23}$$

where it means that the addition of the support of sides $QS_d(a)$ and $QS_d(b)$ is strictly greater than the that of side $QS_d(c)$ in terms of α cut values, for which description α-greater is the shorthand, please refer to Eqs. (11) and (12) and the extension principle. Based on the definition of a fuzzy qualitative triangle and fuzzy partition distribution in Section 3.1, a fuzzy qualitative acute triangle, fuzzy qualitative right-angled triangle and fuzzy qualitative obtuse triangle can be defined, they are given as follows,

Definition II. A fuzzy qualitative acute triangle is a fuzzy qualitative triangle, each of whose angles is strictly α-less than a fuzzy qualitative right angle (i.e., $QS_a(\frac{p}{4}+1)$).

Definition III. A fuzzy qualitative right-angled triangle is a fuzzy qualitative triangle, one of whose angles is α-equal to a fuzzy qualitative right angle.

Definition IV. A fuzzy qualitative obtuse triangle is a fuzzy qualitative triangle, one of whose angles is strictly α-greater than a fuzzy qualitative right angle. This classification of fuzzy qualitative triangles provides a theoretical platform for deriving fuzzy qualitative sine and cosine rules, and triangle theorems, which play a crucial role in the qualitative analysis of trigonometry. With the extension principle, the approximation principle and lemmas (11) and (12) in mind, we can derive fuzzy qualitative sine and cosine rules and triangle theorems.

6.1. Sine and cosine rules

Fuzzy qualitative sine and cosine rules are derived in this section and they are required to play the same role in fuzzy qualitative terms as their counterparts do in conventional trigonometry. The area $QS_\Delta(S)$ of a fuzzy qualitative triangle $QS_\Delta(ABC)$ can be calculated from the perspective of the three sides,

$$\begin{aligned} S_1 &=_\alpha \frac{QS_d(a)QS_d(c)\sin(QS_a(B))}{2} \\ S_2 &=_\alpha \frac{QS_d(b)QS_d(c)\sin(QS_a(A))}{2} \\ S_3 &=_\alpha \frac{QS_d(a)QS_d(b)\sin(QS_a(C))}{2} \end{aligned} \tag{24}$$

Clearly $S_1=_\alpha S_2=_\alpha S_3$ can be reached since all three describe the same area of a fuzzy qualitative triangle. The fuzzy qualitative version of the sine rule can be derived by dividing $QS_d(a)QS_d(b)QS_d(c)$ into Eq. (24),

$$\frac{QS_d(a)}{\sin(QS_a(A))}=_\alpha \frac{QS_d(b)}{\sin(QS_a(B))}=_\alpha \frac{QS_d(c)}{\sin(QS_a(C))} \tag{25}$$

The sine rule relates the sides and angles of a fuzzy qualitative triangle, stating that the ratio of the length of each side and the sine of the angle opposite is α-equal to a fuzzy constant. This allows calculation of any unknown fuzzy qualitative sides and angles, provided that some of the sides and angles in the triangle are known. The cosine rule can be derived using the fuzzy qualitative multiplication shown in Table 1. For instance,

$$\|QS_d(b)\|^2 =_\alpha QS_d(b) \cdot QS_d(b) =_\alpha (QS_d(a) - QS_d(c)) \cdot (QS_d(a) - QS_d(c)) =_\alpha QS_d(a) \cdot QS_d(a) + QS_d(c) \cdot QS_d(c) - 2 \cdot QS_d(a) \cdot QS_d(c) =_\alpha \|QS_d(a)\|^2 + \|QS_d(c)\|^2 - 2 \cdot \|QS_d(a)\| \cdot \|QS_d(c)\| \cdot \cos(\angle B) \tag{26}$$

Likewise, the other two fuzzy qualitative sides can be derived in the same way. The fuzzy qualitative cosine rule also provides the same facility as the sine rule does to calculate any unknown side and angle, provided that some of the sides and angles are known in the triangle.

6.2. Triangle theorems

In this section we present the abstraction of the triangle theorems into FQT. These include *AAA*, *AAS*, *ASA*, *ASS*, *SAS* and *SSS*, where *A* stands for a fuzzy qualitative angle of a fuzzy qualitative triangle, *S* stands for a side. The notation is the same as those in the fuzzy qualitative triangle definition. Recall from Section 2.1, that we use the members of fuzzy partitions to describe the result of each fuzzy qualitative operation in Table 1. It provides better performance of the approximation principle for fuzzy number selection in fuzzy qualitative arithmetic.

(1) *FQT AAA Theorem:* Specifying two angles of a fuzzy qualitative triangle automatically gives the third angle in fuzzy qualitative terms since the sum of angles in such a triangle sums to $QS_a(p/2+2)$ whose center is π. Recall from Section 3.1, p is the number of fuzzy qualitative states of a full orientation. The FQT *AAA* theorem can be described as,

$$QS_a(C) =_\alpha QS_a\left(\frac{p}{2} + 2\right) - QS_a(A) - QS_a(B) \tag{27}$$

(2) *FQT AAS Theorem:* Specifying two fuzzy qualitative angles $QS_a(A)$ and $QS_a(B)$ and a side $QS_d(a)$ determines a fuzzy qualitative triangle with its area,

$$S =_\alpha \frac{QS_d^2(a)\sin(QS_a(B))\sin(QS_a(C))}{2\sin(QS_a(A))} =_\alpha \frac{QS_d^2(a)\sin(QS_a(B))\sin\left(\frac{p}{2} - QS_a(B) - QS_a(B)\right)}{2\sin(QS_a(A))}$$

By applying the sine rule given in Eq. (25), we obtain,

$$QS_d(b) =_\alpha QS_d(a)\frac{\sin(QS_a(B))}{\sin(QS_a(A))} \tag{28}$$

Then

$$QS_d(c) =_\alpha QS_d(b)\cos(QS_a(A)) + QS_d(a)\cos(QS_a(B)) \tag{29}$$

(3) *FQT ASA Theorem:* Specifying two adjacent fuzzy qualitative angles $QS_a(A)$ and $QS_a(B)$ and the fuzzy qualitative side between them $QS_d(c)$ determines a fuzzy qualitative triangle with its area:

$$S =_\alpha \frac{QS_d^2(c)}{2(\cos(QS_a(A)) + \cos(QS_a(B)))}$$

The angle $QS_a(C)$ can be calculated by using the *AAA* theorem. The other two sides are,

$$\begin{aligned} QS_d(a) &=_\alpha \frac{\sin(QS_a(A))}{\sin\left(\frac{p}{2} - QS_a(A) - QS_a(B)\right)} QS_d(c) \\ QS_d(b) &=_\alpha \frac{\sin(QS_a(B))}{\sin\left(\frac{p}{2} - QS_a(A) - QS_a(B)\right)} QS_d(c) \end{aligned} \tag{30}$$

(4) *FQT ASS Theorem:* Specifying two adjacent fuzzy qualitative side lengths $QS_d(a)$ and $QS_d(c)$ of a triangle $(QS_d(a) < QS_d(c))$ and one acute fuzzy qualitative angle $QS_a(A)$ opposite $QS_d(a)$ does not, in general, determine a fuzzy qualitative triangle. The number of possible triangles satisfying the given conditions, n, is given by,

$$n = \begin{cases} 2 & \text{if } \sin(QS_a(A)) <_\alpha \frac{QS_d(a)}{QS_d(c)} \\ 1 & \text{if } \sin(QS_a(A)) =_\alpha \frac{QS_d(a)}{QS_d(c)} \\ 0 & \text{if } \sin(QS_a(A)) >_\alpha \frac{QS_d(a)}{QS_d(c)} \end{cases} \tag{31}$$

(5) *FQT SAS Theorem:* Specifying two fuzzy qualitative sides and the fuzzy qualitative angle between them determines a fuzzy qualitative triangle. The length of the third fuzzy qualitative side is given by the cosine rule,

$$QS_d(b) =_\alpha \sqrt{QS_d^2(a) + QS_d^2(c) - 2QS_d(a)QS_d(c)\cos(QS_a(B))} \tag{32}$$

we obtain the following by employing the sine rule in Eq. (25),

$$
\begin{aligned}
QS_a(A) &=_\alpha \sin^{-1}(QS_d(a))\sin(QS_a(B)) \Big/ \sqrt{QS_d^2(a)+QS_d^2(c)-2QS_d(a)QS_d(c)\cos(QS_a(B))} \\
QS_a(C) &=_\alpha \sin^{-1}(QS_d(c))\sin(QS_a(B)) \Big/ \sqrt{QS_d^2(a)+QS_d^2(c)-2QS_d(a)QS_d(c)\cos(QS_a(B))} \\
QS_d(b) &=_\alpha \sqrt{QS_d(a)+QS_d(c)-2QS_d(a)QS_d(c)\cos(QS_a(B))}
\end{aligned}
\tag{33}
$$

(6) *FQT SSS Theorem:* If the three fuzzy qualitative sides of any fuzzy qualitative triangle $QS_d(a)$, $QS_d(b)$ and $QS_d(c)$ are specified, the area of the triangle is given by Heron's formula and the extension principle,

$$
\begin{aligned}
S &=_\alpha \sqrt{QS_d(s)(QS_d(s)-QS_d(a))}\cdot\sqrt{(QS_d(s)-QS_d(b))(QS_d(s)-QS_d(c))} \\
QS_d(s) &=_\alpha \frac{1}{2}(QS_d(a)+QS_d(b)+QS_d(c))
\end{aligned}
$$

applying the cosine rule in Eq. (26), it yields the following equations,

$$
\begin{aligned}
QS_a(A) &=_\alpha \cos^{-1}\left(\frac{QS_d^2(b)+QS_d^2(c)-QS_d^2(a)}{2QS_d(b)QS_d(c)}\right) \\
QS_a(B) &=_\alpha \cos^{-1}\left(\frac{QS_d^2(a)+QS_d^2(c)-QS_d^2(b)}{2QS_l(a)QS_d(c)}\right) \\
QS_a(C) &=_\alpha \cos^{-1}\left(\frac{QS_d^2(a)+QS_d^2(b)-QS_d^2(c)}{2QS_d(a)QS_d(b)}\right)
\end{aligned}
\tag{34}
$$

7. Concluding remarks

A fuzzy qualitative description of conventional trigonometry has been presented in this paper. The unit circle of conventional trigonometry has been modified by the introduction of fuzzy partitions for orientation and translation. Conventional trigonometric functions (e.g., a sine function) and rules (e.g., the sine rule) have been abstracted to give fuzzy qualitative versions of these operations. In addition, fuzzy qualitative versions of the conventional triangle theorems are also provided in FQT. Examples have been given throughout the paper to demonstrate FQT's ability. One of these focuses on robot kinematics and explains how contributions could be made by a new type of FQT-based kinematics to achieve the intelligent connection of low-level sensing & control tasks to high-level symbolic tasks.

The proposed fuzzy qualitative variables become linguistic variables or symbolic variables when they are modified with descriptive symbols [54–56]. Besides, methods such as aggregation operators [57] can be used to select specific symbols from a set of fuzzy qualitative states. Hence, fuzzy qualitative states in FQT can be denoted by symbolic terms which provide a promising base for behavioral description in robotics [58]. For example, Jenkins and Mataric [59] have provided a skill-level interface named behavior vocabulary for a humanoid robot. However, the connection between the behavior vocabulary and low-level sensing and control tasks is still uncertain. FQT is an abstraction of conventional trigonometry into the domain of fuzzy logic and qualitative reasoning. This version of trigonometry can replace the role that conventional trigonometry plays in robot kinematics; so a general robot kinematics can be derived based on FQT. The new type of kinematics handles fuzzy qualitative states that allow access to both numerical data and symbolic data. Further, a FQT-based robot kinematics could provide a transit layer for communicating variables, even variable-based cognitive functions of knowledge-level tasks and numerical sensing & motion control [45,46]. From an implementation perspective, a robotic system with this type of kinematics can be easily fitted into a conventional fuzzy model which consists of a fuzzification unit, knowledge base, inference engine and defuzzification unit. The FQT-based reasoning could play a crucial role in an inference engine whose variables are supported by a knowledge base. The output of an inference engine is able to access knowledge-based systems, e.g., symbolic planning subsystems. The fuzzification and defuzzification units are able to provide low-level sensing and control tasks, it is obvious this role can be easily replaced by other techniques, e.g., fuzzy clustering.

Fuzzy qualitative reasoning seeks to harness the strengths of fuzzy reasoning and qualitative reasoning. It is in a position to play a crucial role in closing the gap between low-level sensing & control tasks and high-level symbolic description. Our future work will focus on the development of an intelligent architecture using the FQT in the domain of AI robotics. The architecture is composed of a low-level handler, knowledge-based handler, and inference handler. The low-level handler provides an interface to numerical data; it is composed of a fuzzification unit and defuzzification unit, in each unit different techniques can be applied such as fuzzy clustering. A knowledge-based handler not only provides a knowledge base to support the system inferences, but also facilitates human supervision (e.g., cognitive inputs). FQT is the core of the inference handler which should be able to, scalable and in parallel interface both low-level handler and knowledge-based handler.

Acknowledgements

The project was funded by the Engineering and Physical Science Research Council, United Kingdom, under Grant GR/S10773/01, GR/S10766/01 and EP/G041377/1; the authors would like to thank the anonymous reviewers for constructive comments that help improve the quality of this manuscript.

References

[1] P. Bourseau, K. Bousson, J.I. Dormoy, J.M. Evrard, F. Guerrin, L. Leyval, O. Lhomme, B. Lucas, A. Missier, J. Montmain, N. Piera, N. Rakoto-Ravalontsalama, M. Steyer, J.P. Tomasena, I. Trave-Massuyes, M. Vescovi, S. Xanthakis, B. Yannou, Qualitative reasoning: a survey of techniques and applications, AI communications 8 (3/4) (1995) 119–193.
[2] B. Kuipers, Qualitative Reasoning, MIT Press, 1994.
[3] D. Berleant, Qualitative and quantitative simulation: bridging the gap, Artificial Intelligence Journal 95 (2) (1997) 215–255.
[4] A.F. Blackwell, Spatial Reasoning for Robots: A Qualitative Approach, Master Thesis, Victoria University of Wellington.
[5] Q. Shen, R. Leitch, Fuzzy qualitative simulation, IEEE Transactions on Systems, Man and Cybernetics 23 (4) (1993) 1038–1061.
[6] G.M. Coghill, Mycroft: A Framework for Constraint-Based Fuzzy Qualitative Reasoning, Ph.D. Thesis, Heriot-Watt University, 1996.
[7] K.D. Forbus, P. Nielsen, B. Faltings, Qualitative kinematics: a framework, in: Proceedings of the International Joint Conference on Artificial Intelligence, 1987, pp. 430–435.
[8] P.E. Nielsen, A Qualitative Approach to Rigid Body Mechanics, Ph.D. Thesis, University of Illinois at Urbana-Champaign, 1988.
[9] B. Faltings, A symbolic approach to qualitative kinematics, Artificial Intelligence 56 (2–3) (1992) 139–170.
[10] L. Magdalena, F. Monasterio-Huelin, A fuzzy logic controller with learning through the evolution of its knowledge base, International Journal of Approximate Reasoning 16 (3–4) (1997) 335–358.
[11] J.J. Buckley, E. Eslami, An Introduction to Fuzzy Logic and Fuzzy Sets, Springer-Verlag, 2002.
[12] J. Liu, A method of spatial reasoning based on qualitative trigonometry, Artificial Intelligence 98 (1–2) (1998) 137–168.
[13] E. Smith, J. Eloff, Cognitive fuzzy modelling for enhanced risk assessment in a health care institution, IEEE Intelligent Systems (2002) 69–75.
[14] J.W.T. Lee, D.S. Yeung, E.C.C. Tsang, Ordinal fuzzy sets, IEEE Transactions on Fuzzy Systems 10 (6) (2002) 767–778.
[15] A.H. Ali, D. Dubois, H. Prade, Qualitative reasoning based on fuzzy relative orders of magnitude, IEEE Transactions on Fuzzy Systems 11 (1) (2003) 9–23.
[16] Y. Li, S. Li, A fuzzy sets theoretic approach to approximate spatial reasoning, IEEE Transactions on Fuzzy Systems 12 (6) (2004) 745–754.
[17] R. Fuller, On fuzzy reasoning schemes, in: C. Carlsson (Ed.), The State of the Art of Information Systems Application in 2007, vol. 16, TUCS General Publications, 1999, pp. 85–112.
[18] B.S. Chen, Y.S. Yang, B.K. Lee, T.H. Lee, Fuzzy adaptive predictive flow control of atm network traffic, IEEE Transactions on Fuzzy Systems 11 (4) (2003) 568–581.
[19] M. Setnes, H. Roubos, Ga-fuzzy modelling and classification: complexity and performance, IEEE Transactions on Fuzzy Systems 8 (5) (2000) 35–44.
[20] E. Pedrycz, M. Reformat, Evolutionary fuzzy modelling, IEEE Transactions on Fuzzy Systems 11 (5) (2003) 652–655.
[21] A. Fernndez, M.J. Jesus, F. Herrera, Hierarchical fuzzy rule based classification systems with genetic rule selection for imbalanced data-sets, International Journal of Approximate Reasoning 50 (3) (2009) 561–577.
[22] M. Antonelli, P. Ducange, B. Lazzerini, F. Marcelloni, Learning concurrently partition granularities and rule bases of mamdani fuzzy systems in a multi-objective evolutionary framework, International Journal of Approximate Reasoning 50 (2009) 1066–1080.
[23] E.C.C. Tsang, X.Z. Wang, D.S. Yeung, Improving learning accuracy of fuzzy decision trees by hybrid neural networks, IEEE Transactions on Fuzzy Systems 8 (5) (2000) 601–614.
[24] P. Pulkkinen, H. Koivisto, Fuzzy classifier identification using decision tree and multiobjective evolutionary algorithms, International Journal of Approximate Reasoning 48 (2) (2008) 526–543.
[25] P. Vuorimaa, T. Jukarainen, E. Karpanoja, A neuro-fuzzy system for chemical agent detection, IEEE Transactions on Fuzzy Systems 3 (4) (1995) 415–424.
[26] A. Ciaramella, R. Tagliaferri, W. Pedrycz, A. Di Nola, Fuzzy relational neural network, International Journal of Approximate Reasoning 41 (2) (2006) 146–163.
[27] L.C. Hung, H.Y. Chung, Decoupled sliding-mode with fuzzy-neural network controller for nonlinear systems, International Journal of Approximate Reasoning 46 (1) (2007) 74–97.
[28] E. Smith, J. Eloff, Cognitive fuzzy modelling for enhanced risk assessment in a health care institution, IEEE Intelligent Systems 15 (2) (2000) 69–75.
[29] D.G. Schwartz, A system for reasoning with imprecise linguistic information, International Journal of Approximate Reasoning 5 (5) (1991) 463–488.
[30] H.K. Kwan, Y. Cai, A fuzzy neural network and its application to pattern recognition, IEEE Transactions on Fuzzy Systems 2 (30) (1994) 185–193.
[31] B. Ben Ghalia, P.P. Wang, Intelligent system to support judgmental business forecasting: the case of estimating hotel room demand, IEEE Transactions on Fuzzy Systems 8 (4) (2000) 380–397.
[32] S.M. Bae, S.C. Park, S.H. Ha, Fuzzy web ad selector based on web usage mining, IEEE Intelligent Systems 18 (6) (2003) 62–69.
[33] J. Casillas, F. Herrera, R. PTrez, M. Jesus, P. Villar, Special issue on genetic fuzzy systems and the interpretability-accuracy trade-off, International Journal of Approximate Reasoning 44 (1) (2007) 1–3.
[34] D.S. Weld, J. de Kleer (Eds.), Reading in Qualitative Reasoning About Physical Systems, Morgan Kaufman, San Mateo, CA, 1990.
[35] B.C. Williams, J. de Kleer (Eds.), Special issue on qualitative reasoning about physical systems, Artificial Intelligence 51 (1–3) (1991).
[36] B. Faltings, P. Struss (Eds.), Recent Advances in Qualitative Physics, MIT Press, 1992.
[37] B. Bredeweg, P. Struss (Eds.), Special Issues on Qualitative Reasoning, AI Magzine, 2003.
[38] P. Nayak, B. Williams, A model-based approach to reactive self-configuring systems, AAAI-96 (1996) 971–978.
[39] H. de Jong, J.-L. Gouz, C. Hernandez, M. Page, T. Sari, J. Geiselmann, Qualitative simulation of genetic regulatory networks using piecewise-linear models, Bulletin of Mathematical Biology 66 (2) (2004) 301–340.
[40] G.M. Coghill, S.M. Garrett, R.D. King, Learning qualitative metabolic models, in: European Conference on Artificial Intelligence (ECAI'04), 2004.
[41] Honghai Liu, George M. Coghill, David J. Brown, Qualitative kinematics of planar robots: intelligent connection, International Journal of Approximate Reasoning 46 (3) (2007) 525–541.
[42] H. Liu, D.J. Brown, G.M. Coghill, Fuzzy qualitative robot kinematics, IEEE Transactions on Fuzzy Systems 16 (3) (2008) 808–822.
[43] H. Liu, A fuzzy qualitative framework for connecting robot qualitative and quantitative representations, IEEE Transactions on Fuzzy Systems 16 (6) (2008) 1522–1530.
[44] H. Liu, G.M. Coghill, Fuzzy qualitative trigonometry, in: Proceedings of the IEEE International Conference on Systems, Man and Cybernetics, Hawaii, USA, 2005, pp. 1291–1296.
[45] M. Brady, Artificial intelligence and robotics, Artificial Intelligence 26 (1985) 79–121.
[46] P. Dpherty, G. Lakemeyer, A.P. del Pobil (Eds.), Proceedings of the Fourth International Cognitive Robotics Workshop, 2004.
[47] K. Forbus, Qualitative Reasoning CRC Hand-Book of Computer Science and Engineering, CRC Press, 1996.
[48] L.A. Zadeh, The concept of linguistic variable and its application to approximate reasoning, Information Sciences 8 (1975) 199–249.
[49] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[50] R. Degani, G. Bortolan, The problem of linguistic approximation in clinical decision making, International Journal of Approximate Reasoning 2 (2) (1988) 143–162.

[51] Z. Huang, Q. Shen, Fuzzy interpolative reasoning via scale and move transformation, IEEE Transactions on Fuzzy Systems 16 (2006).
[52] J.J. Saade, Mapping convex and normal fuzzy sets, Fuzzy Sets and Systems 81 (1996) 251–256.
[53] L.C. Barros, R.C. Bassanezi, H. Roman-Flores, A note on Zadeh's extension, Fuzzy Sets and Systems 117 (2001) 327–331.
[54] L.A. Zadeh, The concept of a linguistic variable and its applications to approximate reasoning – I, Information Sciences 8 (1975) 199–249.
[55] L.A. Zadeh, The concept of a linguistic variable and its applications to approximate reasoning – II, Information Sciences 8 (1975) 301–357.
[56] L.A. Zadeh, The concept of a linguistic variable and its applications to approximate reasoning – III, Information Sciences 9 (1975) 43–80.
[57] G. Mayor, T. Calvo, R. Mesiar, Aggregation Operators, New Trends and Applications, Physica-Verlag, 2002.
[58] R.C. Arkin, Behavior-Based Robotics, MIT Press, 1998.
[59] O.C. Jenkins, M.J. Mataric, Performance-derived behavior vocabularies: data-driven acquisition of skills from motion, International Journal of Humanoid Robotics 1 (2) (2004) 237–289.